GODUNOV-TYPE WAVE DIFFRACTION COMPUTATIONS VIA ROE’S APPROXIMATE Riemann Solver

Z. D. Skoula and C. I. Moutzouris
Laboratory of Harbour Works, National Technical University of Athens
5 Heroon Polytechniou, Zografou, 15780, Athens, Greece
E-mail: zskoula@hydro.ntua.gr

ABSTRACT

The present paper discusses the main features of an elaborate upwind finite volume Godunov-type model of the hyperbolic Mild-Slope Equation that is capable of accommodating the presence of currents in the wave field. Wave transformation in the interface of unstructured triangular cells is computed using Roe’s approximate Riemann solver whose efficiency had mostly hitherto been tested in modelling severe discontinuous flows. The spatial accuracy of wave fluxes is of second order, a quality achieved through the enhancement of the polynomial approximation of the conserved variables slope within each cell. Time integration is materialised implicitly while the scheme is characterised of trapezoidal second order accuracy in time marching. Results presented herein demonstrate that the solution technique employed is capable of depicting smoothly and accurately the wave climate in the vicinity of a harbour entrance.

ΠΡΟΣΟΜΟΙΩΣΗ ΠΕΡΙΘΛΑΣΗΣ ΚΥΜΑΤΙΣΜΩΝ ΜΕΣΩ ΤΗΣ ΠΡΟΣΕΓΓΙΣΗΣ ΤΟΥ ROE ΓΙΑ ΤΟ ΠΡΟΒΛΗΜΑ Riemann

Z. Δ. Σκουλά και Κ. Ι. Μουτζουρης
Εργαστήριο Αμοιβακών Έργων, Εθνικό Μετσόβιο Πολυτεχνείο
Ηρώων Πολυτεχνείου 5, Ζωγράφου, 15780, Αθήνα
E-mail: zskoula@hydro.ntua.gr

ΠΕΡΙΛΗΨΗ

Στην παρούσα εργασία παρουσιάζονται τα βασικά χαρακτηριστικά μιας μεθόδου επίλυσης διαφορικών εξισώσεων τύπου Godunov, όπως αυτή εφαρμόζεται για την προσομοίωση κυματικής διάταξης μέσω της υπερβολικής κυματικής Εξίσωσης Μικρής Κλίσης. Το ομοίωμα είναι δεύτερης τάξης ακριβείας, τόσο ως προς την πεπελεγμένη χρονική, όσο και ως προς την τριγωνικά µή δομμένη χωρική διακριτοποίηση αυτού. Η μέθοδος επίλυσης του ομοίωματος είναι αυτή των πεπερασμένων όγκων όπου η «Πληροφορία» μεταφέρεται μέσω της προσέγγισης του προβλήματος Riemann που έχει προτείνει ο Roe. Τα αποτελέσματα που συνοδεύουν την παρούσα εργασία καταδεικνύουν το χρησιμοποιημένο εδώ τρόπο επίλυσης διαφορικών εξισώσεων ως δόκιμο για την προσομοίωση της κυματικής διάταξης σε προστατευόμενες περιοχές της παράκτιας ζώνης.
1. INTRODUCTION

Environment-related flow fields are dominated by phenomena at a variety of scales that in the nearshore region are characterised with discontinuities. Godunov [1] managed to deal efficiently with discontinuities of flow fields by assuming piecewise constant distributions of the conserved variables within discrete intervals. Schemes based on this concept are first-order accurate. Van Leer [2] introduced a piecewise linear distribution of the conserved variables at computational cell interfaces, giving rise to schemes of second order, see Fig. 1. Having established the variable values at cell interfaces, using a first or higher order scheme, there remains the problem of calculating the evolution of the variables between time steps $n$ and $n+1$. The technique adopted in the present study for tackling this problem is the solution of the Riemann or shock tube problem, see Fig. 1.

