Asymptotics for the space-time Wigner transform with applications to imaging

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Abstract

We consider the space-time Wigner transform of the solution of the random Schrödinger equation in the white noise limit and for high frequencies. We analyze in particular the strong lateral diversity limit in which the space-time Wigner transform becomes weakly deterministic. We also show how to use these asymptotic results in broadband array imaging in random media.

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1 Introduction

In this paper we analyze the self-averaging property of the space-time Wigner transform for solutions of the random Schrödinger equation, in a particular asymptotic regime. We start with the wave equation in a random medium and then use the parabolic or paraxial approximation, which is valid when waves propagate primarily in a preferred direction and backscattering is negligible. This approximation is widely used in random wave propagation [26, 27, 28, 17, 13] and it is justified in some special cases in [1], in the regime that we consider here. The parabolic wave field satisfies a random Schrödinger equation, which we consider in the white noise limit. White noise limits for random ordinary differential equations have been analyzed extensively [20, 4, 22]. For random partial differential equations, white noise limits are considered in [10] for diffusion equations and, more recently, in [15, 14], for the random Schrödinger equation.

The resulting Itô-Schrödinger equation for the limit wave field is a stochastic partial differential equation of independent interest that is analyzed in [13] and in a wider context in [12, 25]. We consider here the high frequency limit of this equation, using the space-time Wigner transform. This is a slight extension of the high frequency limits analyzed in [23, 24] and in [15], using the spatial Wigner transform. The limit process satisfies an Itô-Liouville partial differential equation that arises from a stochastic flow [21, 25].

We analyze this Itô-Liouville equation in the strong lateral diversity limit, where the propagating wave beam is wide with respect to the correlation length of the random inhomogeneities in the transverse direction, orthogonal to the axis of the beam. The importance of this limit in time reversal was pointed out in [5] and it was analyzed later in [23, 24, 2, 18], using the spatial Wigner transform. Applications to imaging are considered in [6, 7, 8], especially applications of the space-time Wigner transform, but the strong lateral diversity limit is not analyzed there.

We dedicate this work to Boris Rozovskii on the occasion of his 60th birthday.

2 The parabolic approximation

Let $P(\vec{\mathbf{x}},t)$ be the solution of the acoustic wave equation

$$\frac{1}{c^2(\vec{\mathbf{x}})}\frac{\partial^2 P}{\partial t^2} - \Delta P = 0, \quad t > 0, \ \vec{\mathbf{x}} \in \mathbb{R}^3, \tag{2.1}$$

with a given excitation source at time $t = 0$ and in a medium with sound speed $c(\vec{\mathbf{x}})$ that is fluctuating about the mean value c_o, taken as constant for simplicity. We model the fluctuations of $c(\vec{\mathbf{x}})$ as a random process

$$c(\vec{\mathbf{x}}) = c_o \left[1 + \sigma_o \mu \left(\frac{\vec{\mathbf{x}}}{\ell} \right) \right]^{-1/2} \tag{2.2}$$

where μ is a normalized, bounded and statistically homogeneous random field, with mean zero and with smooth and rapidly decaying correlation function

$$E\{\mu(\vec{\mathbf{x}} + \vec{\mathbf{x}}')\mu(\vec{\mathbf{x}}')\} = R(\vec{\mathbf{x}}). \tag{2.3}$$

Here the normalization means that

$$R(\vec{0}) = 1, \qquad \int d\vec{\mathbf{x}}\, R(\vec{\mathbf{x}}) = 1, \tag{2.4}$$

so that ℓ in (2.2) is the correlation length of the fluctuations.

We consider a regime with weak fluctuations ($\sigma_o \ll 1$) where backscattering of the waves by the medium can be neglected and where we can study $P(\vec{\mathbf{x}}, t)$ with the parabolic approximation [26]. For this, we take the z coordinate in the direction of propagation of the waves and we let $\vec{\mathbf{x}} = (z, \mathbf{x})$, with $\mathbf{x}$ the two dimensional vector of coordinates transverse to the direction of propagation. In the parabolic approximation the wave field is given by

$$P(z, \mathbf{x}, t) = \frac{1}{2\pi} \int \widehat{P}(z, \mathbf{x}, \omega) e^{-i\omega t} d\omega, \quad \widehat{P}(z, \mathbf{x}, \omega) \approx e^{ikz} \psi(z, \mathbf{x}, k), \tag{2.5}$$

where $k = \omega / c_o$ is the wavenumber and ψ is a complex valued amplitude satisfying the Schrödinger equation

$$2ik \frac{\partial \psi}{\partial z} + \Delta_{\mathbf{x}} \psi + k^2 \sigma_o \mu \left(\frac{z}{\ell}, \frac{\mathbf{x}}{\ell} \right) \psi = 0, \quad z > 0, \tag{2.6}$$

with $\Delta_{\mathbf{x}}$ denoting the Laplacian in $\mathbf{x}$. This equation is obtained by substituting $e^{ikz}\psi$ in the reduced wave equation for $\widehat{P}$

$$\Delta \widehat{P} + k^2 n^2(\vec{\mathbf{x}}) \widehat{P} = 0,$$

with index of refraction $n(\vec{\mathbf{x}}) = c_o / c(\vec{\mathbf{x}})$ given by

$$n^2(\vec{\mathbf{x}}) = 1 + \sigma_o \mu \left(\frac{\vec{\mathbf{x}}}{\ell} \right), \tag{2.7}$$

and by neglecting the term $\frac{\partial^2 \psi}{\partial z^2}$ under the hypothesis that ψ is slowly varying in z (i.e., $k \left| \frac{\partial \psi}{\partial z} \right| \gg \left| \frac{\partial^2 \psi}{\partial z^2} \right|$).

We now have an initial value problem for the wave amplitude ψ, governed by equation (2.6) with initial condition

$$\psi(0, \mathbf{x}, k) = \psi_o(\mathbf{x}, k). \tag{2.8}$$

We assume that ψ_o is a compactly supported function with frequency dependence in the positive interval

$$\omega \in \left[\omega_o - \frac{B}{2}, \omega_o + \frac{B}{2} \right], \tag{2.9}$$

centered at ω_o and with bandwidth B. The negative image of this interval is also included if the initial data is real.

3 Scaling and the asymptotic regime

To carry out an asymptotic analysis of the wave field (2.5) we write the Schrödinger equation (2.6) in dimensionless form

$$2ik'\frac{\partial\psi}{\partial z'} + \frac{L_z}{k_o L_x^2}\Delta_{\mathbf{x}'}\psi + k_o L_z \sigma_o (k')^2 \mu\left(\frac{L_z z'}{\ell}, \frac{L_x \mathbf{x}'}{\ell}\right)\psi = 0, \tag{3.1}$$

with scaled variables

$$\mathbf{x} = L_x \mathbf{x}', \quad z = L_z z', \quad \omega = \omega_o \omega', \quad k = k_o k', \quad c = c_o c'. \tag{3.2}$$

Here $k_o = \omega_o / c_o$ is the central wavenumber, L_z quantifies the distance of propagation and L_x is a transversal length scale which we take to be the width of the propagating beam. Note that the scaled sound speed has constant mean $< c' >= 1$. Therefore, since the scaled wavenumber k' is the same as the scaled frequency ω', we shall replace ω' by k' from now on.

To simplify notation we drop the primes on the scaled variables and we introduce three dimensionless parameters depending on the random medium

$$\epsilon = \frac{\ell}{L_z}, \quad \delta = \frac{\ell}{L_x}, \quad \sigma = \sigma_o \delta \epsilon^{-\frac{3}{2}}, \tag{3.3}$$

and the reciprocal of the Fresnel number

$$\theta = \frac{L_z}{k_o L_x^2} = \frac{1}{2\pi}\frac{\left(\frac{\lambda_0 L_z}{L_x}\right)}{L_x}. \tag{3.4}$$

Here λ_o is the central wavelength and the reciprocal of the Fresnel number is written as the ratio of the focusing spot size in time reversal imaging, $\lambda_o L_z / L_x$, and the transversal length scale L_x.

