FULLY DISCRETE NONCONFORMING FINITE ELEMENT SCHEMES FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS.

THEODOROS D. KATSAOUNIS

Department of Mathematics University of Crete, Heraklion 71409, Crete and
Institute of Applied and Computational Mathematics, FORTH, Heraklion 71110, Crete, GREECE.

ABSTRACT. Fully discrete approximations to the solution of the incompressible Navier–Stokes equations are introduced and analyzed. Standard elements are used for the pressure approximation, while nonconforming finite element spaces are used for the velocity approximation. These elements are discontinuous across interelement boundaries and satisfy the incompressibility condition pointwise on each "triangle". Implicit Runge–Kutta methods are used for the temporal discretizations. The spatial grids are allowed to change from one time level to the next. Optimal order error estimates are proved for the approximations of velocity and pressure.

1. INTRODUCTION

In this work we study the convergence of a class of high-order finite element/Runge Kutta discretizations approximating smooth solutions of the Navier-Stokes equations of incompressible fluid flow:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\times(0,T] \tag{1.1}$$

$$\text{div } \mathbf{u} = 0 \quad \text{in} \quad \Omega\times[0,T] \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial\Omega \tag{1.3}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{1.4}$$

where Ω is an open bounded domain in $\mathbb{R}^d, d = 2,3$ with smooth boundary $\partial\Omega$ and $0 < T < \infty$. Also $\mathbf{u} = (u_1,\ldots,u_d) : \Omega\times[0,T] \mapsto \mathbb{R}^d$ is the velocity field, $p : \Omega\times[0,T] \mapsto \mathbb{R}$ is the pressure, $\mathbf{f}$ is a given $\mathbb{R}^d$-valued function (body forces) defined on $\Omega\times[0,T]$. $\mathbf{u}_0 : \Omega \mapsto \mathbb{R}^d$, and $\mathbf{g} : \partial\Omega \mapsto \mathbb{R}^d$ are also given and ν is the kinematic viscosity constant. It is worth note that if a (smooth) solution of (1.1) – (1.4) exists it must then satisfy the compatibility condition $\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}\,d\sigma = \int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}\,d\sigma = 0$, i.e. the net flux across the boundary is zero. Throughout this work we shall assume that $\mathbf{g}$ is independent of time. The existence, uniqueness and regularity of solutions of (1.1)-(1.4), are discussed, e.g., in the classic works of R.Temam [28] and O.Ladyzhenskaya [24]. In this paper we work under the assumption that a smooth enough solution of (1.1-4) exists on $[0,T]$.

This paper is an extension of the works of Baker, Jureidini and Karakashian [3],[18] where the stationary problems for the Stokes and Navier-Stokes equations were respectively treated. We consider Implicit Runge-Kutta time discretizations of arbitary order. Techniques for analyzing the convergence of full discretizations based on Implicit Runge-Kutta methods obtaining optimal (classical) rates were introduced in [22] in the

E-mail : thodoros@math.uch.gr

context of the Korteweg-de Vries equation and later extended to the Nonlinear Schrödinger equation in [16]. Even though the techniques introduced in the aforementioned papers are fully exploited here, there were several important issues that required settling. Foremost among these was the fact that it is not obvious how to perform the time-stepping for the pressure since the equation does not contain a p_t term. This is not the case for linear multistep methods e.g. [2]. In the same vein, the analysis of [22] and [16] required a nontrivial extension. Another important detail consisted in understanding the role of the incompressibility condition in the analysis of convergence. Also, the efficiency of computations dictates that one be able to change the spatial grids from time to time as the integration proceeds. As shown by Dupont [9], arbitrarily changed meshes can lead to nonsensical numerical results. Therefore we chose to do the analysis while taking into account such changes and gave sufficient conditions guaranteeing the viability of the results.

Preleminaries. For a subdomain $D \subset \Omega$, let $L^p(D)$, $W^{m,p}(D)$ denote the standard Lebesgue and Sobolev spaces respectively and let $\mathbf{L}^p(D) = \{L^p(D)\}^d$, $\mathbf{W}^{m,p}(D) = \{W^{m,p}(D)\}^d$ and $\mathbf{H}^m(D) = \{H^m(D)\}^d$. We denote the associated norms (of vector and scalar spaces) by $\|\cdot\|_{L^p(D)}$, $\|\cdot\|_{m,p,D}$ and $\|\cdot\|_{m,D}$. In addition, we consider the spaces of solenoidal vector fields:

$$\mathbf{S}^m(D) = \{\mathbf{v} \in \mathbf{H}^m(D) : \operatorname{div}\mathbf{v} = 0 \text{ in } D\} , \qquad m \geq 1.$$

Also if X is a Banach space with norm $\|\cdot\|_X$, then $X/\mathbb{R}$ is the quotient space of equivalent classes of functions differing by constants. $X/\mathbb{R}$ is a Banach space with norm

$$\|v\|_{X/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|v - c\|_X.$$

The finite element spaces. The spatial discretizations will be based on the piecewise solenoidal spaces introduced in Baker, Jureidini and Karakashian [3]. The approximations to $\mathbf{u}$ and p will be sought in appropriate approximation spaces denoted by $\mathbf{V}_k$ and P_h respectively. These spaces will be generated by two (in general different) partitions of Ω namely $\mathcal{T}_k = \{\Omega_i^k\}_{i=1}^{d_k}$ and $\mathcal{T}_h = \{\Omega_\ell^h\}_{\ell=1}^{d_h}$, where k and h are the corresponding mesh discretization parameters.

For an integer $s \geq 1$, let $J^s(D)$ denote the linear space of polynomials in d-variables of degree less than or equal to $s-1$ on D. We also let $J^0 = \{0\}$ and

$$\mathbf{J}^s(D) = \{J^s(D)\}^d, \qquad \mathbf{V}^s(D) = \{\mathbf{v} \in \mathbf{J}^s(D) : \ \operatorname{div}\mathbf{v} = 0 \in D\}.$$

Obviously $\mathbf{V}(D) \subset \mathbf{S}^m(D)$ for $m \geq 1$. For $r_1 \geq 2$ integer we set

$$\mathbf{V}_k = \mathbf{V}^{r_1}(\Omega_1^k) \times \cdots \times \mathbf{V}^{r_1}(\Omega_{d_k}^k) ,$$

$\mathbf{V}_k$ is a space of piecewise solenoidal vector fields in Ω since for $\mathbf{v} \in \mathbf{V}_k, \mathbf{v}^{(i)}$ is a solenoidal vector field in Ω_i^k, i.e. $\mathbf{V}_k$ is a totally nonconforming piecewise polynomial function space and for any $\mathbf{v} \in \mathbf{V}_k$ there holds $\operatorname{div}\mathbf{v}|_{\Omega_i^k} = 0$. In practice these spaces can be constructed easily and posess optimal approximation properties, in relation to the spaces $\mathbf{S}^m(\Omega)$. They also form with the pressure space a stable pair provided the subdivision for the velocity is subordinate to and finer than the subdivision used to generate the pressure space, cf. [3].

Let P_h denote the space of continuous piecewise polynomial functions of degree less than or equal to $r_2 - 1, r_2 \geq 2$, defined on $\mathcal{T}_h$,[7]

$$P_h = \{\phi \in C^0(\Omega) : \phi|_{\Omega_\ell^h} \in J^{r_2}(\Omega_\ell^h),\ \ell = 1, \ldots, d_h\}.$$

Clearly $P_h \subset H^1(\Omega)$; P_h will be the approximating space for the pressure. For the estimates to follow we will denote $r = \min\{r_1, r_2 + 1\}$, and μ, an integer $2 \leq \mu \leq r$. We introduce the "energy" space $\mathbf{E}_k = \mathbf{H}^2(\Omega_1^k) \times \cdots \times \mathbf{H}^2(\Omega_{d_k}^k)$ and on $\mathbf{E}_k \times \mathbf{E}_k$ we define a bilinear form $a_k^\gamma(\cdot,\cdot)$ as follows : for $\mathbf{u}, \mathbf{v} \in \mathbf{E}_k$

$$(1.5) \quad a_k^\gamma(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{d_k} \Big\{ (\nabla \mathbf{u}^{(i)}, \nabla \mathbf{v}^{(i)})_{0,\Omega_i^k} + \sum_{j \in \mathcal{N}_i} \tau_{ij} \Big[-\langle \frac{\partial \mathbf{u}^{(i)}}{\partial n}, \mathbf{v}^{(i)} - \mathbf{v}^{(j)} \rangle_{\partial\Omega_{ij}^k} - \langle \frac{\partial \mathbf{v}^{(i)}}{\partial n}, \mathbf{u}^{(i)} - \mathbf{u}^{(j)} \rangle_{\partial\Omega_{ij}^k}$$

$$+ \gamma k_i^{-1} \langle \mathbf{u}^{(i)} - \mathbf{u}^{(j)}, \mathbf{v}^{(i)} - \mathbf{v}^{(j)} \rangle_{\partial\Omega_{ij}^k} \Big] - \langle \frac{\partial \mathbf{u}^{(i)}}{\partial n}, \mathbf{v}^{(i)} \rangle_{\partial\Omega_i^{*k}} - \langle \frac{\partial \mathbf{v}^{(i)}}{\partial n}, \mathbf{u}^{(i)} \rangle_{\partial\Omega_i^{*k}} + \gamma k_i^{-1} \langle \mathbf{u}^{(i)}, \mathbf{v}^{(i)} \rangle_{\partial\Omega_i^{*k}} \Big\},$$

where $\mathbf{v}^{(i)} = \mathbf{v}|_{\Omega_i^k}$, $\gamma > 0$ is a constant to be chosen appropriately, (τ_{ij}) denotes the matrix

$$\tau_{ij} = \begin{cases} 1 & i < j \\ 0 & i \geq j \end{cases} \qquad i, j = 1, \ldots, d_k,$$

$\partial\Omega_{ij}^k = \Omega_i^k \cap \Omega_j^k$, $\partial\Omega_i^{*k} = \Omega_i^k \cap \partial\Omega$ and $\mathcal{N}_i = \{j : \Omega_i^k$ and Ω_j^k have a common interface $\}$, $i = 1, \ldots, d_k$. We have set also for $D \subset \partial\Omega_{ij}^k$ or $\partial\Omega_i^{*k}$,

$$\langle \mathbf{v}, \mathbf{w} \rangle_D = \sum_{j=1}^{d} \langle v_j, w_j \rangle_D = \sum_{j=1}^{d} \int_D v_j w_j \, d\tau \qquad \text{and} \qquad |\mathbf{v}|_D = [\langle \mathbf{v}, \mathbf{v} \rangle_D]^{1/2}.$$

The "jump" terms are due to the fact that the elements of $\mathbf{V}$ are discontinuous at the interfaces of the elements Ω_i^k. The bilinear form $a_k^\gamma(\cdot,\cdot)$ generalizes the standard Dirichlet form

$$a(\mathbf{u},\mathbf{v}) = \sum_{j,\ell=1}^{d} \int_\Omega \frac{\partial u_j}{\partial x_\ell} \frac{\partial v_j}{\partial x_\ell} dx.$$

Indeed if $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ then $a_k^\gamma(\mathbf{u},\mathbf{v}) = -(\Delta \mathbf{u}, \mathbf{v}), \forall \mathbf{v} \in \mathbf{E}_k$. The $3rd$ and the $6th$ terms in $a_k^\gamma(\cdot,\cdot)$ are added to produce symmetry; also by choosing γ sufficiently large, $a_k^\gamma(\cdot,\cdot)$ will be coercive with respect to an appropriate "energy" norm $\|\cdot\|_{1,k} : \mathbf{E}_k \to \mathbb{R}$ defined by

$$(1.6) \quad \|\mathbf{v}\|_{1,k} = \left\{ \sum_{i=1}^{d_k} \left(\|\nabla \mathbf{v}^{(i)}\|_{0,\Omega_i^k}^2 + \sum_{j \in \mathcal{N}_i} \tau_{ij} \left[k_i \left| \frac{\partial \mathbf{v}^{(i)}}{\partial n} \right|_{\partial\Omega_{ij}^k}^2 + k_i^{-1} |\mathbf{v}^{(i)} - \mathbf{v}^{(j)}|_{\partial\Omega_{ij}^k}^2 \right] + k_i \left| \frac{\partial \mathbf{v}^{(i)}}{\partial n} \right|_{\partial\Omega_i^{*k}}^2 + k_i^{-1} |\mathbf{v}^{(i)}|_{\partial\Omega_i^{*k}}^2 \right) \right\}^{1/2}.$$

To measure the pressure error we introduce the following norm on $H^1(\Omega)/\mathbb{R}$,

$$(1.7) \qquad \|q\|_{0,h} = \left\{ \|q\|_{L^2(\Omega)/\mathbb{R}}^2 + \sum_{\ell=1}^{d_h} h_\ell^2 \|\nabla q^{(\ell)}\|_{0,\Omega_\ell^h}^2 \right\}^{1/2}.$$

Finally we define the energy space $X = \mathbf{E}_k \times H^1(\Omega)/\mathbb{R}$ with norm

$$|\!|\!|[\mathbf{v}, q]|\!|\!| = \{\nu \|\mathbf{v}\|_{1,k}^2 + \frac{1}{\nu}\|q\|_{0,h}^2\}^{1/2},$$

and its finite dimensional subspace $X_{k,h} = \mathbf{V}_k \times P_h/\mathbb{R}$. On $X \times X$ we define the bilinear form

$$\mathcal{B}([\mathbf{u}, p], [\mathbf{v}, q]) = \nu a_k^\gamma(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \nabla q) + (\mathbf{v}, \nabla p),$$

which corresponds to the stationary Stokes problem. In [3] it was proved that $\mathcal{B}$ satisfies the necessary continuity as well as the essential Babuska-Brezzi(inf-sup) condition: Suppose r_1, r_2 are given. If $\mathcal{T}_k$ is sufficiently fine with respect to $\mathcal{T}_h$, then,

$$|\mathcal{B}([\mathbf{u}, p], [\mathbf{v}, q])| \le c |\!|\!|[\mathbf{u}, p]|\!|\!| \, |\!|\!|[\mathbf{v}, q]|\!|\!| \,, \qquad \forall [\mathbf{u}, p], [\mathbf{v}, q] \in X, \tag{1.8a}$$

$$\sup_{\substack{0 \ne [\mathbf{v}, q] \in X_{k,h}^n \\ |\!|\!|[\mathbf{v}, q]|\!|\!| \le 1}} |\mathcal{B}([\mathbf{u}, p], [\mathbf{v}, q])| \ge \frac{c}{(1+\gamma)^2} |\!|\!|[\mathbf{u}, p]|\!|\!| \,, \qquad \forall [\mathbf{u}, p] \in X_{k,h}, \tag{1.8b}$$

where c is independent of k and h. Given $[\mathbf{u}, p] \in X$, we define its Stokes projection $[\mathbf{w}_k, \pi_h] \in X_{k,h}$ by

$$\mathcal{B}([\mathbf{w}_k, \pi_h], [\mathbf{v}, q]) = \mathcal{B}([\mathbf{u}, p], [\mathbf{v}, q]), \qquad \forall [\mathbf{v}, q] \in X_{k,h}. \tag{1.9}$$

Also let P_S be the associated projection operator $P_S : X \mapsto X_{k,h}$ defined by

$$P_S([\mathbf{u}, p]) = [\mathbf{w}_k, \pi_h] \,. \tag{1.10}$$

The Stokes projection is well defined and satisfies an optimal energy error estimate with respect to $|\!|\!|[\cdot,\cdot]|\!|\!|$, see (3.5), cf. [3].

