Active array imaging in free space

In the case of active imaging the array elements act as sources and receivers and the object that we wish to image is a scatterer. The geometry of the problem is depicted in Figure 1. We dispose of a linear array (in 2d) which sends the pulse $f(t, \vec{\mathbf{x}}_s)$ from the sources

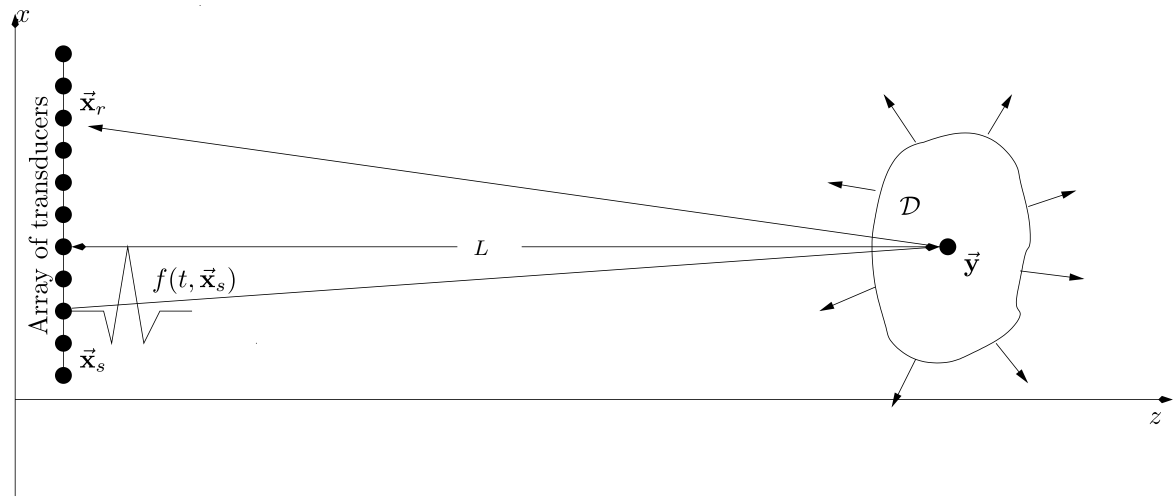

Figure 1: Setup for imaging a distributed scatterer $\mathcal{D}$ with an active array in free space.

and records the data at the receivers. We assume here that the sources and receivers are point transducers located at the same points, denoted $\vec{\mathbf{x}}_s$, $s = 1, \ldots, N$ for the sources and $\vec{\mathbf{x}}_r$, $r = 1, \ldots, N$ for the receivers. The aperture of the array is $a = (N-1)\ h$, with h the array pitch, that is, the distance between the receiver elements. The data recorded at the array is the acoustic pressure field $p(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, t)$ recorder at receiver $\vec{\mathbf{x}}_r$ when the pulse $f(t, \vec{\mathbf{x}}_s)$ is emitted from the source located at $\vec{\mathbf{x}}_s$. In imaging we are interested in solving the following problem:

Problem 1 *Find the support D of the scatterer given the array response matrix $p(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, t)$, for $s, r = 1, \ldots, N$ and t in some time interval $[0, T]$.*

We will assume that the source function $f(t, \vec{\mathbf{x}}_s)$ is of the following form,

$$f(\vec{\mathbf{x}}, t) = f(t)\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}_s). \tag{1}$$

To produce numerically the array data we can solve the wave equation either in the time or in the frequency domain. Note that the data is the scattered field computed by taking the difference between the total field and the incident field. We can also use the following expression (only to test the imaging algorithm, and for the point scatterers, for the extended object the full wave solution should be obtained)

$$\widehat{p}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega) = \widehat{f}(\omega) \int_D d\vec{\mathbf{y}} \widehat{G}_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}, \omega) \widehat{G}_0(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}, \omega), \tag{2}$$

with

$$\widehat{G}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega) = \frac{e^{i\ \omega \frac{|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}{c_0}}}{4\pi\,|\vec{\mathbf{x}}-\vec{\mathbf{y}}|},$$

being the Green's function in the homogeneous background medium. For this project take $c_0 = 1500m/s$.

To produce the images, you should use the Kirchhoff migration imaging functional,

$$\mathcal{I}^{\mathrm{KM}}(\vec{\mathbf{y}}^S) = \int d\omega \sum_{r=1}^{N}\sum_{s=1}^{N} \overline{\widehat{p}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)}\,\widehat{G}_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S, \omega)\widehat{G}_0(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}^S, \omega) \tag{3}$$

Central frequency/Multiple frequencies For the imaging part assume that the array data are known at the frequency range $[f_0 - B/2, f_0 + B/2]$, with f_0 the central frequency (recall that $\omega = 2\pi f$). For the construction of the data you should consider the following source function,

$$\widehat{f}(\omega) = \mathbb{1}_{[\omega_0 - \pi B, \omega_0 + \pi B]}$$

or some (smooth) tapered version of $\widehat{f}(\omega)$ with support in the same bandwidth.