The scaled form of equation (3.1) is

$$2ik\frac{\partial\psi}{\partial z} + \theta\Delta_{\mathbf{x}}\psi + \frac{1}{\epsilon^{1/2}}\mu\left(\frac{z}{\epsilon}, \frac{\mathbf{x}}{\delta}\right)\frac{\sigma k^2 \delta}{\theta}\psi = 0, \quad z > 0 \tag{3.5}$$

and we study it in the asymptotic regime

$$\epsilon \ll \delta \ll 1, \quad \theta \ll 1, \quad \sigma = O(1). \tag{3.6}$$

Thus, we suppose that the waves travel many correlation lengths in the random medium ($\epsilon \ll 1$) and, to be consistent with the parabolic approximation, we take $L_x \ll L_z$ (i.e., $\epsilon \ll \delta$). We also take $\theta \ll 1$, which means that the time reversal imaging spot size is much smaller than L_x

$$\frac{\lambda_o L_z}{L_x} \ll L_x. \tag{3.7}$$

Finally, we scale the strength of the fluctuations in (3.3) and (3.6) so that we can take the white noise limit $\epsilon \to 0$ in (3.5).

The asymptotic regime (3.6) can be realized with several scale orderings. In this paper we assume that

$$\epsilon \ll \theta \ll \delta \ll 1, \tag{3.8}$$

which amounts to taking $\epsilon \to 0$ as the first in a sequence of three limits. This leads to an Itô-Schrödinger equation for the limit ψ. The second limit $\theta/\delta \to 0$ implies that we are in a high frequency regime

$$\frac{\lambda_o}{\ell} \ll \frac{\epsilon}{\delta} \ll 1. \tag{3.9}$$

In imaging, the spot size is small in this limit, when compared with the correlation length

$$\frac{\lambda_o L_z}{L_x} \ll \ell \ll L_x. \tag{3.10}$$

This is a regime in which we can derive an Itô-Liouville equation for the Wigner transform of ψ, under the additional assumption of isotropy of the fluctuations of the sound speed. Finally, we take the strong lateral diversity limit $\delta \ll 1$, which allows us to show that the appropriately smoothed Wigner transform is self-averaging.

Other scale orderings consistent with (3.6) are

$$\theta \ll \epsilon \ll \delta \ll 1 \tag{3.11}$$

and

$$\epsilon \ll \delta \leq \theta \ll 1. \tag{3.12}$$

The ordering (3.11) is considered in [23], in a study of statistical stability of time reversal in random media. It is a high frequency regime and it gives similar results to those obtained here. The scale ordering (3.12) is consistent with $\lambda_o \sim \ell$ and it is used in numerical simulations in [6, 7, 8, 9], in the context of array imaging of sources and reflectors. In the parabolic approximation this scaling is analyzed in [24]. The theory is not so well developed when the parabolic approximation does not apply. Nevertheless, it appears from the numerical simulations in [6, 7, 8, 9] that the statistical stability that we have in regimes (3.8) or (3.11) is valid in the case (3.12).

4 The Itô-Liouville equation for the Wigner transform

In this section we give, without details, the Itô-Liouville equation for the Wigner transform of ψ in the limits $\epsilon \to 0$ and $\theta \to 0$. We then state the main result of this paper, which is that in the limit $\delta \to 0$ we have self-averaging for smooth linear functionals of the space-time Wigner transform. The proof is given in section 5. We consider an application of this self-averaging property in section 6, where we look at coherent interferometric imaging in random media, as introduced in [7, 8].

4.1 The white noise limit

Let us emphasize with the notation $\psi^\epsilon(z,\mathbf{x},k)$ the dependence on ϵ of the wave amplitude satisfying (3.5). This amplitude depends on θ and δ as well, but since these are kept fixed in the first limit we suppress them from the notation. The initial wave field ψ_o is assumed independent of ϵ.

It follows from [15, 14] that as $\epsilon \to 0$, $\psi^\epsilon(z,\mathbf{x},k)$ converges weakly, in law, to the solution $\psi(z,\mathbf{x},k)$ of Itô-Schrödinger equation

$$\begin{aligned} d\psi &= \left[\frac{i\theta}{2k}\Delta_{\mathbf{x}} - \frac{k^2\sigma^2\delta^2}{8\theta^2}R_o(0)\right]\psi dz + \frac{ik\sigma\delta}{2\theta}\psi d\mathcal{B}\left(z,\frac{\mathbf{x}}{\delta}\right), \quad z>0, \\ \psi(0,\mathbf{x},k) &= \psi_o(\mathbf{x},k), \quad \text{at } z=0. \end{aligned} \tag{4.1}$$

Here $\mathcal{B}(z,\mathbf{x})$ is a Brownian random field that is smooth in the transverse variable $\mathbf{x}$. The mean of $\mathcal{B}$ is zero and its correlation is given by

$$E\{\mathcal{B}(z_1,\mathbf{x}_1)\mathcal{B}(z_2,\mathbf{x}_2)\} = z_1\wedge z_2 R_o(\mathbf{x}_1-\mathbf{x}_2), \tag{4.2}$$

where $z_1\wedge z_2 = \min\{z_1,z_2\}$ and

$$R_o(\mathbf{x}) = \int_{-\infty}^{\infty} R(z,\mathbf{x})dz. \tag{4.3}$$

Because of our assumptions on R in section 2 we have that R_o is smooth and rapidly decaying. This is used in section 5 to deduce the statistical stability of the smoothed Wigner transform of ψ, in the limit $\theta/\delta\to 0$ and $\delta\to 0$.

4.2 The high frequency limit and the space-time Wigner transform

As in section 4.1, we now use the notation $\psi^\theta(z,\mathbf{x},k)$ to emphasize the dependence of the solution of Itô-Schrödinger equation (4.1) on the parameter θ. We study the high frequency limit $\theta\to 0$ with the space-time Wigner transform

$$W^\theta(z,\overline{\mathbf{x}},\overline{k},\mathbf{q},r) = \int\frac{d\widetilde{\mathbf{x}}}{(2\pi)^2}\int\frac{d\widetilde{k}}{2\pi}e^{i\mathbf{q}\cdot\widetilde{\mathbf{x}}-i\widetilde{k}r}\psi^\theta\left(z,\overline{\mathbf{x}}+\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}+\frac{\theta\widetilde{k}}{2}\right)\overline{\psi^\theta\left(z,\overline{\mathbf{x}}-\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}-\frac{\theta\widetilde{k}}{2}\right)}, \tag{4.4}$$

where the bar on ψ^θ denotes complex conjugate. The r variable in W^θ is dual to $\widetilde{k}$ and it represents the distance traveled by the waves in a medium with constant speed $<c>=1$, during a travel time $t=r/<c>$. The argument $\mathbf{q}$ in W^θ is a two dimensional vector that is dual to $\widetilde{\mathbf{x}}$.

For an arbitrary but fixed z, the L^2 norm of the Wigner transform W^θ is determined by the space and frequency L^2 norm of the initial wave function ψ_o^θ

$$\left\|W^\theta(z,\cdot)\right\|_{L^2} = \left[\int d\overline{\mathbf{x}} \int d\overline{k} \int d\mathbf{q} \int dr \left|W^\theta(z,\overline{\mathbf{x}},\overline{k},\mathbf{q},r)\right|^2\right]^{1/2}$$
$$= \left[\int d\overline{\mathbf{x}} \int d\overline{k} \int \frac{d\widetilde{\mathbf{x}}}{(2\pi)^2} \int \frac{d\widetilde{k}}{2\pi} \left|\psi^\theta\left(z,\overline{\mathbf{x}}+\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}+\frac{\theta\widetilde{k}}{2}\right)\right|^2 \left|\psi^\theta\left(z,\overline{\mathbf{x}}-\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}-\frac{\theta\widetilde{k}}{2}\right)\right|^2\right]^{1/2}$$
$$= \frac{\left\|\psi^\theta(z,\cdot)\right\|_{L^2}^2}{(2\pi\theta)^{3/2}} = \frac{\left\|\psi_o^\theta\right\|_{L^2}^2}{(2\pi\theta)^{3/2}}, \tag{4.5}$$

because the Itô-Schrödinger equation (4.1) preserves the space and frequency L^2 norm of its solution [13]. This means that with a proper definition and scaling of the initial wave function ψ_o^θ [19], we can bound the L^2 norm of $W^\theta(z,\cdot)$ uniformly with respect to θ.