Remark 1.1. We bring to the reader's attention that although P_S acts on an element of X (a pair), later we shall use the following notation : $\mathbf{w}_k = P_S \mathbf{u}$, $\pi_h = P_S p$.

To handle the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ we define the trilinear form (à la Témam) $b_1(\cdot,\cdot,\cdot) : \mathbf{E}_k \mapsto \mathbb{R}$ by

$$b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i=1}^{d_k} \left\{ -\int_{\Omega_i^k} u_\ell^{(i)} v_m^{(i)} \frac{\partial w_m^{(i)}}{\partial x_\ell} \, dx + \sum_{j \in \mathcal{N}_i} \int_{\partial \Omega_{ij}^k} u_\ell^{(i)} v_m^{(i)} (w_m^{(i)} - w_m^{(j)}) n_\ell^{(i)} \, d\sigma + \int_{\partial \Omega_i^{*k}} u_\ell^{(i)} v_m^{(i)} w_m^{(i)} n_\ell^{(i)} \, d\sigma \right\}, \tag{1.11}$$

where from now on we adopt Einstein's summation convention for repeated indices for components of vectors. We introduce now the trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}[b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) - b_1(\mathbf{u}, \mathbf{w}, \mathbf{v})], \tag{1.12}$$

which will be used in the Galerkin formulation. Note that $b(\cdot, \mathbf{v}, \mathbf{v}) = 0$; also, this definition is slightly different from the one used in [18]. Also we have

$$b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \int_\Omega ((\mathbf{u} \cdot \nabla)\mathbf{u})\mathbf{v} \, dx - \frac{1}{2} \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{v}) \, d\sigma \quad \forall \mathbf{v} \in \mathbf{E}_k.$$

The Implicit Runge-Kutta (IRK) methods. In this work we will consider Implicit Runge-Kutta (IRK) time discretizations. Our schemes will include the Backward Euler and the Crank-Nicholson method. The modern form of an IRK method is the *Tableau*

$$\begin{array}{ccc|c} a_{11} & \dots & a_{1q} & \tau_1 \\ \vdots & & \vdots & \vdots \\ a_{q1} & \dots & a_{qq} & \tau_q \\ \hline b_1 & \dots & b_q & \end{array} .$$

In order to prove existence, stability and convergence of the fully discrete approximations, we shall make a series of assumptions about the IRK methods. These were introduced in the context of o.d.e's by J.C. Butcher and others; cf. [6].

$$\text{(P)} \qquad \begin{cases} A \text{ is invertible and there exists a positive diagonal matrix } D \text{ such that} \\ x^T C x > 0, \forall x \in \mathbb{R}^s, x \neq 0, \text{ where } C = DA^{-1}D^{-1}. \end{cases}$$

This property is used to prove existence of the fully discrete approximations and was introduced in [8]. The following properties (B), (C) and (D) are known in the literature as the "simplifying assumptions" and provide sufficient conditions for estimating the local truncation errors within the limits set by properties (E) and (F).

$$\text{(B)} \qquad \sum_{j=1}^{s} b_j \tau_j^{\ell} = \frac{1}{\ell+1}, \qquad \ell = 0, \dots, \lambda - 1,$$

$$\text{(C)} \qquad \sum_{j=1}^{s} a_{ij} \tau_j^{\ell} = \frac{\tau_i^{\ell+1}}{\ell+1}, \qquad i = 1, \dots, s, \quad \ell = 0, \dots, \bar{p} - 1,$$

$$\text{(D)} \qquad \sum_{i=1}^{s} a_{ij} \tau_i^{\ell} b_i = \frac{b_j}{\ell+1}(1 - \tau_j^{\ell+1}), \quad j = 1, \dots, s, \quad \ell = 0, \dots, \rho - 1 ,$$

for some integers $\lambda, \bar{p}, \rho \geq 1$ satisfying

$$\text{(E)} \qquad \lambda \leq \rho + \bar{p} + 1 ,$$

$$\text{(F)} \qquad \lambda \leq 2\bar{p} + 2 .$$

Finally, the next property provides a powerful sufficient condition for establishing the stability of the time stepping process. It is known as the "Algebraic stability" criterion.

$$\text{(S)} \qquad \begin{cases} b_i \geq 0 \ , i = 1, \dots, s \ , \\ \text{The } s \times s \text{ array } m_{ij} = a_{ij} b_i + a_{ji} bj - b_i b_j \text{ is positive semidefinite .} \end{cases}$$

The fully discrete schemes. We allow our spatial finite element spaces $\mathbf{V}_k, P_h$ to vary in time, denoting them by $\mathbf{V}_k^n, P_h^n$. For $N > 0$ integer let $I_n = (t^n, t^{n+1}]$, $n = 0, \dots, N-1$ be a partition of $[0, T]$ and let $\delta_n = t^{n+1} - t^n$, and $t^{n,j} = t^n + \delta \tau_j$, $j = 1, \dots, s$, $n = 0, \dots, N$. We associate partitions $\mathcal{T}_k^n = \{\Omega_i^{k,n}\}_{i=1}^{d_k^n}$, $\mathcal{T}_h^n = \{\Omega_\ell^{h,n}\}_{\ell=1}^{d_h^n}$ of Ω and finite element spaces $\mathbf{V}_k^n$, P_h^n to each interval I_n. Further we set

$X^n_{k,h} = \mathbf{V}^n_k \times P^n_h/\mathbb{R}$. To simplify the notation, we will omit the indices n on δ_n, the elements and the cardinality of the partitions $\mathcal{T}^n_k, \mathcal{T}^n_h$.

Let $\mathbf{v}^0_k$ be chosen optimally close to $\mathbf{u}^0$ in the sense that

$$\|\mathbf{u}^0 - \mathbf{v}^0_k\|_{0,\Omega} \le ck^{\mu}, \qquad 2 \le \mu \le r. \tag{1.13}$$

The general Implicit Runge-Kutta scheme, that we consider, reads as follows: for $n = 0, \dots, N$ we define the intermediate stages $\{[\mathbf{v}^{n,i}_k, \psi^{n,i}_h]\}^s_{i=1}$ by

$$(\mathbf{v}^{n,i}_k, \mathbf{v}) = (\mathbf{v}^n_k, \mathbf{v}) + \delta \sum_{j=1}^{s} a_{ij} \left\{ -\mathcal{B}([\mathbf{v}^{n,j}_k, \psi^{n,j}_h], [\mathbf{v}, q]) - b(\mathbf{v}^{n,j}_k, \mathbf{v}^{n,j}_k, \mathbf{v}) + F^{n,j}([\mathbf{v}, q]) \right\}, \qquad \forall [\mathbf{v}, q] \in X^n_{k,h}, \tag{1.14}$$

where $F^{n,j}, \ j = 1, \dots, s$ are defined by

$$F^{n,j}([\mathbf{v}, q]) = (\mathbf{f}(t^{n,j}), \mathbf{v}) - \nu \sum_{i=1}^{d_k} \left\{ \left\langle \frac{\partial \mathbf{v}^{(i)}}{\partial n}, \mathbf{g} \right\rangle_{\partial\Omega^{*k}_i} - \gamma k_i^{-1} \left\langle \mathbf{v}^{(i)}, \mathbf{g} \right\rangle_{\partial\Omega^{*k}_i} \right\} + \langle \mathbf{g} \cdot \mathbf{n}, q \rangle_{\partial\Omega} - \frac{1}{2} \langle \mathbf{g} \cdot \mathbf{n}, \mathbf{g} \cdot \mathbf{v} \rangle_{\partial\Omega},$$

and

$$(\mathbf{v}^{n+1}_k, \mathbf{v}) = (\mathbf{v}^n_k, \mathbf{v}) + \delta \sum_{i=1}^{s} b_i \left\{ -\mathcal{B}([\mathbf{v}^{n,i}_k, \psi^{n,i}_h], [\mathbf{v}, q]) - b(\mathbf{v}^{n,i}_k, \mathbf{v}^{n,i}_k, \mathbf{v}) + F^{n,i}([\mathbf{v}, q]) \right\}, \qquad \forall [\mathbf{v}, q] \in X^n_{k,h}, \tag{1.15}$$

where $A = \{a_{ij}\}, \ b = \{b_i\}, \ \tau = \{\tau_i\}$ denote the set of constants of the IRK method. With $\bar{p}, \lambda$ denoting the stage accuracy and the classical order respectively of the IRK method, we set $\sigma = \min\{\bar{p}+1, \lambda\}$. By assuming that A is invertible, then we write

$$\mathbf{v}^{n+1}_k = (1 - b^T A^{-1} e)\mathbf{v}^n_k + b^T A^{-1} V^n_k, \qquad n = 0, \dots, N, \tag{1.16}$$

where $V^n_k = (\mathbf{v}^{n,1}_k, \dots, \mathbf{v}^{n,s}_k)^T, \ e = (1, \dots, 1)^T \in \mathbb{R}^s$.

The pressure approximation will be updated as follows :

$$\psi^{n+1}_h = \xi \Psi^n_h, \qquad n = 0, \dots, N, \tag{1.17}$$

where $\Psi^n_h = (\psi^{n,1}_h, \dots, \psi^{n,s}_h)^T$ and $\xi = (\xi_1, \dots, \xi_s) \in R^s$ are the solution of the following Vandermonde system, [20]

$$\sum_{j=1}^{s} \xi_j \tau_j^{\ell} = 1, \qquad \ell = 0, \dots, s-1.$$

Convergence results. In this paper we study the existence, stability and convergence of the schemes (1.15)-(1.17), while questions related to their efficient implementation and performance on certain benchmark problems are addressed in [19]. In section 2 we list useful properties of the forms introduced above and in particular we derive an estimate for the trilinear form $b(.,.,.)$ which is essential for the forthcoming analysis. Section 3 is devoted to the proof of existence of solutions of (1.14), and Section 4 to the consistency analysis of the approximating scheme. The general plan of the analysis in this section follows that of [22], [1]. Note that one additional difficulty in our case is due, of course, to the presence of the pressure approximation and the corresponding spaces. In Section 5 we prove our main convergence result (Theorem 5.1): under certain assumptions, cf. Section 5 for details, there holds ($k \leq h,\ 2 \leq \mu \leq r$):

$$\max_{0\leq n\leq N} \|\mathbf{v}_k^n - \mathbf{u}(t^n)\|_{0,\Omega} \leq C(h^\mu + \delta^\sigma) + C\sqrt{\mathcal{N}_C}\left(\sum_{m=1}^{n} \|J[\lambda^m]\|_{0,\Omega}^2\right)^{\frac{1}{2}}, \tag{1.18}$$

(1.19)

$$\max_{0\leq n\leq N} \|\psi_h^n - p(t^n)\|_{0,h} \leq C\{h^{\mu-1} + \delta^{-1}h^\mu + \delta^{\sigma-1}\} + C\sqrt{\mathcal{N}_C}\left(\sum_{m=1}^{n} \|J[\lambda^m]\|_{0,\Omega}^2\right)^{\frac{1}{2}} + C\mathcal{N}_C\delta^{-1}\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2 ,$$

where $\mathcal{N}_C$ denotes the number of times where $X_{k,h}^j \neq X_{k,h}^{j-1}, j = 1,\ldots,N$ and $J[\lambda^n] = \mathbf{w}_k^n - \mathbf{w}_k^{n+} = (P_S^{n-1} - P_S^n)\mathbf{u}(t^n)$ denotes the jump of the Stokes projection at t^n. The jump terms appearing in these estimates are similar to those in [17] and their presence indicates that we cannot change the finite element spaces with n arbitrarily, [9].

Fully discrete finite element schemes were analyzed in [10], [2], [28], [14], [21], [11], [12], [23]. A third order accurate in time scheme was proposed and analyzed in [2]. Characteristic Galerkin methods were considered in [26], [27], [5], and nonlinear as well as "two grid" methods in [25]. Nonsmooth data estimates for a Crank-Nicolson scheme were proved in [13].