The length units in the following will be given in terms of the reference wavelength $\lambda_0 = f_0/c_0$.

Questions

1. **Linear array** Consider a linear array with $N = 51$ elements and array pitch $h = \lambda_0/2$. The location of the array elements is $\vec{\mathbf{x}}_r = (x_r, z)$ with $z = 2\lambda_0$m and $x_r = 2\lambda_0 + (r-1)\ h$. We call the direction x the cross-range and z the range.

   (a) **Point target** Consider the case of a point target located at $\vec{\mathbf{y}}^* = (15\lambda_0, 52\lambda_0)$.

      i. Construct the KM image using only one frequency, $f_0 = 1.5$kHz. What do you observe? What is the resolution in range? in cross range? Compare with the theory. You can use either one source on the array or many sources. Do you get any benefit by using more sources?

      ii. Add white noise to the data according to the model described by equations (2.16)-(2.17) of chapter 3 in the notes. Chose $\sigma_{\mathcal{N}}$ so that the SNR of the data takes the values $10, 0, -10$ dB. How does the SNR of the image depends on the SNR of the data? the number of receivers? the number of sources?

      iii. Use multiple frequencies, what is the effect of the bandwidth on the resolution of the image? On the SNR?

      iv. Decrease the array size, what do you observe? Can you find the location of the source with one array element? with two?

(b) **Multiple point targets (use the Foldy-Lax model)** Consider the case of five point targets ($\vec{\mathbf{y}}^*$ plus four). Choose first their configuration so that all of them are visible from the array. Can you find a configuration for which one target becomes invisible? Is this true for multiple frequencies as well?

(c) **Extended object (full wave simulation)** Consider the case of an extended object. Take for example, as domain D a disk with center $\vec{\mathbf{y}}^*$ and radius $b = 2\lambda_0$. Compute the data using $f_0 = 1.5$kHz and $B = 1$kHz. Construct the KM image for one frequency and multiple frequencies. What do you observe? What happens for a different shape of D, a square or a rectangle?

2. **Circular array** Consider a circular array with $N = 101$ (equidistant) elements. The location of the array elements is on a circle centered at $\vec{\mathbf{y}}^*$ with radius $r = 20\lambda_0$. You might need to increase the number of array elements if the results are not very good, make some tests to decide.

   (a) **Point target** Consider the case of a point target located at $\vec{\mathbf{y}}^* = (30\lambda_0, 30\lambda_0)$. Construct the KM image using only one frequency, $f_0 = 1.5$kHz. What do you observe? What is the resolution in range? in cross range? Compare with the theory. You can use either one source on the array or many sources. Do you get any benefit by using more sources?

   (b) **Multiple point targets (use the Foldy-Lax model)** Consider the case of five point targets ($\vec{\mathbf{y}}^*$ plus four). Chose first their configuration so as all of them are visible from the array. Can you find a configuration for which one target becomes invisible?

   (c) **Extended object (full wave simulation)** Consider the case of an extended object. Take for example, as domain D a disk with center $\vec{\mathbf{y}}^*$ and radius $b = 2\lambda_0$. Compute the data using $f_0 = 1.5$kHz and $B = 1$kHz. Construct the KM image for one frequency and multiple frequencies. What do you observe? What happens for a different shape of D, a square or a rectangle?

3. **Kirchhoff-Helmholtz identity** Use white noise sources on the circular array and record the solution of the wave equation $p(\vec{\mathbf{y}}_1, t)$ and $p(\vec{\mathbf{y}}_2, t)$ at two points $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. Compute the empirical cross-correlation between these two recordings

$$C_T(\tau, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2) = \frac{1}{T} \int_0^T p(\vec{\mathbf{y}}_1, t) p(\vec{\mathbf{y}}_2, t + \tau) dt$$

   and compare it with the symmetrized Green's function between these two points. Consider different configurations of $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. What do you observe? What happens when the array is not circular but linear? In what case can you recover the symmetrized Green's function between the two points.
   NOTE: the two points do not need to be in a homogeneous medium.

All images should be computed on a square of size $20\lambda_0 \times 20\lambda_0$ with a discretization of $\lambda_0/4$ and centered at $\vec{\mathbf{y}}^*$.