We formally obtain an Itô-Liouville equation for the high frequency limit W as follows [14]. We use Itô's formula to get from (4.1) an equation for $\psi_1^\theta\overline{\psi_2^\theta} = \psi^\theta(z,\mathbf{x}_1,k_1)\overline{\psi^\theta(z,\mathbf{x}_2,k_2)}$, with

$$\mathbf{x}_1 = \overline{\mathbf{x}} + \frac{\theta\widetilde{\mathbf{x}}}{2}, \quad \mathbf{x}_2 = \overline{\mathbf{x}} - \frac{\theta\widetilde{\mathbf{x}}}{2}, \tag{4.6}$$
$$k_1 = \overline{k} + \frac{\theta\widetilde{k}}{2}, \quad k_2 = \overline{k} - \frac{\theta\widetilde{k}}{2} \tag{4.7}$$

and then we Fourier transform in $\widetilde{\mathbf{x}}$ and $\widetilde{k}$ and take the limit $\theta/\delta \to 0$. The variables $\overline{\mathbf{x}}$, $\widetilde{\mathbf{x}}$, $\overline{k}$ and $\widetilde{k}$ are independent of the small parameters. We have

$$d\left(\psi_1^\theta\overline{\psi_2^\theta}\right) = \Big[\tfrac{i\theta}{2\overline{k}+\theta\widetilde{k}}\left(\tfrac{1}{4}\Delta_{\overline{\mathbf{x}}} + \tfrac{1}{\theta}\nabla_{\overline{\mathbf{x}}}\cdot\nabla_{\widetilde{\mathbf{x}}} + \tfrac{1}{\theta^2}\Delta_{\widetilde{\mathbf{x}}}\right) - \tfrac{i\theta}{2\overline{k}-\theta\widetilde{k}}\left(\tfrac{1}{4}\Delta_{\overline{\mathbf{x}}} - \tfrac{1}{\theta}\nabla_{\overline{\mathbf{x}}}\cdot\nabla_{\widetilde{\mathbf{x}}} + \tfrac{1}{\theta^2}\Delta_{\widetilde{\mathbf{x}}}\right)$$
$$+\frac{\left(\overline{k}^2-\frac{\theta^2}{4}\widetilde{k}^2\right)\sigma^2\delta^2}{4\theta^2}R_0\left(\tfrac{\theta|\widetilde{\mathbf{x}}|}{\delta}\right) - \frac{\left(\overline{k}^2+\frac{\theta^2}{4}\widetilde{k}^2\right)\sigma^2\delta^2}{4\theta^2}R_0(0)\Big]\psi_1^\theta\overline{\psi_2^\theta}dz \tag{4.8}$$
$$+\tfrac{i\sigma\delta}{2\theta}\psi_1^\theta\overline{\psi_2^\theta}\left[\left(\overline{k}+\tfrac{\theta}{2}\widetilde{k}\right)d\mathcal{B}\left(z,\tfrac{\overline{\mathbf{x}}}{\delta}+\tfrac{\theta\widetilde{\mathbf{x}}}{2\delta}\right) - \left(\overline{k}-\tfrac{\theta}{2}\widetilde{k}\right)d\mathcal{B}\left(z,\tfrac{\overline{\mathbf{x}}}{\delta}-\tfrac{\theta\widetilde{\mathbf{x}}}{2\delta}\right)\right]$$

and, using the smoothness of $\mathcal{B}$ and R_o in the transverse variables, we have further

$$d\mathcal{B}\left(z,\frac{\overline{\mathbf{x}}}{\delta}+\frac{\theta\widetilde{\mathbf{x}}}{2\delta}\right) - d\mathcal{B}\left(z,\frac{\overline{\mathbf{x}}}{\delta}-\frac{\theta\widetilde{\mathbf{x}}}{2\delta}\right) = \frac{\theta\widetilde{\mathbf{x}}}{\delta}\cdot\nabla_{\overline{\mathbf{x}}}dB(z,\overline{\mathbf{x}}) + O\left(\frac{\theta}{\delta}\right)^2 \tag{4.9}$$

and

$$R_o\left(\frac{\theta\widetilde{\mathbf{x}}}{\delta}\right) = R_o(\mathbf{0}) + \frac{\theta}{\delta}\widetilde{\mathbf{x}}\cdot\nabla R_o(\mathbf{0}) + \frac{\theta^2}{2\delta^2}\sum_{i,j=1}^{2}\partial_{ij}^2 R_o(\mathbf{0})\tilde{x}_i\tilde{x}_j + O\left(\frac{\theta}{\delta}\right)^3. \tag{4.10}$$

The equation for W follows by Fourier transforming (4.8) in $\widetilde{\mathbf{x}}$ and $\widetilde{k}$, using the expansions (4.9) and (4.10) and letting $\theta/\delta \to 0$. We simplify the result by assuming that the fluctuations are isotropic so that $R_o(\mathbf{x}) = R_o(|\mathbf{x}|)$. This gives

$$\nabla R_o(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad \partial_{ij}^2 R_o(\mathbf{0}) = R''(0)\delta_{ij} \tag{4.11}$$

and we obtain for W the Itô-Liouville equation

$$\begin{aligned} dW = & \left[\frac{\mathbf{q}}{\overline{k}}\cdot\nabla_{\overline{\mathbf{x}}} - \frac{|\mathbf{q}|^2}{2\overline{k}^2}\frac{\partial}{\partial r} + \frac{\overline{k}^2 D_\kappa}{2}\Delta_{\mathbf{q}} + \frac{\delta^2 D_r}{2}\frac{\partial^2}{\partial r^2}\right] W dz \\ & + \frac{\sigma\overline{k}}{2}\nabla_{\mathbf{q}} W\cdot\nabla_{\overline{\mathbf{x}}} d\mathcal{B}\left(z,\frac{\overline{\mathbf{x}}}{\delta}\right) - \frac{\sigma\delta}{2}\frac{\partial W}{\partial r} d\mathcal{B}\left(z,\frac{\overline{\mathbf{x}}}{\delta}\right), \quad z>0 \end{aligned} \tag{4.12}$$

with initial condition

$$W(z=0,\overline{\mathbf{x}},\overline{k},\mathbf{q},r) = W_o(\overline{\mathbf{x}},\overline{k},\mathbf{q},r) \tag{4.13}$$

and with the positive diffusion coefficients

$$D_\kappa = -\frac{\sigma^2 R_o''(0)}{4} \quad \text{and} \quad D_r = \frac{\sigma^2 R_o(0)}{4}. \tag{4.14}$$

Equation (4.12) was also derived in [7] and we note that it is a stochastic flow equation [21, 25] that is the starting point of the analysis in this paper. We want to study the limit of the process W as $\delta \to 0$.

4.3 Statement of the strong lateral diversity limit

Now that we have the Itô-Liouville equation (4.12), we emphasize the dependence of the process on δ by writing $W^\delta(z,\overline{\mathbf{x}},\overline{k},\mathbf{q},r)$. We assume that the initial condition W_o is independent of δ.

The mean $\mathcal{W}^\delta = E\{W^\delta\}$ is considered in section 4.4 and it follows, as is easily seen from (4.12), that as $\delta \to 0$, $\mathcal{W}^\delta$ converges weakly to the solution $\mathcal{W}$ of the phase space advection-diffusion equation

$$\frac{\partial \mathcal{W}}{\partial z} = \mathcal{L}\mathcal{W} \tag{4.15}$$

with initial conditions

$$\mathcal{W}(0,\overline{\mathbf{x}},\overline{k},\mathbf{q},r) = W_o(\overline{\mathbf{x}},\overline{k},\mathbf{q},r), \tag{4.16}$$

where

$$\mathcal{L} = \frac{\mathbf{q}}{\overline{k}}\cdot\nabla_{\overline{\mathbf{x}}} - \frac{|\mathbf{q}|^2}{2\overline{k}^2}\frac{\partial}{\partial r} + \frac{\overline{k}^2 D_\kappa}{2}\Delta_{\mathbf{q}}. \tag{4.17}$$

This deterministic equation is solved explicitly in [7].

However, the point-wise variance of W^δ is not zero for any δ and it does not vanish as $\delta \to 0$. This means that W^δ is randomly fluctuating and it does not converge to a deterministic process as $\delta \to 0$ in the strong, point-wise sense. Nevertheless, we do have convergence in a weak sense as follows.

Theorem 1 *Suppose that W_o is in L^2 and it does not depend on δ. Then, given any smooth and rapidly decaying test function $\phi(\overline{\mathbf{x}},\overline{k},\mathbf{q},r)$, we have that*

$$< W^\delta,\phi > (z) = \int d\overline{\mathbf{x}}\int d\overline{k}\int d\mathbf{q}\int dr\,\phi(\overline{\mathbf{x}},\overline{k},\mathbf{q},r) W^\delta(z,\overline{\mathbf{x}},\overline{k},\mathbf{q},r) \tag{4.18}$$

converges in probability as $\delta \to 0$ to $< \mathcal{W},\phi > (z)$, for any $z > 0$.

