Passive and active array imaging in a waveguide

We consider in this project the problem of passive and active array imaging in a waveguide. For the passive array case the geometry of the problem is depicted in Figure 1. The linear array is composed by point transducers located at fixed range $\vec{\mathbf{x}}_r = (x_r, z_r)$, $r = 1, \ldots, N_r$.

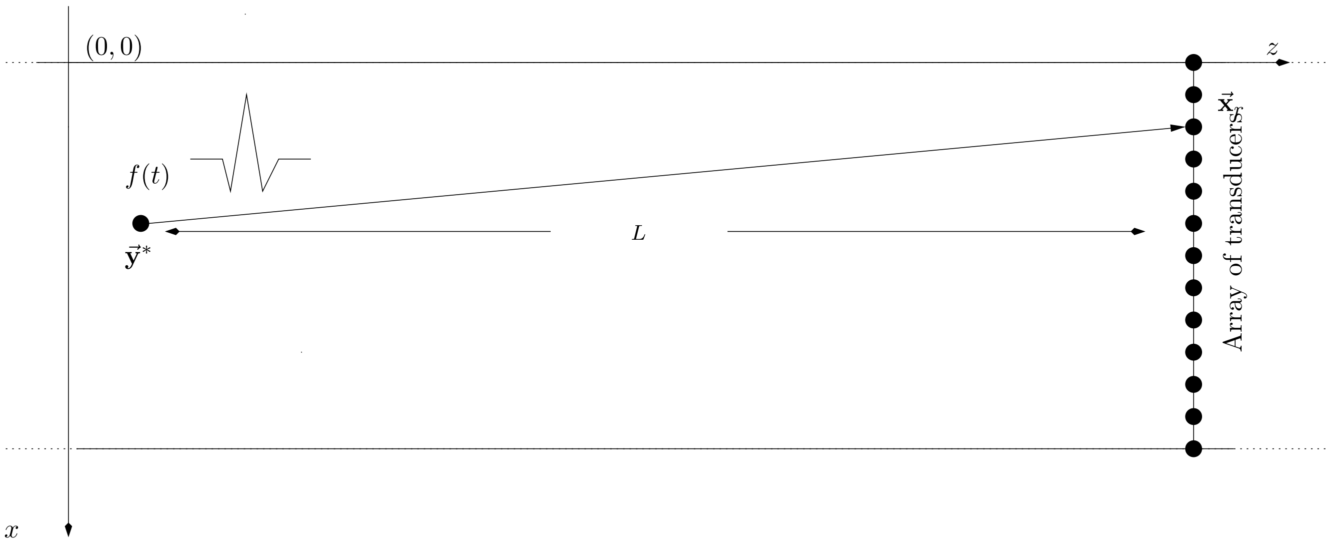

Figure 1: Setup for imaging a point source located at $\vec{\mathbf{y}}^*$ with a passive array of transducers in a waveguide.

The aperture of the array is $a = (N_r - 1)\ h$, with h the array pitch, that is, the distance between two consecutive receiver elements.

The data recorded at the array is the acoustic pressure field $\widehat{p}(\vec{\mathbf{x}}_r, \omega)$ solution of the Helmholtz equation

$$\frac{1}{c_0^2}\widehat{p}(\vec{\mathbf{x}}, \omega) + \Delta\widehat{p}(\vec{\mathbf{x}}, \omega) = f(t)\delta(\vec{\mathbf{x}} - \vec{\mathbf{y}}^*), \tag{1}$$

in a waveguide with constant speed of propagation $c_0(\vec{\mathbf{x}}) = c_0 = 1500$m/s and depth D. At the boundaries of the waveguide we have the boundary conditions,

$$p(x = 0, z, t) = 0, \quad \frac{\partial}{\partial x}p(x = D, z, t) = 0. \tag{2}$$

In (1) we assume that we have a point source located at $\vec{\mathbf{y}}^* = (x^*, z^*)$.

To produce numerically the array data you should solve the wave equation either in the frequency or in the time domain. Note that when the active array problem is considered the data is the scattered field computed by taking the difference between the total field and the incident field. To produce the images, you will use the Kirchhoff migration imaging functional, which for the passive array imaging problem is

$$\mathcal{I}^{\mathrm{KM}}(\vec{\mathbf{y}}^S) = \sum_{r=1}^{N_r} \overline{\widehat{p}(\vec{\mathbf{x}}_r, \omega)} G_{WG}(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S, \omega), \tag{3}$$

where

$$\widehat{G}_{WG}(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S, \omega) = \frac{\imath}{2} \sum_{j=1}^{N(\omega)} \frac{1}{\beta_j(\omega)} \phi_j(x^s)\phi_j(x_r) e^{\imath\beta_j(\omega)(z_r - z^s)}, \tag{4}$$

is the Green's function in the homogeneous background waveguide. Here

$$\begin{aligned} &N(\omega) = \lfloor \frac{1}{2} + \frac{\omega D}{\pi c_0} \rfloor \ ; \ \phi_j(x) = \sqrt{\frac{2}{D}} \sin\left(\sqrt{\lambda_j(\omega)}x\right) \ ; \ \lambda_j = \frac{\left(j - \frac{1}{2}\right)^2 \pi^2}{D^2} \\ &\beta_j(\omega) = \begin{cases} \sqrt{k^2 - \lambda_j}, & j = 1, 2, \dots N(\omega), \\ \imath\sqrt{\lambda_j - k^2} & j \geq N(\omega). \end{cases} \end{aligned} \tag{5}$$