Let us notice finally the asymptotic character of the estimates (1.18) – (1.19) for small values of ν due to the fact that $C = ce^{c\frac{T}{\nu}}$. Without special assumptions on $\mathbf{u}, p$, the exponential dependence on $\frac{1}{\nu}$ appears in all the existing estimates for numerical approximations of (1.1) – (1.4), cf., e.g., the references cited above. Under a (smallness) assumption on the exact solution, (A) in Remark 5.1, we show that $C = c\frac{1}{\nu}$ (Remark 5.1). In [2], under a weaker, but similar, assumption and a mesh condition, the constant in the estimates was of the form $c\frac{1}{\nu}\sqrt{1+T}$. Estimates valid for the approximation of "exponentialy stable" solutions of (1.1-4) with constants $C = ce^{\frac{c}{\nu}}$ were proved in [13]. Also in [15] were proved estimates with constant $C = C_*(T)$, $C_*(T)$ being the strong stability constant of a first order in time discontinuous Galerkin method for a model pipe flow problem.

Remark 1.2. In the case where the partitions $\mathcal{T}_k, \mathcal{T}_h$ of Ω do not change with time, estimates (1.18) and (1.19) simplify to, [20]

$$\max_{0\leq n\leq N} \|\mathbf{v}_k^n - \mathbf{u}(t^n)\|_{0,\Omega} \leq c(h^\mu + \delta^\sigma) ,$$
$$\max_{0\leq n\leq N} \|\psi_h^n - p(t^n)\|_{0,h} \leq c(h^{\mu-1} + \delta^{-1}h^\mu + \delta^{\sigma-1}) .$$

2. Preliminaries

We begin by mentioning briefly the properties of the various linear forms introduced in the previous section. First, for $d = 2,3$ we have that $\mathbf{E}_k \subset \mathbf{L}^p(\Omega),\ \forall p \geq 1$. Moreover for $p = 2,4$ we have, cf. [18]

$$\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq c\|\mathbf{v}\|_{1,k} , \quad \forall \mathbf{v} \in \mathbf{E}_k , \tag{2.1}$$

for some constant c independent of k. The proof of the following continuity and coercivity results for $a_k^\gamma(\cdot,\cdot)$ can be found in [3]

$$|a_k^\gamma(\mathbf{u},\mathbf{v})| \le (1+\gamma)\|\mathbf{u}\|_{1,k}\|\mathbf{v}\|_{1,k}\ , \qquad \forall \mathbf{u},\mathbf{v}\in \mathbf{E}_k. \tag{2.2}$$

Also there exist positive constants c_a, γ_0 such that for all $\gamma \ge \gamma_0$

$$a_k^\gamma(\mathbf{v},\mathbf{v}) \ge c_a\|\mathbf{v}\|_{1,k}^2\ , \qquad \forall \mathbf{v}\in \mathbf{V}_k. \tag{2.3}$$

Further, there is a constant c such that,[3]

$$\sup_{0\ne \mathbf{v}\in \mathbf{V}_k} \frac{(\mathbf{v},\nabla q)}{\|\mathbf{v}\|_{1,k}} \ge c\|q\|_{L^2(\Omega)/\mathbb{R}}\ , \qquad \forall q\in P_h. \tag{2.4}$$

Note that the assumption that $\mathcal{T}_k$ is fine with respect to $\mathcal{T}_h$ is used to prove (2.4) and subsequently to prove $(1.8b)$. The trilinear form b is continuous with respect to appropriate norms. Indeed we have, cf. [18], that there exists a constant $c_b > 0$ such that

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le c_b\|\mathbf{u}\|_{1,k}\|\mathbf{v}\|_{1,k}\|\mathbf{w}\|_{1,k}\ , \qquad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in \mathbf{E}_k\ , \tag{2.5}$$

and $\forall \mathbf{u}\in \mathbf{H}^2(\Omega),\ \forall \mathbf{v},\mathbf{w}\in \mathbf{E}_k$,

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| + |b(\mathbf{v},\mathbf{u},\mathbf{w})| + |b(\mathbf{v},\mathbf{w},\mathbf{u})| \le c_b\|\mathbf{u}\|_{2,\Omega}\|\mathbf{v}\|_{1,k}\|\mathbf{w}\|_{1,k}\ . \tag{2.6}$$

We proceed by deriving two alternative estimates for the trilinear form $b(\cdot,\cdot,\cdot)$. These estimates are sharper than those in (2.5), (2.6) and play a crucial role in our analysis. Without loss of generality we use the same constant c_b.

Proposition 2.1. *For $\mathbf{u}\in \mathbf{H}^2(\Omega)$ with $div\,\mathbf{u}^{(i)} = div\,\mathbf{u}|_{\Omega_i^k} = 0, i = 1,\ldots,d_k$ and $\mathbf{v},\mathbf{w}\in \mathbf{E}_k$ we have*

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le c_b\|\mathbf{u}\|_{2,\Omega}\{\|\mathbf{v}\|_{0,\Omega} + k\|\mathbf{v}\|_{1,k}\}\|\mathbf{w}\|_{1,k}\ , \quad \forall \mathbf{v},\mathbf{w}\in \mathbf{E}_k\ . \tag{2.7}$$

Furthermore for $\mathbf{u}\in \mathbf{W}^{1,\infty}(\Omega)$ and $\mathbf{v},\mathbf{w}\in \mathbf{E}_k$ we have

$$|b(\mathbf{v},\mathbf{u},\mathbf{w})| \le c_b\|\mathbf{u}\|_{1,\infty,\Omega}\{\|\mathbf{v}\|_{0,\Omega} + k\|\mathbf{v}\|_{1,k}\}\|\mathbf{w}\|_{1,k}\ , \quad \forall \mathbf{v},\mathbf{w}\in \mathbf{E}_k. \tag{2.8}$$

Proof. To estimate boundary and interface integrals we shall use the following trace inequality, cf. [3],

$$|v|_{0,\partial\Omega_i^k}^2 \le c\{k_i^{-1}\|v\|_{0,\Omega_i^k}^2 + \|v\|_{0,\Omega_i^k}|v|_{1,\Omega_i^k}\}\ . \tag{2.9}$$

i) We have ,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{d_k}\Bigg\{\int_{\Omega_i^k} u_\ell^{(i)}\frac{\partial v_m^{(i)}}{\partial x_\ell}w_m^{(i)}\,dx - \int_{\Omega_i^k} u_\ell^{(i)}v_m^{(i)}\frac{\partial w_m^{(i)}}{\partial x_\ell}\,dx$$
$$+ \sum_{j\in\mathcal{N}_i}\tau_{ij}\int_{\partial\Omega_{ij}^k}\Big[u_\ell^{(i)}v_m^{(i)}(w_m^{(i)} - w_m^{(j)}) - u_\ell^{(i)}(v_m^{(i)} - v_m^{(j)})w_m^{(i)}\Big]\,n_\ell^{(i)}\,d\sigma\Bigg\}\ .$$

Now using Green's theorem and $\mathrm{div}\mathbf{u}^{(i)} = 0$, it follows that

$$\sum_{i=1}^{d_k} \int_{\Omega_i^k} u_\ell^{(i)} \frac{\partial v_m^{(i)}}{\partial x_\ell} v_m^{(i)}\, dx = \sum_{i=1}^{d_k} \Big\{ \sum_{j\in\mathcal{N}_i} \tau_{ij} \int_{\partial\Omega_{ij}^k} \Big[u_\ell^{(i)} (v_m^{(i)} - v_m^{(j)}) w_m^{(i)} n_\ell^{(i)} + u_\ell^{(i)} v_m^{(j)} (w_m^{(i)} - w_m^{(j)}) n_\ell^{(i)} \Big]\, d\sigma$$
$$+ \int_{\partial\Omega_i^{*k}} u_\ell^{(i)} v_m^{(i)} w_m^{(i)} n_\ell^{(i)}\, d\sigma - \int_{\Omega_i^k} u_\ell^{(i)} v_m^{(i)} \frac{\partial w_m^{(i)}}{\partial x_\ell}\, dx \Big\}.$$

Hence,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{d_k} \Big\{ -2\int_{\Omega_i^k} u_\ell^{(i)} v_m^{(i)} \frac{\partial w_m^{(i)}}{\partial x_\ell}\, dx + \int_{\partial\Omega_i^{*k}} u_\ell^{(i)} v_m^{(i)} w_m^{(i)} n_\ell^{(i)}\, d\sigma$$
$$+ \sum_{j\in\mathcal{N}_i} \tau_{ij} \int_{\partial\Omega_{ij}^k} u_\ell^{(i)} (v_m^{(i)} + v_m^{(j)})(w_m^{(i)} - w_m^{(j)}) n_\ell^{(i)}\, d\sigma \Big\}.$$

Now we estimate each term on the right hand side of the above equality.

For fixed ℓ, m we have

$$\left| \sum_{i=1}^{d_k} \int_{\Omega_i^k} u_\ell^{(i)} v_m^{(i)} \frac{\partial w_m^{(i)}}{\partial x_\ell}\, dx \right| \le \sum_{i=1}^{d_k} \|u_\ell^{(i)}\|_{L^\infty(\Omega_i^k)} \|v_m^{(i)}\|_{0,\Omega_i^k} \left\| \frac{\partial w_m^{(i)}}{\partial x_\ell} \right\|_{0,\Omega_i^k} \le \|\mathbf{u}\|_{L^\infty(\Omega)} \|v_m\|_{0,\Omega} \left\{ \sum_{i=1}^{d_k} \left\| \frac{\partial w_m^{(i)}}{\partial x_\ell} \right\|_{0,\Omega_i^k}^2 \right\}^{\frac{1}{2}}.$$

Summing over ℓ, m and using the imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, $(d \le 3)$ we obtain

$$\left| \sum_{i=1}^{d_k} \int_{\Omega_i^k} u_\ell^{(i)} v_m^{(i)} \frac{\partial v_m^{(i)}}{\partial x_\ell}\, dx \right| \le c\|\mathbf{u}\|_{2,\Omega} \|\mathbf{v}\|_{0,\Omega} \|\mathbf{w}\|_{1,k}. \tag{2.10}$$

Now for fixed ℓ, m using the trace inequality (2.9), we get for the 2nd term

$$\left| \sum_{i=1}^{d_k} \int_{\partial\Omega_i^{*k}} u_\ell^{(i)} v_m^{(i)} w_m^{(i)} n_\ell^{(i)}\, d\sigma \right| \le \sum_{i=1}^{d_k} \|u_\ell^{(i)}\|_{L^\infty(\partial\Omega_i^{*k})} |v_m^{(i)}|_{\partial\Omega_i^{*k}} |w_m^{(i)}|_{\partial\Omega_i^{*k}}$$
$$\le \|\mathbf{u}\|_{L^\infty(\Omega)} \left\{ c \sum_{i=1}^{d_k} \Big[\|v_m^{(i)}\|_{0,\Omega_i^{*k}}^2 + k_i^2 |v_m^{(i)}|_{1,\Omega_i^{*k}}^2 \Big] \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{d_k} k_i^{-1} |w_m^{(i)}|_{\partial\Omega_i^{*k}}^2 \right\}^{\frac{1}{2}}.$$

Using again the imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and summing over ℓ, m we get

$$\left| \sum_{i=1}^{d_k} \int_{\partial\Omega_i^{*k}} u_\ell^{(i)} v_m^{(i)} w_m^{(i)} n_\ell^{(i)}\, d\sigma \right| \le c\|\mathbf{u}\|_{2,\Omega} \Big\{ \|\mathbf{v}\|_{0,\Omega} + k\|\mathbf{v}\|_{1,k} \Big\} \|\mathbf{w}\|_{1,k}. \tag{2.11}$$

Similarly for the 3rd term, using the trace inequality (2.9), we have

$$\left| \sum_{i=1}^{d_k} \sum_{j\in\mathcal{N}_i} \tau_{ij} \int_{\partial\Omega_{ij}^k} u_\ell^{(i)} (v_m^{(i)} + v_m^{(j)})(w_m^{(i)} - w_m^{(j)}) n_\ell^{(i)}\, d\sigma \right| \le c|u_\ell|_{L^\infty(\Omega)} \left\{ \sum_{i=1}^{d_k} \Big[\|v_m^{(i)}\|_{0,\Omega_i^k}^2 + k_i^2 |v_m^{(i)}|_{1,\Omega_i^k}^2 \Big] \right\}^{\frac{1}{2}} \|\mathbf{w}\|_{1,k}.$$

Hence summing over ℓ, m we obtain

$$\left|\sum_{i=1}^{d_k}\sum_{j\in\mathcal{N}_i}\tau_{ij}\int\limits_{\partial\Omega_{ij}^k} u_\ell^{(i)}(v_m^{(i)}+v_m^{(j)})(w_m^{(i)}-w_m^{(j)})n_\ell^{(i)}\,d\sigma\right| \le c\|\mathbf{u}\|_{2,\Omega}\left\{\|\mathbf{v}\|_{0,\Omega}+k\|\mathbf{v}\|_{1,k}\right\}\|\mathbf{w}\|_{1,k}. \tag{2.12}$$

Now (2.7) follows from (2.10), (2.11) and (2.12).

ii) For $\mathbf{u}\in W^{1,\infty}(\Omega)$ we have

$$b(\mathbf{v},\mathbf{u},\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{d_k}\left\{\int\limits_{\Omega_i^k}\frac{\partial u_m^{(i)}}{\partial x_\ell}v_\ell^{(i)}w_m^{(i)}\,dx - \int\limits_{\Omega_i^k} u_m^{(i)}v_\ell^{(i)}\frac{\partial w_m^{(i)}}{\partial x_\ell}\,dx + \sum_{j\in\mathcal{N}_i}\tau_{ij}\int\limits_{\partial\Omega_{ij}^k} u_m^{(i)}v_\ell^{(i)}(w_m^{(i)}-w_m^{(j)})n_\ell^{(i)}\,d\sigma\right\}.$$

Similarly , as it was done for (2.10) , we get using,(2.1) with $p=2$,

$$\left|\sum_{i=1}^{d_k}\int\limits_{\Omega_i^k}\frac{\partial u_m^{(i)}}{\partial x_\ell}v_\ell^{(i)}w_m^{(i)}\,dx\right| \le c\|\mathbf{u}\|_{W^{1,\infty}(\Omega)}\|\mathbf{v}\|_{0,\Omega}\|\mathbf{w}\|_{1,k}\ . \tag{2.13}$$

Similarly we estimate the second term by

$$\left|\sum_{i=1}^{d_k}\int\limits_{\Omega_i^k} u_m^{(i)}v_\ell^{(i)}\frac{\partial w_m^{(i)}}{\partial x_\ell}\,dx\right| \le c\|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathbf{v}\|_{0,\Omega}\|\mathbf{w}\|_{1,k}\ . \tag{2.14}$$

The third term can be estimated as (2.12). Hence (2.12), (2.13), (2.14) establish (2.8). $\square$

3. Existence of solutions

We begin this section with the following useful result which, in the sequel , will be referred to as a *"diagonalization argument"*, cf. [16].