Theorem 1 is proved in section 5. It states that even though W^δ does not have a deterministic point-wise limit, it is weakly self-averaging. That is, smooth linear functionals of W^δ become deterministic in the limit $\delta \to 0$. This is the property that can be exploited in applications such as imaging in random media, as we explain in section 6.

4.4 The mean space-time Wigner transform

Taking expectations in (4.12) we get that $\mathcal{W}^\delta(z,\overline{\mathbf{x}},\mathbf{q},r) = E\{W^\delta(z,\overline{\mathbf{x}},\mathbf{q},r)\}$ satisfies the phase space advection-diffusion equation

$$\frac{\partial \mathcal{W}^\delta}{\partial z} = \mathcal{L}_\delta \mathcal{W}, \tag{4.19}$$

with initial condition $W_o(\overline{\mathbf{x}},\overline{k},\mathbf{q},r)$, where

$$\mathcal{L}_\delta = \mathcal{L} + \frac{\delta^2 D_r}{2}\frac{\partial^2}{\partial r^2}. \tag{4.20}$$

Equivalently, $\mathcal{W}^\delta$ is given as an expectation

$$\mathcal{W}^\delta(z,\overline{\mathbf{x}},\boldsymbol{\kappa},r) = E\left\{W_o\left(\mathbf{X}^\delta(z),\overline{k},\mathbf{Q}^\delta(z),\mathcal{R}^\delta(z)\right)\right\}, \tag{4.21}$$

where $\left\{\mathbf{X}^\delta(z),\mathbf{Q}^\delta(z),\mathcal{R}^\delta(z)\right\}$ is the Itô diffusion process with generator $\mathcal{L}_\delta$ and with initial condition

$$\mathbf{X}^\delta(0) = \overline{\mathbf{x}}, \quad \mathbf{Q}^\delta(0) = \mathbf{q}, \quad \mathcal{R}^\delta(0) = r. \tag{4.22}$$

For $z > 0$, the Itô stochastic differential equations are

$$\begin{aligned} d\mathbf{X}^\delta(z) &= \frac{1}{\overline{k}}\mathbf{Q}^\delta(z)dz, \\ d{Q_j}^\delta(z) &= \overline{k}\sqrt{D_\kappa}\, d\mathcal{B}_j(z), \quad j=1,2, \quad \mathbf{Q}^\delta = \left(Q_1^\delta, Q_2^\delta\right), \\ d\mathcal{R}^\delta(z) &= -\frac{|\mathbf{Q}^\delta(z)|^2}{2\overline{k}^2}dz - \delta\sqrt{D_r}d\mathcal{B}(z), \end{aligned} \tag{4.23}$$

where the driving is with three independent standard Brownian motions $\mathcal{B}(z)$ and $\mathcal{B}_j(z)$, for $j=1,2$. The same process $\left\{\mathbf{X}^\delta(z),\mathbf{Q}^\delta(z),\mathcal{R}^\delta(z)\right\}$ also determines expectations of higher powers of W^δ

$$E\left\{\left|W^\delta(z,\overline{\mathbf{x}},\mathbf{q},r)\right|^n\right\} = E\left\{\left|W_o\left(\mathbf{X}^\delta(z),\overline{k},\mathbf{Q}^\delta(z),\mathcal{R}^\delta(z)\right)\right|^n\right\}, \quad n \geq 1. \tag{4.24}$$

As $\delta \to 0$, we see that $\mathcal{W}^\delta$ converges to the solution of (4.15), computed explicitly in [7]. Actually, all one point moments converge as $\delta \to 0$,

$$E\left\{\left|W^\delta(z,\overline{\mathbf{x}},\mathbf{q},r)\right|^n\right\} \to E\left\{\left|W_o\left(\mathbf{X}(z),\overline{k},\mathbf{Q}(z),\mathcal{R}(z)\right)\right|^n\right\}, \quad n \geq 1, \tag{4.25}$$

where $\{\mathbf{X}(z), \mathbf{Q}(z), \mathcal{R}(z)\}$ is the δ independent Itô diffusion

$$
\begin{aligned}
d\mathbf{X}(z) &= \frac{1}{\overline{k}}\mathbf{Q}(z)dz, \\
dQ_j(z) &= \overline{k}\sqrt{D_\kappa}\, d\mathcal{B}_j(z), \quad j = 1, 2, \quad \mathbf{Q} = (Q_1, Q_2), \\
d\mathcal{R}(z) &= -\frac{|\mathbf{Q}(z)|^2}{2\overline{k}^2}dz, \quad z > 0,
\end{aligned}
\tag{4.26}
$$

with initial conditions

$$
\mathbf{X}(0) = \overline{\mathbf{x}}, \quad \mathbf{Q}(0) = \mathbf{q} \text{ and } \mathcal{R}(0) = r. \tag{4.27}
$$

Clearly, W^δ does not have have a point-wise deterministic limit because the limit variance is not zero

$$
E\left\{\left|W_o\left(\mathbf{X}(z), \overline{k}, \mathbf{Q}(z), \mathcal{R}(z)\right)\right|^2\right\} - \left|E\left\{W_o\left(\mathbf{X}(z), \overline{k}, \mathbf{Q}(z), \mathcal{R}(z)\right)\right\}\right|^2 \neq 0. \tag{4.28}
$$

5 Self-averaging of the smoothed space-time Wigner transform, in the strong lateral diversity limit

In this section we prove Theorem 1. We begin by calculating the form of the infinitesimal generator $\mathcal{A}_\delta$ of Itô-Liouville process $W^\delta(z, \overline{\mathbf{x}}, \overline{k}, \boldsymbol{\kappa}, r)$, considered as a process in the space of continuous functions in z with values in the space $\mathcal{S}'$ of Schwartz distributions w over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$.

Let F be a real valued test function on $\mathbb{R}$ and define for each test function ϕ in $\mathcal{S}$ over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ the function $f(w)$ by

$$
f(w) = F(< w, \phi >). \tag{5.1}
$$

We have that

$$
\begin{aligned}
\mathcal{A}_\delta f(w) &= \frac{d}{dz}E\{F(< W^\delta(z), \phi >)|W^\delta(0) = w\}|_{z=0} \\
&= \mathcal{D}^\delta(w)F^{'}(< w, \phi >) + \mathcal{M}^\delta(w)F^{''}(< w, \phi >)
\end{aligned}
\tag{5.2}
$$

where

$$
\mathcal{D}^\delta(w) =< w, \mathcal{L}_\delta^\star \phi >, \tag{5.3}
$$

with $\mathcal{L}_\delta^\star$ the adjoint of $\mathcal{L}_\delta$ in (4.20). The $\mathcal{M}^\delta$ can be written as the sum of three terms

$$
\begin{aligned}
\mathcal{M}_1^\delta(w) = \frac{\sigma^2\delta^2}{8}\int d\overline{\mathbf{x}}\int d\overline{k}\int d\mathbf{q}\int dr\int d\overline{\mathbf{x}}'\int d\overline{k}'\int d\mathbf{q}'\int dr'\, w(\overline{\mathbf{x}}, \overline{k}, \mathbf{q}, r)w(\overline{\mathbf{x}}', \overline{k}', \mathbf{q}', r') \times \\
R_o\left(\frac{\overline{\mathbf{x}} - \overline{\mathbf{x}}'}{\delta}\right)\frac{\partial\phi(\overline{\mathbf{x}}, \overline{k}, \mathbf{q}, r)}{\partial r}\frac{\partial\phi(\overline{\mathbf{x}}', \overline{k}', \mathbf{q}', r')}{\partial r'},
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
\mathcal{M}_2^\delta(w) = -\frac{\sigma^2\delta}{4}\int d\overline{\mathbf{x}}\int d\overline{k}\int d\mathbf{q}\int dr\int d\overline{\mathbf{x}}'\int d\overline{k}'\int d\mathbf{q}'\int dr'\, w(\overline{\mathbf{x}}, \overline{k}, \mathbf{q}, r)w(\overline{\mathbf{x}}', \overline{k}', \mathbf{q}', r')\overline{k}' \times \\
\nabla_{\mathbf{x}'}R_o\left(\frac{\overline{\mathbf{x}} - \overline{\mathbf{x}}'}{\delta}\right) \cdot \nabla_{\mathbf{q}'}\phi(\overline{\mathbf{x}}', \overline{k}', \mathbf{q}', r')\frac{\partial\phi(\overline{\mathbf{x}}, \overline{k}, \mathbf{q}, r)}{\partial r},
\end{aligned}
\tag{5.5}
$$

$$\mathcal{M}_3^\delta(w) = -\frac{\sigma^2}{8}\int d\overline{\mathbf{x}}\int d\overline{k}\int d\mathbf{q}\int dr\int d\overline{\mathbf{x}}'\int d\overline{k}'\int d\mathbf{q}'\int dr'\, w(\overline{\mathbf{x}},\overline{k},\mathbf{q},r)w(\overline{\mathbf{x}}',\overline{k}',\mathbf{q}',r')\overline{k}\overline{k}' \times \sum_{j,l=1}^{2}\partial^2_{lj}R_o\left(\frac{\overline{\mathbf{x}}-\overline{\mathbf{x}}'}{\delta}\right)\frac{\partial\phi(\overline{\mathbf{x}},\overline{k},\mathbf{q},r)}{\partial q_j}\frac{\partial\phi(\overline{\mathbf{x}}',\overline{k}',\mathbf{q}',r')}{\partial q_l'}. \quad (5.6)$$