The solution of the Riemann problem may employ exact or approximate solvers. Exact Riemann solvers suffer from the fact that they are computationally expensive. Hence, many approximate Riemann solvers have been proposed in literature, see Toro [3] for a comprehensive review. Among them there is the HLL one proposed by Harten et al. [4], which was later modified by Toro et al. [5] to incorporate contact wave behaviour through the HLLC approximation. Osher & Solomon [6] as well as Roe [7] have proposed approximate Riemann solvers that have been extensively used for predicting shocks in compressible aerodynamic flows. Roe’s approach (RA) utilises the eigenstructure of the flux Jacobian matrix of the governing equations, while generally speaking RA is an approximation to the original problem because the Jacobian matrix is replaced by a constant matrix, which is a function of the initial data states. The original problem is therefore replaced by an approximate Riemann solver that is solved exactly. The present study focuses on applying RA to a hyperbolic set of the Mild-Slope Equation and results demonstrate the versatile and smooth nature it can offer in computing the computational wave climate.

2. APPLICATION OF ROE’S APPROXIMATE RIEMANN SOLVER ON A CONSERVATION-LAW FORMULATED SYSTEM WITH IMPLICIT TIME UPDATES

A conservation law formulated system of equations that includes source terms can be written in vector form as

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{H} \quad (1)$$

After introducing integration and usage of Gauss’s divergence theorem, this form yields

$$\frac{\partial}{\partial t} \int_{\Omega} \mathbf{U} d\Omega + \oint_{S} \mathbf{F} n dS = \int_{\Omega} \mathbf{H} d\Omega \quad (2)$$

where $\Omega$ is the two-dimensional domain of interest, $S$ is the boundary surrounding $\Omega$, $\mathbf{n}$ is the unit vector normal to $S$ in the outward direction, $\mathbf{U}$ is the vector of the conserved variables, $\mathbf{H}$ is the vector of source terms, $\mathbf{F}$ is the vector of flux functions through $S$, while $\mathbf{F} \cdot \mathbf{n}$ is decomposed to

$$\mathbf{F} \cdot \mathbf{n} = f \cdot \mathbf{n}_x + g \cdot \mathbf{n}_y \quad (3)$$

with $\mathbf{n}_x$ and $\mathbf{n}_y$ denoting the components of the outward normal vector $\mathbf{n}$ in the $x$ and $y$ directions respectively. Given that a certain set of equations is to be discretised upon cell-centred triangular control volumes, Eq. 2 transforms into the following integral form for each of the control volumes.
\[
\frac{\partial U_i^\Omega_i}{\partial t} = -\oint_{\partial C_i} \mathbf{F} \cdot \mathbf{n} dS + \mathbf{H}_i \Omega_i - \sum_{j \in k(i)} F_{i,j} \Delta l_{i,j} + \mathbf{H}_i \Omega_i = -\text{RHS}(U_i) \tag{4}
\]

where \(U_i, H_i\) are the average quantities of cell \(i\) stored at the cell centre, while \(\Omega_i\) denotes the area of cell \(i\) and \(\partial C_i\) denotes the boundary of cell \(i\). Moreover, \(k(i)\) is a list of the neighbouring cells to cell \(i\), while \(F_{i,j}\) and \(\Delta l_{i,j}\) are the numerical flux through the interface of cells \(i\) and \(j\) and the length of the \(i,j\) edge, respectively.

In order to evaluate the wave fluxes \(F_{i,j}\) a one-dimensional Riemann problem is assumed locally in the direction normal to each cell edge. Roe’s Godunov-type approximation to this problem yields:

\[
F_{i,j} = \frac{1}{2} \left[ F\left(U_{i,j}^{+}\right) + F\left(U_{i,j}^{-}\right) - |A|\left(U_{i,j}^{+} - U_{i,j}^{-}\right) \right] \tag{5}
\]

where \(|A| = R \cdot |\Lambda| \cdot L\) is Roe’s matrix and \(U_{i,j}^{+}, U_{i,j}^{-}\) are the right and left states respectively of the cell face between cells \(i\) and \(j\). Matrices \(R\) and \(L\) are the right and left eigenvector matrices of flux Jacobian \(A\), while \(|\Lambda|\) is a diagonal matrix of the absolute values of the eigenvalues of \(A\). In order to determine the left and right Riemann states at each interface a piecewise linear variation of the conserved variables vector is assumed within each cell rendering the scheme second order accurate in spatial accuracy, see Fig. 1.