Lemma 3.1. *Let H be a finite dimensional Hilbert space and let $\phi^i, z^i, \psi^i \in H$ be given by*

$$\phi^i = z^i + \delta\sum_{j=1}^{s} a_{ij}\psi^j\ , \qquad i=1,\dots,s, \tag{3.1}$$

for an IRK satisfying (P). Then (3.1) is equivalent to

$$\sum_{i=1}^{s} d_m^2(A^{-1})_{mi}\phi_i = \sum_{i=1}^{s} d_m^2(A^{-1})_{mi}z^i + \delta d_m^2\psi^m, \quad m=1,\dots,s, \tag{3.2}$$

where d_m, $m=1,\dots,s$ are the diagonal elements in (P). Further there is a constant $c>0$ depending only on the IRK method such that

$$\sum_{i=1}^{s}\|\phi^i\|_H^2 \le c\sum_{i=1}^{s}\|z^i\|_H^2 + c\delta\sum_{i=1}^{s}(\psi^i,\phi^i)_H. \tag{3.3}$$

We next show existence of solutions of (1.14). First, we note that as a consequence of the linearity of the Stokes projection operator, (see (1.10), [3]) the following result holds,

Lemma 3.2. *Let $t \in [0,T]$ and $[\mathbf{u}(t), p(t)] \in \mathbf{H}^{\mu}(\Omega) \times H^{\mu-1}(\Omega)/\mathbb{R}$ for $2 \le \mu \le r$. Then there exists a unique pair $[\mathbf{w}_k(t), \pi_h(t)] \in X_{k,h}$ satisfying*

$$\mathcal{B}([\mathbf{w}_k(t), \pi_h(t)], [\mathbf{v}, q]) = \mathcal{B}([\mathbf{u}(t), p(t)], [\mathbf{v}, q]), \ \forall [\mathbf{v}, q] \in X_{k,h} \ , \tag{3.4}$$

and

$$|||[\mathbf{u}(t) - \mathbf{w}_k(t), p(t) - \pi_h(t)]||| \le c\left\{k^{\mu-1}\|\mathbf{u}(t)\|_{\mu,\Omega} + h^{\mu-1}\|p(t)\|_{H^{\mu-1}(\Omega)/\mathbb{R}}\right\}, \tag{3.5a}$$

furthermore

$$\|\mathbf{u}(t) - \mathbf{w}_k(t)\|_{0,\Omega} \le ck^{\mu}\|\mathbf{u}(t)\|_{\mu,\Omega}. \quad \square \tag{3.5b}$$

Let $\mathbf{V}_k = \mathbf{V}_1 \oplus \mathbf{V}_2$ be the L^2-orthogonal decomposition of $\mathbf{V}_k$ with

$$\mathbf{V}_1 = \{\mathbf{v} \in \mathbf{V} : (\mathbf{v}, \nabla\ q) = 0, \quad \forall q \in P_h\}.$$

Then one can show by using a variation of Browder's fixed point theorem,cf. [4] the following

Proposition 3.1. (i) *Let $[\mathbf{u}(t), p(t)]$ be the solution of (1.1) – (1.4). For $t \in [0,T]$ there exists $\boldsymbol{\chi}(t) \in \mathbf{V}_k$ such that*

$$(\boldsymbol{\chi}(t), \nabla\ q) = (\mathbf{u}(t), \nabla\ q), \qquad \forall q \in P_h. \tag{3.6}$$

Also there is a constant c , independent of k, h , such that

$$\|\boldsymbol{\chi}(t)\|_{1,k} \le c\|\mathbf{u}(t)\|_{1,k} \ , \tag{3.7}$$

and

$$\|\mathbf{u}(t) - \boldsymbol{\chi}(t)\|_{1,k} \le k\|\mathbf{u}(t)\|_{2,\Omega} \ . \tag{3.8}$$

(ii) *Assume that*

$$\nu c_a - c_b\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))} > 0. \tag{3.9}$$

Then for k sufficiently small, there exists $\left\{\mathbf{v}_1^{n,i}\right\}_{i=1}^{s} \in \mathbf{V}_1, n = 0, \ldots, N$ satisfying

$$\begin{aligned}\sum_{j=1}^{s} d_i d_j c_{ij}\Big[(\mathbf{v}_1^{n,j}, \mathbf{v}_1) + (\boldsymbol{\chi}^{n,j}, \mathbf{v}_1)\Big] &+ \delta d_i^2\Big\{\nu a_k^\gamma(\mathbf{v}_1^{n,i}, \mathbf{v}_1) + \nu a_k^\gamma(\boldsymbol{\chi}^{n,i}, \mathbf{v}_1) + b(\mathbf{v}_1^{n,i}, \boldsymbol{\chi}^{n,i}, \mathbf{v}_1)\\ &+ b(\boldsymbol{\chi}^{n,i}, \mathbf{v}_1^{n,i}, \mathbf{v}_1) + b(\boldsymbol{\chi}^{n,i}, \boldsymbol{\chi}^{n,i}, \mathbf{v}_1) + b(\mathbf{v}_1^{n,i}, \mathbf{v}_1^{n,i}, \mathbf{v}_1)\Big\}\\ &= \sum_{j=1}^{s} d_i d_j c_{ij}(\mathbf{v}_k^n, \mathbf{v}_1) + \delta d_i^2 F^{n,i}([\mathbf{v}_1, 0]), \ \forall \mathbf{v}_1 \in \mathbf{V}_1,\end{aligned} \tag{3.10}$$

where $\boldsymbol{\chi}^{n,i} = \boldsymbol{\chi}(t^{n,i}) = \boldsymbol{\chi}(t^n + \delta\tau_i), \quad i = 1, \ldots, s, \quad n = 0, \ldots, N.$

(iii) *Let $\mathbf{v}_k^{n,i} = \mathbf{v}_1^{n,i} + \boldsymbol{\chi}^{n,i}, \quad i = 1, \ldots, s, \quad n = 0, \ldots N.$ Then there exists $\left\{\psi_h^{n,i}\right\}_{i=1}^{s} \in P_h$ satisfying*

$$\begin{aligned}\delta d_i^2(\mathbf{v}_2, \nabla\ \psi_h^{n,i}) &= \sum_{j=1}^{s} d_i d_j c_{ij}\Big[(\mathbf{v}_k^n, \mathbf{v}_2) - (\mathbf{v}_k^{n,j}, \mathbf{v}_2)\Big]\\ &+ \delta d_i^2\left\{F^{n,i}([\mathbf{v}_2, 0]) - \nu a_k^\gamma(\mathbf{v}_k^{n,i}, \mathbf{v}_2) - b(\mathbf{v}_k^{n,i}, \mathbf{v}_k^{n,i}, \mathbf{v}_2)\right\} \ , \quad n = 0, \ldots N, \ \forall \mathbf{v}_2 \in \mathbf{V}_2.\end{aligned} \tag{3.11}$$

Proof. (i) The problem of finding $[\boldsymbol{\chi}(t),\varphi(t)] \in X_{k,h},\ 0 \le t \le T$ satisfying

$$\mathcal{B}([\boldsymbol{\chi}(t),\varphi(t)],[\mathbf{v},q]) = \mathcal{B}([\mathbf{u}(t),0],[\mathbf{v},q]),\ \forall\,[\mathbf{v},q] \in X_{k,h}, \tag{3.12}$$

has a unique solution by *Lemma* 3.2. Letting $\mathbf{v}=0$ in (3.12) we obtain

$$(\boldsymbol{\chi}(t),\nabla\ q) = (\mathbf{u}(t),\nabla\ q)\ , \qquad \forall q \in P_h\ , \quad 0 \le t \le T\ . \tag{3.13}$$

Also from (3.12) using $(1.8a), (1.8b)$ we get

$$\begin{aligned}\|\boldsymbol{\chi}(t)\|_{1,k} &\le |\!|\!|[\boldsymbol{\chi}(t),\varphi(t)]|\!|\!| \le c \sup_{|\!|\!|[\mathbf{v},q]|\!|\!|\le 1} |\mathcal{B}([\boldsymbol{\chi}(t),\varphi(t)],[\mathbf{v},q])| \\ &= c \sup_{|\!|\!|[\mathbf{v},q]|\!|\!|\le 1} |\mathcal{B}([\mathbf{u}(t),0],[\mathbf{v},q])| \le c|\!|\!|[\mathbf{u}(t),0]|\!|\!| = c\|\mathbf{u}(t)\|_{1,k}\ ,\end{aligned}$$

which shows (3.7). Now (3.8) follows from $(3.5a)$.

(ii) If $\mathbf{V}_1=\{0\}$ then (3.10) holds trivially. For fixed $\left\{\mathbf{v}_1^{n,i}\right\}_{i=1}^{s} \in \mathbf{V}_1,\ n=0,\ldots,N$, we define the map :

$$\begin{aligned}\mathbf{v}_1 \mapsto \ell^i(\mathbf{v}_1) \equiv \sum_{j=1}^{s} d_i d_j c_{ij} \Big[(\mathbf{v}_1^{n,j},\mathbf{v}_1) + (\boldsymbol{\chi}^{n,j},\mathbf{v}_1)\Big] &+ \delta d_i^2 \Big\{\nu a_k^\gamma(\mathbf{v}_1^{n,i},\mathbf{v}_1) + \nu a_k^\gamma(\boldsymbol{\chi}^{n,i},\mathbf{v}_1) \\ &+ b(\mathbf{v}_1^{n,i},\boldsymbol{\chi}^{n,i},\mathbf{v}_1) + b(\boldsymbol{\chi}^{n,i},\mathbf{v}_1^{n,i},\mathbf{v}_1) + b(\boldsymbol{\chi}^{n,i},\boldsymbol{\chi}^{n,i},\mathbf{v}_1) + b(\mathbf{v}_1^{n,i},\mathbf{v}_1^{n,i},\mathbf{v}_1)\Big\} \\ &- \sum_{j=1}^{s} d_i d_j c_{ij} (\mathbf{v}_k^n,\mathbf{v}_1) - \delta d_i^2 F^{n,i}([\mathbf{v}_1,0]), \quad i=1,\ldots,s,\end{aligned}$$

which is a bounded linear functional. By the Riesz representation theorem there exists $\zeta^i = \zeta^i(\mathbf{v}_1^{n,1},\ldots,\mathbf{v}_1^{n,s})$ $\in \mathbf{V}_1$ such that

$$(\zeta^i(\mathbf{v}_1^{n,i}),\mathbf{v}_1) = \ell^i(\mathbf{v}_1), \quad \forall\ ,\mathbf{v}_1 \in \mathbf{V}_1\ , \quad i=1,\ldots,s\ , \quad n=0,\ldots,N.$$

Then, from $(P), (3.7), (3.8), (2.1)$ with $K = \sup_{0\le t\le T} \|\mathbf{u}(t)\|_{1,k}$ and $\eta = \left(\sum_{i=1}^{s} \|\mathbf{v}_1^{n,i}\|_{1,k}^2\right)^{1/2}$ one can verify that,

$$\begin{aligned}\sum_{i=1}^{s} (\zeta^i(\mathbf{v}_1^{n,i}),\mathbf{v}_1^{n,i}) &\ge \delta\eta^2 \left\{\nu c_a - c_b\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))} - kcc_b\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))}\right\} \\ &\quad - \delta\eta \left\{\nu(1+\gamma)K + cc_bK^2 + c_F + d_i^{-2}\delta^{-1}(c_2K + c_3\|\mathbf{v}_k^n\|_{0,\Omega})\right\}\end{aligned}$$

In view of (3.9) for k sufficiently small we have $\nu c_a - c_b\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))} - kcc_b\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))} > 0$. Thus we can find $\eta > 0$, sufficiently large so that

$$\sum_{i=1}^{s} (\zeta^i,\mathbf{v}_1^{n,i}) > 0, \qquad \forall \left\{\mathbf{v}_1^{n,i}\right\}_{i=1}^{s} \in \mathbf{V}_1.$$

Considering $\mathbf{V}_1^s$, equipped with the standard L^2 inner product, as a (finite dimensional) Hilbert space it follows that there exists $\{\mathbf{u}_1^{n,i}\}_{i=1}^{s} \in \mathbf{V}_1$ satisfying $\zeta^i(\mathbf{v}_1^{n,i}) = 0,\ i=1,\ldots,s$.