Now we get from (5.3) that as $\delta \to 0$,

$$\lim_{\delta\to 0}\mathcal{D}^\delta(w) = \mathcal{D}(w) =< w, \mathcal{L}^\star\phi >, \quad (5.7)$$

uniformly for w bounded in L^2. Here $\mathcal{L}^\star$ is the adjoint of $\mathcal{L}$ defined by (4.17). Furthermore,

$$\lim_{\delta\to 0}\mathcal{M}_j^\delta(w) = 0, \quad \text{for } j = 1, 2, 3,$$

uniformly for w bounded in L^2, as we show next.

From (5.4)-(5.6) we see that it is enough to show that $\mathcal{M}_3^\delta(w) \to 0$. By the Cauchy-Schwartz inequality, we have

$$|\mathcal{M}_3^\delta(w)| \le \|w\|^2_{L^2} J^\delta(\phi), \quad (5.8)$$

where

$$[J^\delta(\phi)]^2 = \left(\frac{\sigma^2}{8}\right)^2\int d\overline{\mathbf{x}}\int d\overline{k}\int d\mathbf{q}\int dr\int d\overline{\mathbf{x}}'\int d\overline{k}'\int d\mathbf{q}'\int dr' \left[\overline{k}\overline{k}'\sum_{j,l=1}^{2}\partial^2_{lj}R_o\left(\frac{\overline{\mathbf{x}}-\overline{\mathbf{x}}'}{\delta}\right)\frac{\partial\phi(\overline{\mathbf{x}},\overline{k},\mathbf{q},r)}{\partial q_j}\frac{\partial\phi(\overline{\mathbf{x}}',\overline{k}',\mathbf{q}',r')}{\partial q_l'}\right]^2. \quad (5.9)$$

Since R_o is rapidly decaying at infinity, we see that for any fixed test function ϕ, $J^\delta(\phi)$ tends to zero as $\delta \to 0$.

We have shown therefore that for functions $f(w)$ of the form (5.1), with ϕ in $\mathcal{S}$ fixed and uniformly for w bounded in L^2,

$$\mathcal{A}^\delta f(w) \to \mathcal{A}f(w) = \mathcal{D}(w)F'(< w, \phi >), \quad (5.10)$$

where

$$\mathcal{D}(w) =< w, \mathcal{L}^\star\phi > . \quad (5.11)$$

The operator $\mathcal{A}$ is the generator of the deterministic process $\mathcal{W}(z, \overline{\mathbf{x}}, \overline{k}, \mathbf{q}, r)$, that is the solution of (4.15). Since the limit process is deterministic, it follows that convergence in law implies convergence in probability, weakly in $\mathcal{S}'$. It also follows that functions of the form (5.1) are sufficient again since the limit is deterministic [25]. The moment condition needed for tightness for processes in $C([0, Z], \mathcal{S}')$ or in $D([0, Z], \mathcal{S}')$ [16] is easily obtained as in [18], and we omit it here.

Figure 1: Setup for imaging a distributed source with a planar array of transducers

6 Application to imaging

In this section we consider applications to imaging a source in a random medium from measurements of the wave field P at an array of transducers. The setup is shown in Figure 1, where we introduce a new coordinate system, with scaled range $\zeta = L - z$ measured from the array and with cross range (transverse) coordinates $\mathbf{x}$ defined with respect to the center $\vec{\mathbf{y}}^\star = (L, \mathbf{0})$ of the source, which can be small or spatially distributed. The array is in the plane $\zeta = 0$ and it consists of receivers at N discrete locations $\vec{\mathbf{x}}_r = (0, \mathbf{x}_r)$, where we record the traces $P(\vec{\mathbf{x}}_r, t)$ over a time window $t \in [0, T]$ that we suppose is long enough for

$$P(\vec{\mathbf{x}}_r, t) \approx 0 \quad \text{for } t > T$$

to hold. This allows us to simplify the analysis by neglecting the effect of a finite measurement time window.

The goal in imaging is to estimate the support of the source from the traces at the array and this is done very efficiently with Kirchhoff migration [11, 3], if there are no fluctuations of the sound speed. However, in random media Kirchhoff migration gives noisy and unpredictable results, in the sense that they lack statistical stability. As an alternative to Kirchhoff migration we introduced in [7, 8] a new, coherent interferometric (CINT) imaging functional, which is a statistically smoothed migration. The resolution analysis of CINT is given in [7]. Here we use the result of section 4.3 to show that it is statistically stable in the asymptotic regime (3.8).

6.1 Migration

In this and the following sections all variables are scaled as in section 3. The wave field $\widehat{P}$ at receiver location $\vec{\mathbf{x}}_r = (0, \mathbf{x}_r)$ is

$$\widehat{P}(\mathbf{x}_r, \omega) \approx e^{i(\delta/\epsilon)^2 k/\theta L} \psi(L, \mathbf{x}_r, k), \tag{6.1}$$

where we used that $k_o L_z = \delta^2/(\theta\epsilon^2)$ and we dropped the range coordinate in the argument of $\widehat{P}$, as it is always $\zeta = 0$ at the array. We also kept the definition $\psi = \psi(z, \mathbf{x}, k)$, with $z = L - \zeta$, which means that at the array the range coordinate in ψ is $z = L$. The parabolic amplitude ψ solves equation (3.5) with initial data $\psi_o(\mathbf{x}, k)$ and it depends on the three small parameters ϵ, θ and δ. In the previous sections we emphasized this dependence using superscripts, before taking limits. Here we don't use the superscripts to simplify notation and we keep ϵ, θ and δ fixed until the very end where we apply the theoretical results of section 4.3.

Classic Kirchhoff migration imaging [11, 3] estimates the support of the source by migrating (back propagating) the traces $P(\mathbf{x}_r, t)$ to search points $\vec{\mathbf{y}}^S = (L + \eta^S, \boldsymbol{\xi}^S)$, in a fictitious, homogeneous medium, with scaled sound speed $< c >= 1$ and by summing over the receivers. The scaled distance from $\vec{\mathbf{x}}_r$ to $\vec{\mathbf{y}}^S$ is

$$\left[(L + \eta^S)^2 + \left(\frac{L_x}{L_z}\right)^2 \left|\boldsymbol{\xi}^S - \mathbf{x}_r\right|^2\right]^{\frac{1}{2}} \approx L + \eta^S + \frac{\epsilon^2}{2\delta^2} \frac{\left|\boldsymbol{\xi}^S - \mathbf{x}_r\right|^2}{(L + \eta^S)} \tag{6.2}$$

and it equals the scaled travel time $\tau(\mathbf{x}_r, \vec{\mathbf{y}}^S)$, since the scaled mean speed is $< c >= 1$. This gives the migration phase

$$(k_o L_z)\omega\tau(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S) \approx \frac{\delta^2}{\epsilon^2\theta} k(L + \eta^S) + \frac{k}{2\theta} \frac{\left|\boldsymbol{\xi}^S - \mathbf{x}_r\right|^2}{(L + \eta^S)} \tag{6.3}$$

and the migrated wave field to $\vec{\mathbf{y}}^S$

$$\widehat{P}(\mathbf{x}_r, \omega) e^{-i(k_o L_z)\omega\tau(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S)} \approx \psi(L, \mathbf{x}_r, k) \exp\left\{-i\frac{\delta^2}{\epsilon^2}\frac{k}{\theta}\eta^S - i\frac{k}{2\theta}\frac{\left|\boldsymbol{\xi}^S - \mathbf{x}_r\right|^2}{(L + \eta^S)}\right\}. \tag{6.4}$$

The Kirchhoff migration image is given by

$$\mathcal{I}^{\mathrm{KM}}(\vec{\mathbf{y}}^S) = \sum_{r=1}^{N} \int d\omega\, \widehat{P}(\mathbf{x}_r, \omega) e^{-i(k_o L_z)\omega\tau(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S)} \tag{6.5}$$

and, as shown in [6, 7, 8], it lacks statistical stability with respect to the realizations of the random medium and it gives noisy results that are difficult to interpret.