Time evolution updates are computed from Eq. 4, which may be expressed as

\[
\left[ (U_i^\Omega_i)^{n+1} - (U_i^\Omega_i)^{n} \right] / \Delta t = -\left[ \alpha_t \text{RHS}(U_i^{n+1}) + (1 - \alpha_t) \text{RHS}(U_i^n) \right] \tag{6}
\]

where \(U_i^{n+1}\) is the vector of variables for cell \(i\) at time level \(n+1\), \(U_i^n\) is the known state at time level \(n\), \(\Delta t\) is the time step, \(\Omega_i\) is the area of cell \(i\) (not constant with respect to time for an adaptive scheme) and \(\text{RHS}(U_i^n)\) is the right hand side of Eq. 4. When \(\alpha_t = 0\) Eq. 6 is the Euler explicit scheme, when \(\alpha_t = 1\) it is the first-order Euler implicit scheme, while when \(\alpha_t = 0.5\) it is a second-order trapezoidal implicit scheme.

**Fig. 1.** Left and right Riemann states at an interface (solid constant piecewise distribution stands for a first-order spatially accurate scheme, while dashed linear piecewise distribution stands for a second-order spatially accurate scheme).
3. THE MILD-SLOPE EQUATION

Skoula [8] formulated a conservation law for Dong’s [9] hyperbolic mild-slope formulation, where the following definitions for \( U \), \( f \), \( g \) and \( H \) hold (in accordance with the information included in the preceding section):

\[
U = \begin{bmatrix} q_x + \eta U_{cx} \\ \frac{\sigma}{\omega} \cdot g \cdot \eta \\ q_y \\ \end{bmatrix}, \quad f = \begin{bmatrix} \frac{\sigma}{\omega} \cdot g \cdot \eta \cdot \frac{q_x + \eta U_{cx}}{\lambda} \\ 0 \\ \frac{\sigma}{\omega} \cdot g \cdot \eta \cdot \frac{q_y}{\lambda} \\ \end{bmatrix}, \quad g = \begin{bmatrix} \frac{\sigma}{\omega} \cdot g \cdot \eta \cdot \frac{(q_x + U_{cx} \eta) \frac{\partial}{\partial x} (\frac{1}{\lambda}) + (q_y + U_{cy} \eta) \frac{\partial}{\partial y} (\frac{1}{\lambda})}{\lambda} \\ \frac{\eta \frac{\partial}{\partial x} (c \cdot c_g \cdot \omega)}{\sigma} \\ \frac{\eta \frac{\partial}{\partial y} (c \cdot c_g \cdot \omega)}{\sigma} \\ \end{bmatrix}, \quad H = \begin{bmatrix} c \cdot c_g \cdot \omega \cdot \sigma \cdot \lambda \\ c \cdot c_g \cdot \omega \cdot \sigma \cdot \lambda \\ c \cdot c_g \cdot \omega \cdot \sigma \cdot \lambda \\ \end{bmatrix}
\]

where \( \lambda = 1 - \frac{\sigma^2 - k^2 c c_g}{\omega \sigma} \), \( \omega = \sigma + k \cdot U_c \), \( \sigma^2 = g k \cdot \tanh(kd) \), \( k = |k| \), \( c = \frac{\sigma}{k} \) and

\[
c_g = \sqrt{g} \left( \frac{\sqrt{\tanh(kd)}}{2 \sqrt{k}} + \frac{d \sqrt{k}}{2 \cosh(kd)^2 \sqrt{\tanh(kd)}} \right)
\]