(iii) The proof of this part is the same as the proof of the corresponding part of [3, Proposition 4.13]. Indeed, if $\mathbf{v}_2 = \{0\}$, then (3.11) holds trivially for any $\psi_h^{n,i} \in P_h$. Now for fixed $\mathbf{v}_2 \in \mathbf{V}_2,\ \mathbf{v}_2 \neq 0$, the map $q \longmapsto (\mathbf{v}_2, \nabla q)$ defines a nonzero linear functional on $P_h/\mathbb{R}$. It then follows that $\dim \mathbf{V}_2 \leq \dim P_h/\mathbb{R}$.

Let bases $\{\varphi_i\}_{i=1}^n$ and $\{\chi_i\}_{i=1}^m$ be given for $\mathbf{V}_2$ and $P_h^n/\mathbb{R}$ respectively. We shall show that $n = m$ and that the matrix $(M)_{ij} = (\varphi_i, \nabla \chi_j)\ , i,j = 1, \cdots, m$ is nonsingular. Indeed, if $n < m$ or if $n = m$ and M is singular, then for some nonzero vector $(\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m$

$$\sum_{j=1}^m (\varphi_i, \nabla \chi_j)\alpha_j = 0\ , \quad i = 1, \cdots, n.$$

Letting $\overline{p} = \sum_{j=1}^m \alpha_j \chi_j$, we see that $\overline{p} \neq 0$ and $(\mathbf{v}_2, \nabla \overline{p}) = 0,\ \forall \mathbf{v}_2 \in \mathbf{V}_2$, and hence $(\mathbf{v}_2, \nabla \overline{p}) = 0,\ \forall \mathbf{v} \in \mathbf{V}_k$. Since $(\mathbf{v}, \nabla \overline{p}) = \mathcal{B}([0,\overline{p}],[\mathbf{v},q])\, \forall [\mathbf{v},q] \in X_{k,h}$, it follows from (1.8*b*) and (2.4) that $\overline{p} = 0$, leading to a contradiction. As a consequence, M is nonsingular and $\psi_h^{n,i} \in P_h$ satisfying (3.11) exists. This completes the proof of the proposition. $\square$

Proposition 3.2. *Let* $\mathbf{v}_k^{n,i} = \mathbf{v}_1^{n,i} + \boldsymbol{\chi}^{n,i}, \psi_h^{n,i}, \quad i = 1,\ldots,s, \quad n = 0,\ldots,N$ *be as in Proposition* 3.1. *Then* $\left\{\left[\mathbf{v}_k^{n,i}, \psi_h^{n,i}\right]\right\}_{i=1}^s, \quad n = 0,\ldots,N$, *satisfy*

$$\sum_{j=1}^s d_i d_j c_{ij}(\mathbf{v}_k^{n,j}, \mathbf{v}) + \delta d_i^2 \left\{\mathcal{B}([\mathbf{v}_k^{n,i}, \psi_h^{n,i}],[\mathbf{v},q]) + b(\mathbf{v}_k^{n,i}, \mathbf{v}_k^{n,i}, \mathbf{v}) - F^{n,i}([\mathbf{v},q])\right\} = \sum_{j=1}^s d_i d_j c_{ij}(\mathbf{v}_k^n, \mathbf{v}), \quad i = 1,\ldots,s,\ \forall [\mathbf{v},q] \in X_{k,h}. \tag{3.14}$$

Proof. Let $\mathbf{v} \in \mathbf{V}_k$ be given and write $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 \in \mathbf{V}_1, \mathbf{v}_2 \in \mathbf{V}_2$. Then by (3.10), (3.11) we get

$$\sum_{j=1}^s d_i d_j c_{ij}(\mathbf{v}_k^{n,j}, \mathbf{v}) + \delta d_i^2 \left\{\nu a_k^\gamma(\mathbf{v}_k^{n,i}, \mathbf{v}) + b(\mathbf{v}_k^{n,i}, \mathbf{v}_k^{n,i}, \mathbf{v}) + (\mathbf{v}_2, \nabla\ \psi_h^{n,i}) - F^{n,i}([\mathbf{v},0])\right\} = \sum_{j=1}^s d_i d_j c_{ij}(\mathbf{v}_k^n, \mathbf{v}), \quad i = 1,\ldots,s.$$

Now $(\mathbf{v}, \nabla\ \psi_h^{n,i}) = (\mathbf{v}_1, \nabla\ \psi_h^{n,i}) + (\mathbf{v}_2, \nabla\ \psi_h^{n,i}) = (\mathbf{v}_2, \nabla\ \psi_h^{n,i})$. Further $F^{n,i}([\mathbf{v},0]) = F^{n,i}([\mathbf{v},q]) - \langle \mathbf{g}\cdot\mathbf{n}, q\rangle_{\partial\Omega}$, and by (3.6)

$$(\mathbf{v}_k^{n,i}, \nabla\ q) = (\boldsymbol{\chi}^{n,i}, \nabla\ q) + (\mathbf{v}_1^{n,i}, \nabla\ q) = (\boldsymbol{\chi}^{n,i}, \nabla\ q) = (\mathbf{u}(t^{n,i}), \nabla\ q) = \langle \mathbf{g}\cdot\mathbf{n}, q\rangle_{\partial\Omega}\ ,$$

so $\left\{\left[\mathbf{v}_k^{n,i}, \psi_h^{n,i}\right]\right\}_{i=1}^s, \quad n = 0,\ldots,N$ satisfy (3.14), equivalently (1.14). $\square$

Remark 3.1. If $\mathbf{g} = 0$ then the existence of solutions can be established without the condition, cf. [20],

$$\nu c_a - c_b \|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))} > 0\ .$$

4. Consistency

To prove our main convergence results (1.18) and (1.19) we employ a similar strategy to that reported in [22]. In particular we construct approximations $[\mathbf{u}^{n,j}, p^{n,j}]$ to $[\mathbf{u}(t^{n,j}), p(t^{n,j})]$, $j = 1, \ldots, s$ of the solution of (1.1)-(1.4) at the intermediate time $t = t^{n,j} = t^n + \delta\tau_j$. The $[\mathbf{u}^{n,j}, p^{n,j}]$ are combined to produce approximations $[\mathbf{u}^{n+1}, p^{n+1}]$ to $[\mathbf{u}(t^{n+1}), p(t^{n+1})]$. The intermediate stages $[\mathbf{u}^{n,j}, p^{n,j}]$ are expressed as polynomials in δ. More specifically we have,

$$\mathbf{u}^{n,j} = \sum_{\ell=0}^{\sigma} \delta^\ell \boldsymbol{\alpha}_{j,\ell} \quad , \quad p^{n,j} = \sum_{\ell=0}^{\sigma} \delta^\ell \beta_{j\ell} \ ,$$

where $\boldsymbol{\alpha}_{j\ell}$, $\beta_{j\ell}$ are smooth functions depending on the solution $[\mathbf{u}, p]$ and the Runge-Kutta method. We start by estimating the local truncation errors generated by the time stepping. To accomplish this, for $n = 0, \ldots, N$ we define the set of vector-valued functions $\{\boldsymbol{\alpha}_{i\ell}(x)\}, i = 1, \ldots, s,\ \ell = 0, \ldots, \lambda$ recursively by

$$\left\{ \begin{aligned} \boldsymbol{\alpha}_{i0} =& \mathbf{u}(t^n), \qquad i = 1, \ldots, s, \\ \boldsymbol{\alpha}_{i,\ell+1} =& \sum_{j=1}^{s} a_{ij} \left\{ \nu \Delta \boldsymbol{\alpha}_{j\ell} - \sum_{|m|=\ell} (\boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} - \nabla \beta_{j\ell} + \boldsymbol{\gamma}_{j\ell} \right\}, \\ & i = 1, \ldots, s, \qquad \ell = 0, \ldots, \lambda - 1, \end{aligned} \right. \tag{4.1}$$

where

$$\beta_{j\ell}(x) = \frac{\tau_j^\ell}{\ell!} \frac{\partial^\ell}{\partial t^\ell} p(x, t^n), \qquad \boldsymbol{\gamma}_{j\ell}(x) = \frac{\tau_j^\ell}{\ell!} \frac{\partial^\ell}{\partial t^\ell} \mathbf{f}(x, t^n), \qquad j = 1, \ldots, s,\ \ell = 0, \ldots \lambda. \tag{4.2}$$

We proceed by giving a series of results which characterize the functions $\{\boldsymbol{\alpha}_{i\ell}\}$.

Lemma 4.1. *We assume that (C) and (F) hold. Let $\boldsymbol{\alpha}_\ell$ denotes the s $d-$vectors $(\boldsymbol{\alpha}_{1\ell}^T, \ldots, \boldsymbol{\alpha}_{s\ell}^T)^T$ with $D_t^\ell \mathbf{u} = \frac{\partial^\ell}{\partial t^\ell} \mathbf{u}(x,t)|_{t=t^n}$, $\ell = 0, \ldots, \lambda - 1$, $n = 0, \ldots, N$, and consider the $s \times s$ matrix $T = diag(\tau_1, \ldots, \tau_s)$. Then*

$$\boldsymbol{\alpha}_\ell = D_t^\ell \mathbf{u} \, \frac{T^\ell e}{\ell!}, \qquad \ell = 0, \ldots, \bar{p}, \ \textit{if} \ \bar{p} \le \lambda, \tag{4.3}$$

$$\boldsymbol{\alpha}_{\bar{p}+1} = D_t^{\bar{p}+1} \mathbf{u} \, \frac{A T^{\bar{p}} e}{\bar{p}!}, \qquad \textit{if} \ \bar{p} \le \lambda - 1, \tag{4.4}$$

$$\begin{aligned} \boldsymbol{\alpha}_{\ell+1} =& D_t^{\ell+1} \mathbf{u} \, \frac{A T^\ell e}{\ell!} + \nu A \Delta \left[\boldsymbol{\alpha}_\ell - \frac{T^\ell e}{\ell!} D_t^\ell \mathbf{u} \right] \\ & - A \sum_{m_1=\bar{p}+1}^{\ell} \left((T^{\ell-m_1} \boldsymbol{\alpha}_{m_1} - \frac{T^\ell e}{m_1!} D_t^{m_1} \mathbf{u}) \cdot \nabla \right) \frac{D_t^{\ell-m_1} \mathbf{u}}{(\ell - m_1)!} \\ & - A \sum_{m_2=\bar{p}+1}^{\ell} \left(\frac{D_t^{\ell-m_2} \mathbf{u}}{(\ell - m_2)!} \cdot \nabla \right) \left[T^{\ell-m_2} \boldsymbol{\alpha}_{m_2} - \frac{T^\ell e}{\ell!} D_t^{m_2} \mathbf{u} \right] \ , \end{aligned} \tag{4.5}$$

for $\ell = \bar{p} + 1, \ldots, \lambda - 1$, if $\bar{p} \le \lambda - 2$.

Proof. We show (4.3) by induction. It holds obviously for $\ell = 0$. We assume that it holds for all indices up to some ℓ where $0 \le \ell \le \bar{p} - 1$. Then using (4.1),(4.2),(C) and (1.1) we get

$$\begin{aligned} \boldsymbol{\alpha}_{i,\ell+1} =& \sum_{j=1}^{s} a_{ij} \left\{ \nu \Delta \left(\frac{\tau_j^\ell D_t^\ell \mathbf{u}}{\ell!} \right) - \sum_{|m|=\ell} \left(\frac{\tau_j^{m_1}}{m_1!} D_t^{m_1} \mathbf{u} \cdot \nabla \right) \left(\frac{\tau_j^{m_2}}{m_2!} D_t^{m_2} \mathbf{u} \right) - \nabla \left(\frac{\tau_j^\ell}{m_2!} D_t^\ell p \right) + \frac{\tau_j^\ell}{\ell!} D_t^\ell \mathbf{f} \right\} \\ =& \sum_{j=1}^{s} a_{ij} \frac{\tau_j^\ell}{\ell!} \left\{ \nu \Delta (D_t^\ell \mathbf{u}) - (D_t^\ell \mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \left(D_t^\ell p \right) + D_t^\ell \mathbf{f} \right\} = \frac{\tau_i^{\ell+1}}{(\ell+1)!} D_t^{\ell+1} \mathbf{u}, \end{aligned}$$

i.e. (4.3) holds. Now for (4.4) we have from (4.1), (4.2), (4.3) and (1.1),

$$\begin{aligned}\boldsymbol{\alpha}_{i,\bar{p}+1} &= \sum_{j=1}^{s} a_{ij}\left\{\nu\Delta(D_t^{\bar{p}}\mathbf{u}\frac{T^{\bar{p}}e}{\bar{p}!}) - \sum_{|m|=\ell}(\frac{\tau_j^{m_1}}{m_1!}D_t^{m_1}\mathbf{u}\cdot\nabla)(\frac{\tau_j^{m_2}}{m_2!}D_t^{m_2}\mathbf{u}) - \nabla\left(\frac{\tau_j^{\bar{p}}}{\bar{p}!}D_t^{\bar{p}}p\right) + \frac{\tau_j^{\bar{p}}}{\bar{p}!}D_t^{\bar{p}}\mathbf{f}\right\}\\ &= \left(\sum_{j=1}^{s} a_{ij}\frac{\tau_j^{\bar{p}}}{\bar{p}!}\right)D_t^{\bar{p}+1}\mathbf{u},\end{aligned}$$

which establishes (4.4). To prove (4.5) we define for $\ell,\ \bar{p}+1 \le \ell \le \lambda-1$ the following sets:

$$\begin{aligned}&M^\ell = \{m = (m_1, m_2) : 0 \le m_i \le \ell, i = 1,2, |m| = \ell\}\ ,\\ &M_0^\ell = \left\{m \in M^\ell : 0 \le m_i \le \bar{p}, i = 1,2\right\}\ , M_i^\ell = \left\{m \in M^\ell : \bar{p}+1 \le m_i \le \ell\right\}\ ,\quad i = 1,2\ .\end{aligned}$$