We consider next coherent interferometric imaging, which is a statistically smoothed version of migration [7, 8]. Before describing this method, let us make the assumption that the array receivers are placed on a square mesh, in a square aperture of area a^2. The scaled mesh size is h and we suppose that it is small, so we can write

$$\sum_{r=1}^{N} \approx \frac{1}{h^2} \int d\mathbf{x} \sim \int d\mathbf{x}, \tag{6.6}$$

with $\mathbf{x}$ varying continuously in the array aperture and with symbol $\sim$ denoting approximate, up to a multiplicative constant.

6.2 Coherent Interferometric Imaging

The coherent interferometric imaging technique was introduced in [7, 8] and it uses the coherence in the data traces $P(\mathbf{x}, t)$ to obtain reliable images in random media. There are two characteristic coherent parameters in the data:

- The decoherence frequency Ω_d, which is the difference in the frequencies ω_1 and ω_2 over which $\psi(z, \mathbf{x}, k_1)$ and $\psi(z, \mathbf{x}, k_2)$ become uncorrelated.
- The decoherence length X_d, which is the distance $|\mathbf{x}_1 - \mathbf{x}_2|$ over which $\psi(z, \mathbf{x}_1, k)$ and $\psi(z, \mathbf{x}_2, k)$ become uncorrelated.

These decoherence parameters depend on the statistics of the random medium and the range z and they are described in detail in the next section, for the asymptotic regime (3.8).

Coherent interferometry (CINT) is a migration technique that works with cross correlations of the traces, instead of the traces themselves. These cross correlations are computed locally over space-time windows of size $X_d \times \Omega_d$ and they are called coherent interferograms. We give in section 6.2.2 the mathematical expression of the CINT imaging function and then we study its statistical stability. The CINT functional and its resolution properties are motivated and analyzed in [7, 8].

6.2.1 The decoherence length and frequency

The decoherence length and frequency can be determined from the decay over $\widetilde{\mathbf{x}} = (\mathbf{x}_1 - \mathbf{x}_2)/\theta$ and $\widetilde{k} = (k_1 - k_2)/\theta$ of the expectation

$$\left\langle \psi\left(z, \overline{\mathbf{x}} + \frac{\theta\widetilde{\mathbf{x}}}{2}, \overline{k} + \frac{\theta\widetilde{k}}{2}\right) \overline{\psi\left(z, \overline{\mathbf{x}} - \frac{\theta\widetilde{\mathbf{x}}}{2}, \overline{k} - \frac{\theta\widetilde{k}}{2}\right)} \right\rangle,$$

which we calculated explicitly in [7], by solving equation (4.19). The moment formula is given by

$$\begin{aligned}\left\langle \psi\left(z, \overline{\mathbf{x}} + \tfrac{\theta\widetilde{\mathbf{x}}}{2}, \overline{k} + \tfrac{\theta\widetilde{k}}{2}\right) \overline{\psi\left(z, \overline{\mathbf{x}} - \tfrac{\theta\widetilde{\mathbf{x}}}{2}, \overline{k} - \tfrac{\theta\widetilde{k}}{2}\right)} \right\rangle &\approx \tfrac{-\overline{k}^2 \varphi_1(z,\widetilde{k})}{4\pi^2 z^2} \exp\left\{ -\tfrac{\widetilde{k}^2 \delta^2 D_r z}{2} - \tfrac{\overline{k}^2 D_\kappa \varphi_2(z,\widetilde{k}) z |\widetilde{\mathbf{x}}|^2}{6} \right\} \\ \int d\overline{\boldsymbol{\xi}} \int d\tilde{\boldsymbol{\xi}} \exp\left\{ \frac{i\widetilde{k}|\overline{\mathbf{x}} - \overline{\boldsymbol{\xi}}|^2}{2z} + \frac{i\overline{k}}{z}(\overline{\mathbf{x}} - \overline{\boldsymbol{\xi}}) \cdot (\widetilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}) + \overline{k}^2 \varphi_3(z, \widetilde{k}) \widetilde{\mathbf{x}} \cdot \tilde{\boldsymbol{\xi}} - \overline{k}^2 \varphi_4(z, \widetilde{k}) |\tilde{\boldsymbol{\xi}}|^2 \right. \\ \left. - \tfrac{\overline{k}^2 D_\kappa \varphi_2(z,\widetilde{k}) \varphi_5(z,\widetilde{k}) z}{6} \left[\widetilde{\mathbf{x}} \cdot \tilde{\boldsymbol{\xi}} + \varphi_5(z, \widetilde{k}) |\tilde{\boldsymbol{\xi}}|^2 \right] \right\} \psi_o\left(\overline{\boldsymbol{\xi}} + \tfrac{\theta\tilde{\boldsymbol{\xi}}}{2}, \overline{k} + \tfrac{\theta\widetilde{k}}{2} \right) \overline{\psi_o\left(\overline{\boldsymbol{\xi}} - \tfrac{\theta\tilde{\boldsymbol{\xi}}}{2}, \overline{k} - \tfrac{\theta\widetilde{k}}{2} \right)}. \end{aligned} \tag{6.7}$$

This result is obtained in the white noise limit and the approximation involves a simplification of the exact formula for small θ. The coefficients in this moment formula (6.7) are given by

$$\varphi_1(z,\widetilde{k}) = \frac{z\sqrt{-i\widetilde{k}D_\kappa}}{\sinh^{1/2}\left(z\sqrt{-i\widetilde{k}D_\kappa}\right)}\coth^{1/2}\left(z\sqrt{-i\widetilde{k}D_\kappa}\right), \tag{6.8}$$

$$\varphi_2(z,\widetilde{k}) = \frac{3i}{\widetilde{k}zD_\kappa}\left(\frac{\sqrt{-i\widetilde{k}D_\kappa}}{\tanh(z\sqrt{-i\widetilde{k}D_\kappa})} - \frac{1}{z}\right), \tag{6.9}$$

$$\varphi_3(z,\widetilde{k}) = \frac{i}{2\widetilde{k}z}\left(\frac{3z\sqrt{-i\widetilde{k}D_\kappa}}{\sinh(z\sqrt{-i\widetilde{k}D_\kappa})} - \frac{1}{\cosh(z\sqrt{-i\widetilde{k}D_\kappa})} - 2\right), \tag{6.10}$$

$$\varphi_4(z,\widetilde{k}) = \frac{D_\kappa\tanh(z\sqrt{-i\widetilde{k}D_\kappa})}{2\sqrt{-i\widetilde{k}D_\kappa}}\left(1 - \frac{\tanh(z\sqrt{-i\widetilde{k}D_\kappa})}{z\sqrt{-i\widetilde{k}D_\kappa}}\right) \tag{6.11}$$

$$\varphi_5(z,\widetilde{k}) = \frac{1}{\cosh(z\sqrt{-i\widetilde{k}D_\kappa})}. \tag{6.12}$$

To determine the decoherence length we let $\widetilde{k} \to 0$ and then study the decay over $|\widetilde{\mathbf{x}}|$ of the right hand side in (6.7).