In equations (7)-(8) term \( k \) is the wave number vector while \( c \), \( c_g \) are the wave celerity and the group celerity respectively, as defined by small-amplitude wave theory, \( \eta \) is the surface elevation, \( \sigma \) is the relative frequency and \( \omega \) the absolute frequency. Moreover, \( d \) stands for the water depth and \( t \) stands for time. Function \( U_c \) is defined as

\[ U_c = U_{cx} \mathbf{i} + U_{cy} \mathbf{j} \]

where \( U_{cx} \) and \( U_{cy} \) stand for the depth averaged velocity components of the wave or otherwise induced current, while \( q_x \) and \( q_y \) components of

\[ \mathbf{Q} = q_x \mathbf{i} + q_y \mathbf{j} \]

are described in the following way:

\[ q_x = c_g \cdot c \cdot \frac{u_o}{g} \]

\[ q_y = c_g \cdot c \cdot \frac{v_o}{g} \]

where \( u_o \) and \( v_o \) stand for the cell-centred mean water level values of horizontal velocity components \( u \), \( v \) in the \( x \), \( y \) directions respectively. It is noted that mean water level in the absence of currents coincides with still water level.
4. APPLICATION: WAVE DIFFRACTION IN THE VICINITY OF A HARBOUR ENTRANCE

Diffraction takes place in the vicinity of barriers such as breakwaters, islets or submerged shoals. The phenomenon occurs due to lateral, along the wave crests, transfer of energy and is of paramount importance for the design of harbours or other structures in coastal regions. In order to test the ability of the scheme to simulate the phenomenon of diffraction in the presence of two semi-infinite breakwaters, a relatively narrow gap of $2L$ between the breakwaters arms has been chosen, $L$ being the wavelength. This is due to the interest the case provides, for there exists not only the phenomenon of diffraction but there exist the diffraction patterns due to the interaction of the wave fields between the breakwater arms as well. The wave used in present experiment is of height $H=0.03$ m and period $T=0.8$ seconds whilst a constant water depth of 1 m is supposed throughout the computational domain (these conditions allow for wave propagation without the waves sensing the bottom).

The numerical results for the developed wave height and the wave phase contours are depicted in Fig. 2, while Fig. 3 depicts the numerical normalised wave height that further illustrates the smoothness of the numerical results stemming from the present scheme. These compare very well with the analytical solution for the present test case (depicted in Fig. 4) and Walker’s [10] numerical results, see Fig. 5. In particular, the comparison between Walker’s results and present ones is in favour of results stemming from the present scheme since these are characterised by roughness, especially in the vicinity of the domain boundaries. It is worth noting that Walker’s set of results has been chosen for comparison since the hyperbolic set of equations utilised for both present and Walker’s scheme is the same. However, Walker’s scheme did not form a conservation law and the solution technique was based on the finite differences method. Hence, it is concluded that the accuracy of environment-related numerical simulations does not only depend on the set of equations employed, which may suffer from shortcomings due to the approximations made for their derivation, but also depends on the solution technique employed. The adopted herein upwind piecewise Roe’s approximate Riemann solver based solution technique, as it was described in section 2 of the present paper, has provided a solid numerical base for calculating the wave climate in coastal regions accurately and smoothly.

5. CONCLUSIONS

A second order accurate, both in time and space, Godunov-type cell-centred upwind finite volume solver, has been utilised for the simulation of nearshore environment-related phenomena, i.e.: wave diffraction in the vicinity of a harbour entrance. The results presented demonstrate the versatile nature of Roe’s approximate Riemann solver adopted for calculating the wave fluxes at the interface of two adjacent states, which are computed to be linearly varying within each computational cell. Moreover, the present results do demonstrate a smooth character that can be attributed to the solution technique adopted as the comparison with other researchers’ results indicates. Overall, a robust scheme for numerical computations in coastal zones has been described herein which can lead to more accurate results for the wave-induced currents and the closely associated phenomenon of sediment transport.

6. ACKNOWLEDGEMENTS

This work has been supported by the U.K. Engineering and Physical Science Research Council (EPSRC) through grant GR/L92907 and co-investigated by Dr. K. Anastasiou of Imperial College of Science, Technology and Medicine, University of London.
**Fig. 2.** Numerical normalised wave height contours and wave phase for normal incidence onto a $2L$ breakwater gap.

**Fig. 3.** Numerical normalised wave height for normal incidence onto a $2L$ breakwater gap.
**Fig. 4.** Analytical normalised wave height contours and wave phase for normal incidence onto a $2L$ breakwater gap, after [11].

**Fig. 5.** Numerical normalised wave height contours and wave phase for normal incidence onto a $2L$ breakwater gap, after [10].
7. REFERENCES