First we notice that $M_i^\ell \cap M_j^\ell = \emptyset$ for $i \neq j,\ i,j = 0,1,2$ and secondly that $M^\ell = \bigcup_{i=0}^{2} M_i^\ell$. Then using :

$$\sum_{m_2=\ell-m_1}\frac{\tau_j^{m_2}}{m_2!}\frac{\partial}{\partial x_r}(D_t^{m_2}\mathbf{u}) = \frac{\tau_j^{\ell-m_1}}{(\ell-m_1)!}D_t^{\ell-m_1}\left(\frac{\partial \mathbf{u}}{\partial x_r}\right),\quad \sum_{m_1=\ell-m_2}\frac{\tau_j^{m_1}}{m_1!}D_t^{m_1}\mathbf{u} = \frac{\tau_j^{\ell-m_2}}{(\ell-m_2)!}D_t^{\ell-m_2}\mathbf{u},$$

we get

$$\begin{aligned}\boldsymbol{\alpha}_{i,\ell+1} &= \sum_{j=1}^{s} a_{ij}\left\{\nu\Delta\left(\frac{\tau_j^\ell}{\ell!}D_t^\ell\mathbf{u}\right) + \nu\Delta\left(\boldsymbol{\alpha}_{j\ell} - \frac{\tau_j^\ell}{\ell!}D_t^\ell\mathbf{u}\right) - \sum_{m\in M_0^\ell}(\frac{\tau_j^{m_1}}{m_1!}D_t^{m_1}\mathbf{u}\cdot\nabla)(\frac{\tau_j^{m_2}}{m_2!}D_t^{m_2}\mathbf{u})\right.\\ &\left. - \sum_{m_1=\bar{p}+1}^{\ell}(\frac{\tau_j^{\ell-m_1}}{(\ell-m_1)!}\boldsymbol{\alpha}_{jm_1}\cdot\nabla)(D_t^{\ell-m_1}\mathbf{u}) - \sum_{m_2=\bar{p}+1}^{\ell}(\frac{\tau_j^{\ell-m_2}}{(\ell-m_2)!}D_t^{\ell-m_2}\mathbf{u}\cdot\nabla)\boldsymbol{\alpha}_{jm_2} - \nabla\left(\frac{\tau_j^\ell}{\ell!}D_t^\ell p\right) + \frac{\tau_j^\ell}{\ell!}D_t^\ell\mathbf{f}\right\}.\end{aligned}$$

Further using the fact that $\sum_{m\in M_0^\ell} = \sum_{|m|=\ell} - \sum_{\substack{m_1=\bar{p}+1\\ m_2=\ell-m_1}}^{\ell} - \sum_{\substack{m_2=\bar{p}+1\\ m_1=\ell-m_2}}^{\ell}$ we obtain

$$\begin{aligned}\boldsymbol{\alpha}_{i,\ell+1} &= \sum_{j=1}^{s} a_{ij}\left\{\frac{\tau_j^\ell}{\ell!}D_t^\ell\left[\nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \mathbf{f}\right]\right\} + \sum_{j=1}^{s} a_{ij}\left\{\nu\Delta\left[\boldsymbol{\alpha}_{j\ell} - \frac{\tau_j^\ell}{\ell!}D_t^\ell\mathbf{u}\right]\right.\\ &\left. - \sum_{m_1=\bar{p}+1}^{\ell}(\boldsymbol{\alpha}_{jm_1} - \frac{\tau_j^{m_1}}{m_1!}D_t^{m_1}\mathbf{u}\cdot\nabla)(\frac{\tau_j^{\ell-m_1}}{(\ell-m_1)!}D_t^{\ell-m_1}\mathbf{u}) - \sum_{m_2=\bar{p}+1}^{\ell}(\frac{\tau_j^{\ell-m_2}}{(\ell-m_2)!}D_t^{\ell-m_2}\mathbf{u}\cdot\nabla)(\boldsymbol{\alpha}_{jm_2} - \frac{\tau_j^{m_2}}{m_2!}D_t^{m_2}\mathbf{u})\right\}.\end{aligned}$$

Now (1.1) completes the proof of the lemma. $\square$

Lemma 4.2. *Assume that* $(B), (C), (D), (E)$ *and* (F) *hold. Then for each* $\ell,\ \ell = 0, \ldots, \lambda-1$ *we have*

$$b^T T^j \boldsymbol{\alpha}_\ell = \frac{D_t^\ell \mathbf{u}}{\ell!(j+\ell+1)}\ ,\quad \textit{for } j+\ell \le \lambda-1\ . \tag{4.6}$$

Proof. First let $0 \le \ell \le \bar{p},\ 0 \le j \le \lambda-1$ with $j+\ell \le \lambda-1$. Then from (B), (4.3) it follows that

$$b^T T^j \boldsymbol{\alpha}_\ell = b^T T^j D_t^\ell \mathbf{u}\frac{T^\ell e}{\ell!} = b^T\frac{T^{\ell+j}e}{\ell!}D_t^\ell\mathbf{u} = \frac{D_t^\ell\mathbf{u}}{\ell!(j+\ell+1)}\ ,$$

i.e. (4.6) holds. Now we let $\ell = \bar{p}+1$ and $0 \le j \le \lambda - 1$ with $j + \bar{p} + 1 \le \lambda - 1$. Since $\lambda \le \rho + \bar{p} + 1$ we get $j \le \rho - 1$ so by (4.4), (D), (B) we obtain :

$$b^T T^j \boldsymbol{\alpha}_{\bar{p}+1} = \frac{b^T T^j A T^{\bar{p}} e}{\bar{p}!} D_t^{\bar{p}+1} \mathbf{u} = \frac{b^T (I - T^{j+1})}{j+1} \frac{T^{\bar{p}} e}{\bar{p}!} D_t^{\bar{p}+1} \mathbf{u} = \frac{D_t^{\bar{p}+1} \mathbf{u}}{(\bar{p}+1)!(\bar{p}+j+2)},$$

which shows (4.6) for $\ell = \bar{p}+1$.

For the remaining values of ℓ we use induction. Assume that (4.6) holds up to some ℓ with $\bar{p}+1 \le \ell \le \lambda - 1$. Then for $j = 0, \dots, \lambda-1$ with $j + \ell + 1 \le \lambda - 1$, using (4.5) we get

$$\begin{aligned} b^T T^j \boldsymbol{\alpha}_{\ell+1} = {} & \frac{b^T T^j A T^\ell e}{\ell!} D_t^{\ell+1} \mathbf{u} + b^T T^j A \nu \Delta \left[\boldsymbol{\alpha}_\ell - \frac{T^\ell e}{\ell!} D_t^\ell \mathbf{u} \right] \\ & - b^T T^j A \sum_{m_1 = \bar{p}+1}^{\ell} ((T^{\ell - m_1} \boldsymbol{\alpha}_{m_1} - \frac{T^\ell e}{m_1!} D_t^{m_1} \mathbf{u}) \cdot \nabla) \frac{D_t^{\ell - m_1} \mathbf{u}}{(\ell - m_1)!} \\ & - b^T T^j A \sum_{m_2 = \bar{p}+1}^{\ell} (D_t^{\ell - m_2} \mathbf{u} \cdot \nabla)(T^{\ell - m_1} \boldsymbol{\alpha}_{m_1} - \frac{T^\ell e}{m_1!} D_t^{m_1} \mathbf{u}). \end{aligned} \tag{4.7}$$

As before $j \le \rho - 1$ so by (D), (B) we get

$$\frac{b^T T^j A T^\ell e}{\ell!} D_t^{\ell+1} \mathbf{u} = \frac{1}{(\ell+1)!(j+\ell+2)} D_t^{\ell+1} \mathbf{u} \ . \tag{4.8}$$

Now we show that the last three terms in (4.7) are zero. From the induction hypothesis and (D), it follows that

$$b^T T^j A \boldsymbol{\alpha}_\ell = \frac{b^T}{j+1} (I - T^{j+1}) \boldsymbol{\alpha}_\ell = \frac{D_t^\ell \mathbf{u}}{(\ell+1)!(j+\ell+2)} \ . \tag{4.9}$$

Now (4.8) and (4.9) imply that the second term in (4.7) is zero.

Similarly for the third term using (D) and the induction hypothesis, $\bar{p}+1 \le m_1 \le \ell$, we get

$$b^T T^j A T^{\ell - m_1} \boldsymbol{\alpha}_{m_1} = \frac{b^T}{j+1} (I - T^{j+1}) T^{\ell - m_1} \boldsymbol{\alpha}_{m_1} = \frac{D_t^{m_1} \mathbf{u}}{m_1!(\ell+1)(j+\ell+2)} \ ,$$

also

$$\frac{b^T T^j A T^\ell e}{m_1!} D_t^{m_1} \mathbf{u} = \frac{1}{m_1!(\ell+1)(j+\ell+2)} D_t^{m_1} \mathbf{u} \ ,$$

i.e. the third term of (4.7) is zero. Similarly we see that the fourth term is also zero. The proof of the lemma is now complete. □

Corollary 4.1. *Assume that $(B), (C), (D), (E)$ and (F) hold. Then*

$$b^T A^{-1} \boldsymbol{\alpha}_\ell = \frac{D_t^\ell \mathbf{u}}{\ell!} \ , \qquad \ell = 1, \dots, \lambda \ . \tag{4.10}$$

Proof. For $\ell = 1, \dots, \bar{p}$ from (4.3), (C), (B) we have

$$b^T A^{-1} \boldsymbol{\alpha}_\ell = b^T A^{-1} \frac{T^\ell e}{\ell!} D_t^\ell \mathbf{u} = b^T A^{-1} \frac{A T^{\ell-1} e}{(\ell-1)!} D_t^\ell \mathbf{u} = \frac{1}{\ell!} D_t^\ell \mathbf{u} \ ,$$

i.e. (4.10) holds. Now for $\ell = \bar{p}+1$ it follows from (4.4) and (B) that

$$b^T A^{-1} \boldsymbol{\alpha}_{\bar{p}+1} = b^T A^{-1} \frac{AT^{\bar{p}} e}{\bar{p}!} D_t^{\bar{p}+1} \mathbf{u} = \frac{1}{(\bar{p}+1)!} D_t^{\bar{p}+1} \mathbf{u},$$

so again (4.10) holds. For $\ell = \bar{p}+1, \ldots, \lambda-1$ we have from (4.5)

$$\begin{aligned} b^T A^{-1} \boldsymbol{\alpha}_{\ell+1} = {} & \frac{b^T T^\ell e}{\ell!} D_t^{\ell+1} \mathbf{u} + \nu b^T \Delta \left[\boldsymbol{\alpha}_\ell - \frac{T^\ell e}{\ell!} D_t^\ell \mathbf{u} \right] \\ & - b^T \sum_{m_1 = \bar{p}+1}^{\ell} ((T^{\ell - m_1} \boldsymbol{\alpha}_{m_1} - \frac{T^\ell e}{m_1!} D_t^{m_1} \mathbf{u}) \cdot \nabla) \frac{D_t^{\ell - m_1} \mathbf{u}}{(\ell - m_1)!} \\ & - b^T \sum_{m_2 = \bar{p}+1}^{\ell} (D_t^{\ell - m_2} \mathbf{u} \cdot \nabla)(T^{\ell - m_1} \boldsymbol{\alpha}_{m_1} - \frac{T^\ell e}{m_1!} D_t^{m_1} \mathbf{u}). \end{aligned}$$

By (B) and (4.6) we get

$$b^T \boldsymbol{\alpha}_\ell - \frac{b^T T^\ell e}{\ell!} D_t^\ell \mathbf{u} = \frac{D_t^\ell \mathbf{u}}{(\ell+1)!} - \frac{D_t^\ell \mathbf{u}}{(\ell+1)!} = 0 \ .$$

Similarly the other two terms are also zero . This completes the proof. $\square$

If $\mathbf{u}$ is sufficiently smooth then it follows from (4.1) that $\boldsymbol{\alpha}_\ell$, $\ell = 0, \ldots, \lambda - 1$ are smooth. It is important also for the analysis that the components of the $\boldsymbol{\alpha}_\ell' s$ satisfy the incompressibility condition (1.2) since we shall need to approximate them optimally by elements of $\mathbf{V}_k$. Specifically we have the following result.