In the limit $\widetilde{k} \to 0$, we get from (6.8)-(6.12) that

$$\phi_j(z,\widetilde{k}) = \begin{cases} 1 + O(\widetilde{k}), & j = 1,2,5 \\ O(\widetilde{k}^{1/2}), & j = 3,4 \end{cases} \tag{6.13}$$

and equation (6.7) simplifies to

$$\begin{aligned}\left\langle \psi\left(z,\overline{\mathbf{x}}+\tfrac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}\right)\overline{\psi\left(z,\overline{\mathbf{x}}-\tfrac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}\right)}\right\rangle &\approx \tfrac{-\overline{k}^2}{4\pi^2 z^2}\exp\left\{-\tfrac{\overline{k}^2 D_\kappa z|\widetilde{\mathbf{x}}|^2}{6}\right\}\int d\overline{\boldsymbol{\xi}}\int d\tilde{\boldsymbol{\xi}}\psi_o\left(\overline{\boldsymbol{\xi}}+\frac{\theta\tilde{\boldsymbol{\xi}}}{2},\overline{k}\right) \\ &\overline{\psi_o\left(\overline{\boldsymbol{\xi}}-\tfrac{\theta\tilde{\boldsymbol{\xi}}}{2},\overline{k}\right)}\exp\left\{\tfrac{i\overline{k}}{z}(\overline{\mathbf{x}}-\overline{\boldsymbol{\xi}})\cdot(\widetilde{\mathbf{x}}-\tilde{\boldsymbol{\xi}}) - \tfrac{\overline{k}^2 D_\kappa z}{6}\left[\widetilde{\mathbf{x}}\cdot\tilde{\boldsymbol{\xi}}+|\tilde{\boldsymbol{\xi}}|^2\right]\right\},\end{aligned} \tag{6.14}$$

with the explicit integration depending on the spatial support of the wave source function ψ_o. For example, in the case of a spatially distributed source, where

$$\psi_o(\overline{\boldsymbol{\xi}} \pm \theta\tilde{\boldsymbol{\xi}}/2,\overline{k}) \approx \psi_o(\overline{\boldsymbol{\xi}},\overline{k}),$$

the integration over $\tilde{\boldsymbol{\xi}}$ gives

$$\begin{aligned}\left\langle \psi\left(z,\overline{\mathbf{x}}+\tfrac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}\right)\overline{\psi\left(z,\overline{\mathbf{x}}-\tfrac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}\right)}\right\rangle &\approx \tfrac{-3}{2\pi z^3 D_\kappa}\exp\left\{-\tfrac{\overline{k}^2 D_\kappa z|\widetilde{\mathbf{x}}|^2}{2}\right\}\int d\overline{\boldsymbol{\xi}}\,|\psi_o(\overline{\boldsymbol{\xi}},\overline{k})|^2 \\ &\exp\left\{-\tfrac{3}{2z^3 D_\kappa}\left|\overline{\mathbf{x}}-\overline{\boldsymbol{\xi}}-\tfrac{i\overline{k}z^2 D_k}{2}\widetilde{\mathbf{x}}\right|^2\right\}.\end{aligned}$$

In the case of a small source with support of $O(\theta)$, where

$$\psi_o \rightsquigarrow \theta^{-2}\psi_o\left(\frac{\overline{\boldsymbol{\xi}}}{\theta}, \overline{k}\right),$$

we can let $\overline{\boldsymbol{\xi}} \rightsquigarrow \theta\overline{\boldsymbol{\xi}}$ in (6.14) and obtain

$$\left\langle \psi\left(z, \overline{\mathbf{x}} + \tfrac{\theta\widetilde{\mathbf{x}}}{2}, \overline{k}\right) \overline{\psi\left(z, \overline{\mathbf{x}} - \tfrac{\theta\widetilde{\mathbf{x}}}{2}, \overline{k}\right)} \right\rangle \sim \tfrac{-\overline{k}^2}{4\pi^2 z^2} \exp\left\{-\tfrac{\overline{k}^2 D_\kappa z|\widetilde{\mathbf{x}}|^2}{8} + \tfrac{i\overline{k}}{z}\overline{\mathbf{x}}\cdot\widetilde{\mathbf{x}}\right\} \int d\overline{\boldsymbol{\xi}} \int d\tilde{\boldsymbol{\xi}}\, \psi_o\left(\overline{\boldsymbol{\xi}} + \frac{\tilde{\boldsymbol{\xi}}}{2}, \overline{k}\right) \overline{\psi_o\left(\overline{\boldsymbol{\xi}} - \tfrac{\tilde{\boldsymbol{\xi}}}{2}, \overline{k}\right)} \exp\left\{-\tfrac{\overline{k}^2 D_\kappa z|\widetilde{\mathbf{x}}|^2}{6}\left|\tilde{\boldsymbol{\xi}} + \tfrac{\widetilde{\mathbf{x}}}{2}\right|^2 - \tfrac{i\overline{k}}{z}\tilde{\boldsymbol{\xi}}\cdot\overline{\mathbf{x}}\right\}.$$

In either case, the decay in $\widetilde{\mathbf{x}}$ occurs as a Gaussian function, with standard deviation of $O\left(\frac{1}{\overline{k}\sqrt{zD_\kappa}}\right)$. This means that the scaled decoherence length is

$$X_d(\overline{k}) \sim \frac{\theta}{\overline{k}\sqrt{zD_\kappa}} \tag{6.15}$$

and it corresponds to the scaled expected time reversal spot size derived in [23, 7]. From the analysis in [5, 23, 7] we know that the effective aperture is given, in scaled variables, by

$$a_e = \sqrt{D_\kappa z^3} \tag{6.16}$$

We can now write

$$X_d(\overline{k}) \sim \frac{\theta}{\overline{k}\kappa_d} \tag{6.17}$$

where

$$\kappa_d = \frac{a_e}{z} = \sqrt{D_\kappa z}. \tag{6.18}$$

The uncertainty in the direction of arrival of the waves in the random medium [8] is κ_d/θ.

Next, we estimate the decoherence frequency by setting $\widetilde{\mathbf{x}} \to 0$ in (6.7)

$$\begin{aligned} \left\langle \psi\left(z, \overline{\mathbf{x}}, \overline{k} + \tfrac{\theta\widetilde{k}}{2}\right) \overline{\psi\left(z, \overline{\mathbf{x}}, \overline{k} - \tfrac{\theta\widetilde{k}}{2}\right)} \right\rangle &= \tfrac{-\overline{k}^2 \varphi_1(z,\widetilde{k})}{4\pi^2 z^2} \exp\left\{-\tfrac{\widetilde{k}^2\delta^2 D_r z}{2}\right\} \int d\overline{\boldsymbol{\xi}} \int d\tilde{\boldsymbol{\xi}} \\ \exp\left\{\tfrac{i\widetilde{k}|\overline{\mathbf{x}} - \overline{\boldsymbol{\xi}}|^2}{2z} - \tfrac{i\overline{k}}{z}(\overline{\mathbf{x}} - \overline{\boldsymbol{\xi}})\cdot\tilde{\boldsymbol{\xi}} - \overline{k}^2\varphi_4(z,\widetilde{k})|\tilde{\boldsymbol{\xi}}|^2 - \tfrac{\overline{k}^2 D_\kappa \varphi_2(z,\widetilde{k})\varphi_5^2(z,\widetilde{k}) z|\tilde{\boldsymbol{\xi}}|^2}{6}\right\} \\ \psi_o\left(\overline{\boldsymbol{\xi}} + \tfrac{\theta\tilde{\boldsymbol{\xi}}}{2}, \overline{k} + \tfrac{\theta\widetilde{k}}{2}\right) \overline{\psi_o\left(\overline{\boldsymbol{\xi}} + \tfrac{\theta\tilde{\boldsymbol{\xi}}}{2}, \overline{k} - \tfrac{\theta\widetilde{k}}{2}\right)} \end{aligned} \tag{6.19}$$

and by taking the large $\widetilde{k}$ approximation in (6.19). We obtain from (6.8)-(6.12) that

$$\phi_1(z,\widetilde{k}) \approx z\sqrt{-2i\widetilde{k}D_\kappa}\, e^{-\frac{z}{2}\sqrt{\frac{\widetilde{k}D_\kappa}{2}}(1-i)}, \tag{6.20}$$

$$\phi_j(z,\widetilde{k}) = O(\widetilde{k}^{-1/2}), \quad j = 2, 4 \tag{6.21}$$

$$\phi_5(z,\widetilde{k}) \approx e^{-z\sqrt{\frac{\widetilde{k}D_\kappa}{2}}(1-i)}, \tag{6.22}$$

which means that as $\widetilde{k}$ increases, the decay in (6.19) is determined by the factor

$$\exp\left\{-\frac{z}{2}\sqrt{\frac{\widetilde{k}D_\kappa}{2}}-\frac{\widetilde{k}^2\delta^2 D_r z}{2}\right\}$$

and therefore, that the scaled decoherence frequency is

$$\Omega_d \sim \min\left\{\frac{\theta}{z^2 D_\kappa}, \frac{\theta}{\delta\sqrt{D_r z}}\right\} \approx \frac{\theta}{z^2 D_\kappa} \quad \text{as } \delta \to 0. \tag{6.23}$$

In conclusion, both X_d and Ω_d are small, of order θ in our scaling, which means that we can cover many decoherence lengths with an array aperture of $O(1)$ and we can fit many frequency intervals of width Ω_d in a broad bandwidth $B/\omega_o = O(1)$. This is a key point for achieving the self-averaging property of the CINT imaging function discussed below.