To test your code you can use the following analytical expression for the array data in the case of one point source located at $\vec{\mathbf{y}}^* = (x^*, z^*)$

$$\widehat{p}(\vec{\mathbf{x}}_r, \omega) = \frac{\imath}{2} \widehat{f}(\omega) \sum_{j=1}^{N(\omega)} \frac{1}{\beta_j(\omega)} \phi_j(x^*)\phi_j(x_r) e^{\imath\beta_j(\omega)(z_r - z^*)}. \tag{6}$$

In the active array case the KM imaging functional becomes,

$$\mathcal{I}^{\mathrm{KM}}(\vec{\mathbf{y}}^S) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} \overline{\widehat{p}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)} G_{WG}(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S, \omega) G_{WG}(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}^S, \omega). \tag{7}$$

To test your code you can use the following analytical expression for the array data in the case of one point scatterer located at $\vec{\mathbf{y}}^* = (x^*, z^*)$,

$$\widehat{p}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega) = \frac{-1}{4} \widehat{f}(\omega) \sum_{j=1}^{N(\omega)} \sum_{i=1}^{N(\omega)} \frac{1}{\beta_j(\omega)} \frac{1}{\beta_i(\omega)} \phi_j(x^*)\phi_j(x_r)\phi_i(x^*)\phi_i(x_s) e^{\imath\beta_j(\omega)(z_r - z^*)} e^{\imath\beta_i(\omega)(z_r - z^*)}. \tag{8}$$

Central frequency/Multiple frequencies For the imaging part assume that the array data are known at the frequency range $[f_0 - B/2, f_0 + B/2]$, with f_0 the central frequency (recall that $\omega = 2\pi f$). For the construction of the data you should consider the following source function,

$$\widehat{f}_1(\omega) = \mathbb{I}_{[\omega_0 - \pi B, \omega_0 + \pi B]}$$

or some (smooth) tapered version of $\widehat{f}_1(\omega)$ with support in the same bandwidth.

The length units in the following will be given in terms of the reference wavelength $\lambda_0 = f_0/c_0$. Assume a reference frequency of 75Hz, for $c_0 = 1500$m/s the central wavelength is $\lambda_0 = 20$m.

Questions

1. **Full aperture** Consider a waveguide of depth $D = 10\lambda_0$ and a linear array with $N_r = 51$ elements that span the entire depth of the waveguide. The location of the array elements is $\vec{\mathbf{x}}_r = (x_r, z)$ with $z = 22\lambda_0$ and $x_r = (r-1)\ h$, $h = D/(N_r - 1)$.

   (a) **One point source** Consider the case of one point source located at $\vec{\mathbf{y}}^* = (6\lambda_0 + \lambda_0/4, \lambda_0)$. Construct the KM image using the single frequency $f = 73$Hz. *What is the resolution in range? in cross-range ?* Compare with the theory. Here you should also do a theoretical analysis of the point spread function for KM imaging. *Explore your algorithm to image one point source at different locations. What determines the characteristics of the image that you obtain?*

   (b) Use multiple frequencies, what is the effect of the bandwidth on the resolution of the image? On the SNR?

   (c) **Two point sources** *Design an experiment that exploits the resolution limits to image two objects: consider two sources at the minimum distance at which they can be separated.*

2. **Partial aperture** Consider now a smaller array composed by $N_r = 26$ receivers with the same inter-element distance h as before. Take first one and then two point sources and answer the above questions (in blue). How do the results compare with the full aperture case?

3. **Noise** Add white noise to the data according to the model described by equations (1.2)-(1.3) of chapter 3 in the notes. Chose $\sigma_{\mathcal{N}}$ so that the SNR of the data takes the values $10, 0, -10$ dB. How does the SNR of the image depends on the SNR of the data? the number of receivers?

4. **Consider an active configuration where the sources become point targets.**

   (a) Is there a difference between the active and passive case in terms of resolution?

   (b) To construct the image use either the full array data or just one column of the response matrix. Do you observe a difference?(in terms of resolution, SNR?)

   (c) Consider the partial array aperture case can you cook up a geometric configuration where one point target becomes invisible? Do to this use either the full array data or just one column of the response matrix to construct the image.

5. **Consider imaging an extended object in a waveguide (full wave solution)**

   (a) Construct the KM image with an active array.

   (b) The source and the receiver array need not be collocated. Put the receiver array on one side of the object and the source array on the other side. What do you observe? Do you get a better image.

(c) What is the effect of multiple source illuminations.

6. **Kirchhoff-Helmholtz identity** Use white noise sources on the array and record the solution of the wave equation $p(\vec{\mathbf{y}}_1, t)$ and $p(\vec{\mathbf{y}}_2, t)$ at two points $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. Compute the empirical cross-correlation between these two recordings

$$C_T(\tau, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2) = \frac{1}{T} \int_0^T p(\vec{\mathbf{y}}_1, t) p(\vec{\mathbf{y}}_2, t + \tau) dt$$

and compare it with the Green's function between these two points. Consider different configurations of $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. What do you observe?

All images should be computed on a square of size $D \times D$ with a discretization of $\lambda_0/4$ and centered at $\vec{\mathbf{y}}^*$.

The report acmac-0232 and references therein should be useful for this project.