Proposition 4.1. *We assume the solution* $\mathbf{u}$ *of* (1.1) *is sufficiently smooth. Then for* $n = 0, \ldots, N-1$ *we have*

$$div\, \boldsymbol{\alpha}_\ell = (div\, \boldsymbol{\alpha}_{1\ell}, \ldots, div\, \boldsymbol{\alpha}_{s\ell}) = 0, \qquad \ell = 0, \ldots, \min\{\bar{p}+1, \lambda\}. \tag{4.11}$$

In fact if $\lambda > \bar{p}+1$ *then* $div\, \boldsymbol{\alpha}_{\bar{p}+2} \neq 0$ *in general.*

Proof. It follows from (4.3) and (4.4) that for $0 \le \ell \le \bar{p}+1$, $\operatorname{div} \boldsymbol{\alpha}_\ell = 0$. Now from (4.1) we obtain,

$$\operatorname{div} \boldsymbol{\alpha}_{i,\bar{p}+2} = \sum_{j=1}^{s} a_{ij} \left\{ - \sum_{|m| = \bar{p}+1} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} - \Delta \beta_{j,\bar{p}+1} + \operatorname{div} \boldsymbol{\gamma}_{j,\bar{p}+1} \right\}.$$

As before we write

$$\begin{aligned} \sum_{|m| = \bar{p}+1} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} &= \sum_{m \in M_0^{\bar{p}+1}} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} + \sum_{m \in M_1^{\bar{p}+1}} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} + \sum_{m \in M_2^{\bar{p}+1}} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} \\ &= \sum_{m \in M_0^{\bar{p}+1}} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} + (\nabla \boldsymbol{\alpha}_{j,\bar{p}+1} \cdot \nabla) \boldsymbol{\alpha}_{j0} + (\nabla \boldsymbol{\alpha}_{j0} \cdot \nabla) \boldsymbol{\alpha}_{j,\bar{p}+1}. \end{aligned}$$

Now from (4.3), (4.4) we get

$$\sum_{|m| = \bar{p}+1} (\nabla \boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} = \sum_{m \in M_0^{\bar{p}+1}} (\nabla (\frac{\tau_j^{m_1}}{m_1!} D_t^{m_1} \mathbf{u}) \cdot \nabla)(\frac{\tau_j^{m_2}}{m_2!} D_t^{m_2} \mathbf{u}) + \frac{1}{\bar{p}!} \left(\sum_{\ell=1}^{s} a_{j\ell} \tau_\ell^{\bar{p}} \right) D_t^{\bar{p}+1} \Big((\nabla \mathbf{u} \cdot \nabla) \mathbf{u} \Big).$$

Hence using (4.1) we obtain

$$\operatorname{div}\boldsymbol{\alpha}_{i,\bar p+2} = \sum_{j=1}^{s} a_{ij}\Big\{ -\Big[\sum_{m\in M_0^{\bar p+1}} \frac{\tau_j^{\bar p+1}}{m!}((\nabla(D_t^{m_1}\mathbf{u})\cdot\nabla)(D_t^{m_2}\mathbf{u}))$$
$$+\frac{1}{\bar p!}\Big(\sum_{\ell=1}^{s} a_{j\ell}\tau_\ell^{\bar p}\Big) D_t^{\bar p+1}\Big((\nabla\mathbf{u}\cdot\nabla)\mathbf{u}\Big)\Big] - \frac{\tau_j^{\bar p+1}}{(\bar p+1)!}\Big[D_t^{\bar p+1}(\Delta p) - D_t^{\bar p+1}(\operatorname{div}\mathbf{f})\Big]\Big\}.$$

Hence, (1.1) implies,

$$\operatorname{div}\boldsymbol{\alpha}_{\bar p+2} = -AT^{\bar p+1}e \sum_{m\in M_0^{\bar p+1}} \frac{1}{m!}((\nabla(D_t^{m_1}\mathbf{u})\cdot\nabla)D_t^{m_2}\mathbf{u}) - \frac{1}{\bar p!}AT^{\bar p}\left(Ae - \frac{Te}{\bar p+1}\right) D_t^{\bar p+1}((\nabla\mathbf{u}\cdot\nabla)\mathbf{u}),$$

which is not zero in general. □

We proceed by estimating the temporal consistency errors. Due to the stiffness, the stage accuracy will predominate the classical order. To that effect and in view of (4.11) we define $\sigma = \min\{\bar p+1, \lambda\}$. Now given $n\,, n = 0,\dots,N-1$ we define the intermediate stages $\{\mathbf{u}^{n,j}\}_{j=1}^{s}$ and $\mathbf{u}^{n+1}$ by

$$\mathbf{u}^{n,j} = \sum_{\ell=0}^{\sigma} \delta^\ell \boldsymbol{\alpha}_{j\ell}, \qquad j = 1,\dots,s, \tag{4.12}$$

$$\mathbf{u}^{n+1} = (1 - b^T A^{-1} e)\mathbf{u}(t^n) + b^T A^{-1} \begin{pmatrix} \mathbf{u}^{n,1} \\ \vdots \\ \mathbf{u}^{n,s} \end{pmatrix}. \tag{4.13}$$

Similarly for $n\,, 0 \le n \le N-1$ we define the "intermediate" quantities $p^{n,j}$, $\mathbf{f}^{n,j}$ by

$$p^{n,j} = \sum_{\ell=0}^{\sigma} \delta^\ell \beta_{j\ell} \quad , \quad \mathbf{f}^{n,j} = \sum_{\ell=0}^{\sigma} \delta^\ell \boldsymbol{\gamma}_{j\ell}, \qquad j = 1,\dots,s. \tag{4.14}$$

Now the temporal consistency result follows :

Proposition 4.2. *Let the truncation errors* $\mathbf{e}^{n,i}, i = 1,\dots,s$ $\mathbf{e}^{n+1}$ *be given by*

$$\mathbf{u}^{n,i} = \mathbf{u}(t^n) + \delta \sum_{j=1}^{s} a_{ij}\left\{\nu\,\Delta\mathbf{u}^{n,j} - (\mathbf{u}^{n,j}\cdot\nabla)\mathbf{u}^{n,j} - \nabla p^{n,j} + \mathbf{f}^{n,j}\right\} + \mathbf{e}^{n,i}, \tag{4.15}$$

$$\mathbf{u}^{n+1} = \mathbf{u}(t^n) + \delta \sum_{i=1}^{s} b_i\left\{\nu\,\Delta\mathbf{u}^{n,i} - (\mathbf{u}^{n,i}\cdot\nabla)\mathbf{u}^{n,i} - \nabla p^{n,i} + \mathbf{f}^{n,i}\right\} + \mathbf{e}^{n+1}. \tag{4.16}$$

Assuming that the hypotheses of Corollary 4.1 hold , then there exists a constant c independent of δ *such that*

$$\max_{1\le i\le s} \|\mathbf{e}^{n,i}\|_{0,\Omega} \le c\delta^{\sigma+1}, \tag{4.17}$$

$$\|\mathbf{e}^{n+1}\|_{0,\Omega} \le c\delta^{\sigma+1}, \tag{4.18}$$

$$\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_{0,\Omega} \le c\delta^{\sigma+1}. \tag{4.19}$$

Proof. From (4.15), (4.12), (4.14) we have

$$\mathbf{e}^{n,i} = \sum_{\ell=0}^{\sigma-1} \delta^{\ell+1} \boldsymbol{\alpha}_{i,\ell+1} - \delta \sum_{j=1}^{s} a_{ij} \Big\{ \nu \sum_{\ell=0}^{\sigma} \delta^{\ell} \Delta \boldsymbol{\alpha}_{j\ell} + - \sum_{\ell=0}^{\sigma-1} \delta^{\ell} \sum_{|m|=\ell} (\boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2}$$
$$- \sum_{\ell=\sigma}^{\sigma^2} \delta^{\ell} \sum_{|m|=\ell} C_{m_1 m_2;\ell} (\boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} - \sum_{\ell=0}^{\sigma} \delta^{\ell} \nabla \beta_{j\ell} + \sum_{\ell=0}^{\sigma} \delta^{\ell} \boldsymbol{\gamma}_{j\ell} \Big\},$$

where the constants $C_{m_1 m_2;\ell}$ are zero or one. Hence using (4.1) we get

$$\mathbf{e}^{n,i} = -\delta^{\sigma+1} \sum_{j=1}^{s} a_{ij} \Big\{ \nu \, \Delta \boldsymbol{\alpha}_{j\sigma} - \sum_{\ell=\sigma}^{\sigma^2} \delta^{\ell-\sigma} \sum_{|m|=\ell} C_{m_1 m_2;\ell} (\boldsymbol{\alpha}_{jm_1} \cdot \nabla) \boldsymbol{\alpha}_{jm_2} - \nabla \beta_{j\sigma} + \boldsymbol{\gamma}_{j\sigma} \Big\},$$

from which (4.17) follows.

Now with $a_{ij}^{-1} = (A^{-1})_{ij}$ and from (4.15), (4.13) we get $\mathbf{e}^{n+1} = \sum_{i,j=1}^{s} b_i a_{ij}^{-1} \mathbf{e}^{n,j}$, so (4.17) now implies (4.18).

Using (4.13), (4.12), (4.1) and (4.10) we obtain

$$\mathbf{u}^{n+1} = (1 - \sum_{i,j=1}^{s} b_i a_{ij}^{-1}) \mathbf{u}(t^n) + \sum_{i,j=1}^{s} b_i a_{ij}^{-1} \mathbf{u}^{n,j} = \sum_{\ell=0}^{\sigma} \frac{\delta^{\ell}}{\ell!} D_t^{\ell} \mathbf{u}(t^n).$$

Taylor's theorem now yields (4.19). □

5. Stability and Convergence

For n, j, $n = 0, \ldots, N$, $j = 1, \ldots, s$ we set

$$\mathbf{w}_k^{n,j} = P_S^n \mathbf{u}^{n,j} \ , \ \mathbf{w}_k^{n+} = P_S^n \mathbf{u}(t^n) \ , \ \mathbf{w}_k^n = P_S^{n-1} \mathbf{u}(t^n) \ ,$$
$$\pi_h^{n,j} = P_S^n p^{n,j} \ , \ \pi_h^n = P_S^n p(t^n) \ .$$

We also set

$$\boldsymbol{\theta}^{n,j} = \mathbf{v}_k^{n,j} - \mathbf{w}_k^{n,j} \quad , \quad \boldsymbol{\rho}^{n,j} = \mathbf{u}^{n,j} - \mathbf{w}_k^{n,j} \quad , \quad \boldsymbol{\theta}^n = \mathbf{v}_k^n - \mathbf{w}_k^n \ ,$$
$$\zeta^{n,j} = \psi_h^{n,j} - \pi_h^{n,j} \quad , \quad \eta^{n,j} = \pi_h^{n,j} - p^{n,j} \quad , \quad \boldsymbol{\rho}^n = \mathbf{u}^n - \mathbf{w}_k^n.$$

Lemma 5.1. *We assume that the IRK-method satisfies (P). Then for h and δ sufficiently small we have*

$$\sum_{i=1}^{s} \|\boldsymbol{\theta}^{n,i}\|_{0,\Omega}^2 + c\delta\nu \sum_{i=1}^{s} a_k^{\gamma}(\boldsymbol{\theta}^{n,i}, \boldsymbol{\theta}^{n,i}) \le \bar{c} \|\boldsymbol{\theta}^n\|_{0,\Omega}^2 + c^* \delta \left\{ h^{2\mu} + \delta^{2\sigma} \right\} + \bar{c} \|J[\lambda^n]\|_{0,\Omega}^2 \ , \tag{5.1}$$

for $\bar{c}$ depending on the IRK-method and $c^* = \frac{c}{\nu}$.

Proof. Using (1.14), (3.4) and (4.15) we have for $i = 1, \ldots, s$,

$$(\boldsymbol{\theta}^{n,i}, \mathbf{v}) + \delta \sum_{j=1}^{s} a_{ij} \Big\{ \mathcal{B}([\boldsymbol{\theta}^{n,j}, \zeta^{n,j}], [\mathbf{v}, q]) \Big\} = (\mathbf{v}_k^n, \mathbf{v}) - (\mathbf{w}_k^{n,i}, \mathbf{v}) - (\mathbf{u}(t^n), \mathbf{v}) - (\mathbf{e}^{n,i}, \mathbf{v}) + (\mathbf{u}^{n,i}, \mathbf{v})$$
$$+ \delta \sum_{j=1}^{s} a_{ij} \left\{ b(\mathbf{u}^{n,j}, \mathbf{u}^{n,j}, \mathbf{v}) - b(\mathbf{v}_k^{n,j}, \mathbf{v}_k^{n,j}, \mathbf{v}) + F^{n,j}([\mathbf{v}, q]) - \tilde{F}^{n,j}([\mathbf{v}, q]) \right\} \ ,$$

where $\tilde F^{n,j},\ j=1,\dots,s$ are defined by

$$\tilde F^{n,j}([\mathbf{v},q]) = (\mathbf{f}(t^{n,j}),\mathbf{v}) - \nu\sum_{i=1}^{d_k}\Big\{\Big\langle \frac{\partial \mathbf{v}^{(i)}}{\partial n},\mathbf{u}^{n,j}\Big\rangle_{\partial\Omega_i^{*k}} - \gamma k_i^{-1}\Big\langle \mathbf{v}^{(i)},\mathbf{u}^{n,j}\Big\rangle_{\partial\Omega_i^{*k}}\Big\}$$
$$+\left\langle \mathbf{u}^{n,j}\cdot\mathbf{n},q\right\rangle_{\partial\Omega} - \frac{1}{2}\left\langle \mathbf{u}^{n,j}\cdot\mathbf{n},\mathbf{u}^{n,j}\cdot\mathbf{v}\right\rangle_{\partial\Omega}\ .$$

Now we have,

$$\mathbf{u}^{n,i}-\mathbf{w}_k^{n,i}-\mathbf{e}^{n,i}+\mathbf{v}_k^n-\mathbf{u}(t^n)=\boldsymbol{\rho}^{n,i}-\mathbf{e}^{n,i}+(P_S^n-I)\mathbf{u}(t^n)+\boldsymbol{\theta}^n+(\mathbf{w}_k^n-\mathbf{w}_k^{n+}).$$