6.2.2 The coherent interferometric imaging function as a smoothed space-time Wigner transform

Consider a smooth window $\chi(r;\rho)$ of length $O(\rho)$, with Fourier transform

$$\widehat{\chi}(\widetilde{k};\rho^{-1}) = \int \chi(r;\rho)e^{i\widetilde{k}r}dr, \tag{6.24}$$

supported in the wavenumber interval $|\widetilde{k}| \le \rho^{-1}$, where

$$\rho \sim \frac{\theta}{\Omega_d} = O(1). \tag{6.25}$$

Let also $\Phi(\boldsymbol{\kappa};\kappa_d)$ be a smooth function of two dimensional vectors $\boldsymbol{\kappa}$, with support in a disk of radius $O(\kappa_d)$, with κ_d quantifying the uncertainty in the direction of arrival of the waves in the random medium, as explained in section 6.2.1. The Fourier transform of Φ is denoted by

$$\widehat{\Phi}(k\widetilde{\mathbf{x}};\kappa_d^{-1}) = \int \Phi(\boldsymbol{\kappa};\kappa_d)e^{-ik\boldsymbol{\kappa}\cdot\widetilde{\mathbf{x}}}d\boldsymbol{\kappa} \tag{6.26}$$

and it is supported in the disk $k|\widetilde{\mathbf{x}}| \le \kappa_d^{-1}$.

Using the windows (6.24), (6.26) and the migrated wave field (6.4), we define the coherent interferometric imaging function [8]

$$\begin{aligned}\mathcal{I}^{\rm CINT}(\vec{\mathbf{y}}^S;\rho,\kappa_d) \sim & \int d\overline{k}\int d\overline{\mathbf{x}}\int d\widetilde{k}\;\widehat{\chi}(\widetilde{k};\rho^{-1})\,e^{-i(\delta/\epsilon)^2\widetilde{k}\eta^S}\int d\widetilde{\mathbf{x}}\,\widehat{\Phi}(\overline{k}\widetilde{\mathbf{x}};\kappa_d^{-1}) \\ & \psi\left(L,\overline{\mathbf{x}}+\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}+\frac{\theta\widetilde{k}}{2}\right)\overline{\psi\left(L,\overline{\mathbf{x}}-\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}-\frac{\theta\widetilde{k}}{2}\right)} \\ & \exp\left\{-i\frac{(\overline{k}+\frac{\theta\widetilde{k}}{2})}{2\theta}\frac{\left|\boldsymbol{\xi}^S-\overline{\mathbf{x}}-\frac{\theta\widetilde{\mathbf{x}}}{2}\right|^2}{(L+\eta^S)}+i\frac{(\overline{k}-\frac{\theta\widetilde{k}}{2})}{2\theta}\frac{\left|\boldsymbol{\xi}^S-\overline{\mathbf{x}}+\frac{\theta\widetilde{\mathbf{x}}}{2}\right|^2}{(L+\eta^S)}\right\}\end{aligned} \tag{6.27}$$

that becomes after simplifying the exponent,

$$\begin{aligned}\mathcal{I}^{\rm CINT}(\vec{\mathbf{y}}^S;\rho,\kappa_d) \quad \sim \quad & \int d\overline{k} \int d\overline{\mathbf{x}} \int d\widetilde{k}\ \widehat{\chi}(\widetilde{k};\rho^{-1})\, e^{-i(\delta/\epsilon)^2\widetilde{k}\eta^S} \int d\widetilde{\mathbf{x}}\, \widehat{\Phi}(\overline{k}\widetilde{\mathbf{x}};\kappa_d^{-1}) \\ & \psi\left(L,\overline{\mathbf{x}}+\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}+\frac{\theta\widetilde{k}}{2}\right)\overline{\psi\left(L,\overline{\mathbf{x}}-\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}-\frac{\theta\widetilde{k}}{2}\right)} \\ & \exp\left\{-i\overline{k}\widetilde{\mathbf{x}}\cdot\frac{(\overline{\mathbf{x}}-\boldsymbol{\xi}^S)}{(L+\eta^S)} - i\widetilde{k}\frac{\left|\boldsymbol{\xi}^S-\overline{\mathbf{x}}\right|^2}{2(L+\eta^S)}\right\}.\end{aligned} \tag{6.28}$$

Now note that because the support of $\chi(r;\rho)$ is $O(1)$, the imaging function is nonzero if the range offset satisfies

$$\eta^S \le O\left(\epsilon^2/\delta^2\right) \ll 1.$$

We set then $\eta^S \rightsquigarrow \frac{\epsilon^2}{\delta^2}\eta^S$ and write approximately for $\vec{\mathbf{y}}^S = \left(L+\frac{\epsilon^2}{\delta^2}\eta^S,\boldsymbol{\xi}^S\right)$,

$$\begin{aligned}\mathcal{I}^{\rm CINT}(\vec{\mathbf{y}}^S;\rho,\kappa_d) \quad \sim \quad & \int d\overline{k} \int d\overline{\mathbf{x}} \int d\widetilde{k}\ \widehat{\chi}(\widetilde{k};\rho^{-1}) \int d\widetilde{\mathbf{x}}\, \widehat{\Phi}(\overline{k}\widetilde{\mathbf{x}};\kappa_d^{-1}) \\ & \psi\left(L,\overline{\mathbf{x}}+\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}+\frac{\theta\widetilde{k}}{2}\right)\overline{\psi\left(L,\overline{\mathbf{x}}-\frac{\theta\widetilde{\mathbf{x}}}{2},\overline{k}-\frac{\theta\widetilde{k}}{2}\right)} \\ & \exp\left\{i\overline{k}\widetilde{\mathbf{x}}\cdot\frac{(\boldsymbol{\xi}^S-\overline{\mathbf{x}})}{L} - i\widetilde{k}\left(\eta^S+\frac{\left|\boldsymbol{\xi}^S-\overline{\mathbf{x}}\right|^2}{2L}\right)\right\}.\end{aligned} \tag{6.29}$$

Next, we use the definition (4.4) of the Wigner transform in (6.29) and obtain

$$\begin{aligned}\mathcal{I}^{\rm CINT}(\vec{\mathbf{y}}^S;\rho,\kappa_d) \quad \sim \quad & \int d\overline{k} \int d\overline{\mathbf{x}} \int d\mathbf{q} \int dr\, W(L,\overline{\mathbf{x}},\overline{k},\mathbf{q},r) \\ & \int d\widetilde{k}\, \widehat{\chi}(\widetilde{k};\rho^{-1}) \exp\left[-i\widetilde{k}\left(\eta^S+\frac{\left|\boldsymbol{\xi}^S-\overline{\mathbf{x}}\right|^2}{2L}-r\right)\right] \\ & \int d\widetilde{\mathbf{x}}\, \widehat{\Phi}(\overline{k}\widetilde{\mathbf{x}};\kappa_d^{-1}) \exp\left[i\overline{k}\widetilde{\mathbf{x}}\cdot\frac{\boldsymbol{\xi}^S-\overline{\mathbf{x}}}{L} - i\mathbf{q}\cdot\widetilde{\mathbf{x}}\right].\end{aligned} \tag{6.30}$$

Finally, changing variables $\mathbf{q}=\overline{k}\boldsymbol{\kappa}$, we get

$$\begin{aligned}\mathcal{I}^{\rm CINT}(\vec{\mathbf{y}}^S;\rho,\kappa_d) \quad \sim \quad & \int d\overline{\mathbf{x}} \int d\boldsymbol{\kappa} \int dr\, \chi\left(\eta^S+\frac{\left|\boldsymbol{\xi}^S-\overline{\mathbf{x}}\right|^2}{2L}-r;\rho\right) \\ & \Phi\left(\frac{\boldsymbol{\xi}^S-\overline{\mathbf{x}}}{L}-\boldsymbol{\kappa};\kappa_d\right) \int d\overline{k}\, W(L,\overline{\mathbf{x}},\overline{k},\overline{k}\boldsymbol{\kappa},r).\end{aligned} \tag{6.31}$$

In conclusion, the coherent imaging function is given by the Wigner transform, smoothed by convolution over directions $\boldsymbol{\kappa}$ and range r and by integration over the array locations $\overline{\mathbf{x}}$ and wavenumbers $\overline{k}$. The self-averaging of $\mathcal{I}^{\rm CINT}$ follows from Theorem 1.

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