Hence, using a *"diagonalization argument"*, see (3.2), we get

(5.2)
$$\sum_{i=1}^{s} d_m^2 a_{mi}^{-1}(\boldsymbol{\theta}^{n,i},\mathbf{v}) = \sum_{i=1}^{s} d_m^2 a_{mi}^{-1}\Big\{(\boldsymbol{\theta}^n,\mathbf{v})-(\mathbf{e}^{n,i},\mathbf{v})+(\boldsymbol{\rho}^{n,i},\mathbf{v})+((P_S^n-I)\mathbf{u}(t^n),\mathbf{v})+((\mathbf{w}_k^n-\mathbf{w}_k^{n+}),\mathbf{v})\Big\}$$
$$-\,\delta d_m^2\Big\{\mathcal{B}([\boldsymbol{\theta}^{n,m},\zeta^{n,m}],[\mathbf{v},q]) - b(\mathbf{u}^{n,m},\mathbf{u}^{n,m},\mathbf{v}) + b(\mathbf{v}_k^{n,m},\mathbf{v}_k^{n,m},\mathbf{v}) - F^{n,j}([\mathbf{v},q]) + \tilde F^{n,j}([\mathbf{v},q])\Big\},$$
$$m=1,\dots,s,\ \forall[\mathbf{v},q]\in X_{k,h}^n.$$

Letting $\mathbf{v}=0$ in (5.2) and recalling that $\mathbf{g}$ is independent of time we obtain

$$(\boldsymbol{\theta}^{n,m},\nabla q)=0\ ,\quad \forall q\in P_h\ ,\quad m=1,\dots,s. \tag{5.3}$$

Further, putting $\mathbf{v}=\boldsymbol{\theta}^{n,m}$ in (5.2) using (5.3) and summing over m we obtain

(5.4)
$$\sum_{i,m=1}^{s} c_{m,i}(d_m\boldsymbol{\theta}^{n,m},d_i\boldsymbol{\theta}^{n,i}) =$$
$$\sum_{i,m=1}^{s} d_m^2 a_{mi}^{-1}\Big\{(\boldsymbol{\theta}^n,\boldsymbol{\theta}^{n,m})-(\mathbf{e}^{n,i},\boldsymbol{\theta}^{n,m})+(\boldsymbol{\rho}^{n,i},\boldsymbol{\theta}^{n,m})+((P_S^n-I)\mathbf{u}(t^n),\boldsymbol{\theta}^{n,m})+(\mathbf{w}_k^n-\mathbf{w}_k^{n+},\boldsymbol{\theta}^{n,m})\Big\}$$
$$-\delta\sum_{m=1}^{s} d_m^2\Big\{\nu a_k^\gamma(\boldsymbol{\theta}^{n,m},\boldsymbol{\theta}^{n,m})+b(\mathbf{v}_k^{n,m},\mathbf{v}_k^{n,m},\boldsymbol{\theta}^{n,m})-b(\mathbf{u}^{n,m},\mathbf{u}^{n,m},\boldsymbol{\theta}^{n,m})+(\mathbf{f}(t^{n,j})-\mathbf{f}^{n,j},\boldsymbol{\theta}^{n,m})\Big\}.$$

Now for any $\mathbf{v}\in\mathbf{V}_k$ we have

$$\begin{aligned}&b(\mathbf{u}^{n,m},\mathbf{u}^{n,m},\mathbf{v})-b(\mathbf{v}_k^{n,m},\mathbf{v}_k^{n,m},\mathbf{v})=\\ &b(\mathbf{u}^{n,m},\boldsymbol{\rho}^{n,m},\mathbf{v})-b(\mathbf{u}^{n,m},\boldsymbol{\theta}^{n,m},\mathbf{v})+b(\boldsymbol{\rho}^{n,m},\mathbf{u}^{n,m},\mathbf{v})-b(\boldsymbol{\rho}^{n,m},\boldsymbol{\rho}^{n,m},\mathbf{v})\\ &+b(\boldsymbol{\theta}^{n,m},\boldsymbol{\rho}^{n,m},\mathbf{v})-b(\boldsymbol{\theta}^{n,m},\mathbf{u}^{n,m},\mathbf{v})+b(\boldsymbol{\rho}^{n,m},\boldsymbol{\theta}^{n,m},\mathbf{v})-b(\boldsymbol{\theta}^{n,m},\boldsymbol{\theta}^{n,m},\mathbf{v})\ .\end{aligned} \tag{5.5}$$

In particular for $\mathbf{v}=\boldsymbol{\theta}^{n,m}$ we have

$$\begin{aligned}&b(\mathbf{u}^{n,m},\mathbf{u}^{n,m},\boldsymbol{\theta}^{n,m})-b(\mathbf{v}_k^{n,m},\mathbf{v}_k^{n,m},\boldsymbol{\theta}^{n,m})=\\ &b(\mathbf{u}^{n,m},\boldsymbol{\rho}^{n,m},\boldsymbol{\theta}^{n,m})+b(\boldsymbol{\rho}^{n,m},\mathbf{u}^{n,m},\boldsymbol{\theta}^{n,m})\\ &-b(\boldsymbol{\rho}^{n,m},\boldsymbol{\rho}^{n,m},\boldsymbol{\theta}^{n,m})+b(\boldsymbol{\theta}^{n,m},\boldsymbol{\rho}^{n,m},\boldsymbol{\theta}^{n,m})-b(\boldsymbol{\theta}^{n,m},\mathbf{u}^{n,m},\boldsymbol{\theta}^{n,m})\ .\end{aligned} \tag{5.6}$$

Hence from (2.7), (2.8), (2.3) and, see (3.5b)

$$\|\boldsymbol{\rho}^{n,j}\|_{0,\Omega} = \|\mathbf{u}^{n,j} - \mathbf{w}_k^{n,j}\|_{0,\Omega} \le ck^{\mu} \quad , \quad \|\boldsymbol{\rho}^{n,j}\|_{1,k} \le ck^{\mu-1},$$

with $\mathcal{C} = \|\mathbf{u}\|_{L^\infty(\mathbf{W}^{1,\infty}(\Omega))}$, it follows that

$$\begin{aligned}\delta|b(\mathbf{u}^{n,m}, \mathbf{u}^{n,m}, \boldsymbol{\theta}^{n,m}) - b(\mathbf{v}_k^{n,m}, \mathbf{v}_k^{n,m}, \boldsymbol{\theta}^{n,m})| &\le \epsilon_1 \delta \frac{1}{c_a} a_k^\gamma(\boldsymbol{\theta}^{n,m}, \boldsymbol{\theta}^{n,m}) \\ + \delta \frac{1}{c_a} a_k^\gamma(\boldsymbol{\theta}^{n,m}, \boldsymbol{\theta}^{n,m}) \left\{ \frac{1}{\epsilon} c_b \delta \mathcal{C}^2 + c_b h \mathcal{C} \right\} &+ \epsilon \|\boldsymbol{\theta}^{n,m}\|_{0,\Omega}^2 + c\frac{1}{\epsilon_1} \delta h^{2\mu}.\end{aligned} \tag{5.7}$$

Furthermore by (4.14) and Taylor's theorem, it follows that

$$|(\mathbf{f}(t^{n,j}) - \mathbf{f}^{n,j}, \boldsymbol{\theta}^{n,m})| \le \epsilon_2^{-1} c \delta^{2\sigma+2} + \epsilon_2 \|\boldsymbol{\theta}^{n,m}\|_{0,\Omega}^2 \ . \tag{5.8}$$

Now using (P), (5.7), (5.8) and applying the arithmetic-geometric mean inequality in (5.4) we get

$$\begin{aligned} c\sum_{m=1}^{s} \|\boldsymbol{\theta}^{n,m}\|_{0,\Omega}^2 + \delta\nu \sum_{m=1}^{s} d_m^2 a_k^\gamma(\boldsymbol{\theta}^{n,m}, \boldsymbol{\theta}^{n,m}) &\le \sum_{m,i=1}^{s} d_m^2 a_{mi}^{-1} \Big\{ (4\epsilon_0 + \epsilon_2 + \epsilon) \|\boldsymbol{\theta}^{n,m}\|_{0,\Omega}^2 + \epsilon_0^{-1} \|\boldsymbol{\theta}^n\|_{0,\Omega}^2 \\ + \epsilon_0^{-1} \|\mathbf{e}^{n,m}\|_{0,\Omega}^2 + \epsilon_0^{-1} \|(I - P_S^n) \sum_{\ell=1}^{\sigma} \delta^\ell \boldsymbol{\alpha}_{i\ell}\|_{0,\Omega}^2 &+ \epsilon_0^{-1} \|J[\lambda^n]\|_{0,\Omega}^2 + \epsilon_1 \delta \frac{1}{c_a} a_k^\gamma(\boldsymbol{\theta}^{n,m}, \boldsymbol{\theta}^{n,m}) \\ + \delta \frac{1}{c_a} a_k^\gamma(\boldsymbol{\theta}^{n,m}, \boldsymbol{\theta}^{n,m}) \left\{ \frac{1}{\epsilon} c_b \delta \mathcal{C}^2 + c_b h \mathcal{C} \right\} \Big\} &+ \frac{1}{\epsilon_1} c \delta h^{2\mu}. \end{aligned}$$

Now choosing $\epsilon_0,\ \epsilon,\ \epsilon_2$ appropriately, depending on the RK-method and not on ν, $\epsilon_1 \le c\nu$, and taking $h,\ \delta$ sufficiently small, $(h, \delta \le C\nu)$, and using (4.17), $(3.5b)$ we get (5.1). Now (2.3) and (5.1) imply that

$$\sum_{i=1}^{s} \|\boldsymbol{\theta}^{n,i}\|_{0,\Omega}^2 + c\delta\nu \sum_{i=1}^{s} \|\boldsymbol{\theta}^{n,i}\|_{1,k}^2 \le \bar{c} \|\boldsymbol{\theta}^n\|_{0,\Omega}^2 + c^* \delta \left\{ h^{2\mu} + \delta^{2\sigma} \right\} + \bar{c} \|J[\lambda^n]\|_{0,\Omega}^2 \ . \tag{5.9}$$

The proof of the Lemma is now complete. □

Now we prove our main stability result.

Lemma 5.2. *We assume that the IRK-method satisfies* (S). *Then*

$$\|\boldsymbol{\theta}^{n+1}\|_{0,\Omega} \le C_{n+1} h^\mu + c C_{n+1} \delta^{\frac{1}{2}} \{h^\mu + \delta^\sigma\} + C_{n+1} \sqrt{\mathcal{N}_C(n)} \left(\sum_{m=1}^{n} \|J[\lambda^m]\|_{0,\Omega}^2 \right)^{\frac{1}{2}} , \tag{5.10}$$

with c *independent of* h , δ , $C_n = (3e^{2c^* t_n + 8})^{1/2}$ *and* $\boldsymbol{\theta}^{n+1} = \mathbf{v}_k^{n+1} - \mathbf{w}_k^{n+1} = \mathbf{v}_k^{n+1} - P_S^n \mathbf{u}(t^{n+1})$ *and* $c^* = \frac{c}{\nu}$. *Also* $\mathcal{N}_C(n)$ *is the number of times where* $X_{k,h}^j \ne X_{k,h}^{j-1}, j = 1, \dots, n$.

Proof. Analogous to (5.2) we see that $\forall [\mathbf{v}, q] \in X_{k,h}^n$, $\boldsymbol{\theta}^{n+1}$ satisfies

$$\begin{aligned} (\boldsymbol{\theta}^{n+1}, \mathbf{v}) - (\boldsymbol{\theta}^n, \mathbf{v}) + \delta \sum_{j=1}^{s} b_j \left\{ \mathcal{B}([\boldsymbol{\theta}^{n,j}, \zeta^{n,j}], [\mathbf{v}, q]) \right\} &= (\boldsymbol{\rho}^{n+1}, \mathbf{v}) - (\mathbf{e}^{n+1}, \mathbf{v}) + ((P_S^n - I)\mathbf{u}(t^n), \mathbf{v}) \\ + ((\mathbf{w}_k^n - \mathbf{w}_k^{n+}), \mathbf{v}) + \delta \sum_{j=1}^{s} b_j &\left\{ b(\mathbf{u}^{n,j}, \mathbf{u}^{n,j}, \mathbf{v}) - b(\mathbf{v}_k^{n,j}, \mathbf{v}_k^{n,j}, \mathbf{v}) + F^{n,j}([\mathbf{v}, q]) - \tilde{F}^{n,j}([\mathbf{v}, q]) \right\} \ . \end{aligned} \tag{5.11}$$

Since $\mathbf{u}|_{\partial\Omega}$ is independent of time, letting $\mathbf{v} = 0$ in (5.11) we get

$$\sum_{j=1}^{s} (\boldsymbol{\theta}^{n,j}, \nabla q) = 0 \ , \qquad \forall q \in P_h \ .$$

To simplify matters we set

$$(T^{n,j}, \mathbf{v}) = -\nu a_k^{\gamma}(\boldsymbol{\theta}^{n,j}, \mathbf{v}) - (\mathbf{v}, \nabla \zeta^{n,j}) + b(\mathbf{u}^{n,j}, \mathbf{u}^{n,j}, \mathbf{v}) - b(\mathbf{v}_k^{n,j}, \mathbf{v}_k^{n,j}, \mathbf{v}) + (\mathbf{f}(t^{n,j}) - \mathbf{f}^{n,j}, \mathbf{v}).$$

Also, we let

$$\boldsymbol{\theta}^{n+1} = \boldsymbol{\theta}^n + \tilde{G}^{n+1} + \delta \sum_{j=1}^{s} b_j T^{n,j} \ , \qquad \text{where} \tag{5.12}$$

$$\tilde{G}^{n+1} = G^{n+1} + J[\lambda^n] \ , \qquad \text{with} \tag{5.12a}$$

$$G^{n+1} = (P_S^n - I)\mathbf{u}(t^n) + \boldsymbol{\rho}^{n+1} - \mathbf{e}^{n+1} \ . \tag{5.12b}$$

Hence

$$\begin{aligned} \|\boldsymbol{\theta}^{n+1}\|_{0,\Omega}^2 = \|\boldsymbol{\theta}^n\|_{0,\Omega}^2 + \|\tilde{G}^{n+1}\|_{0,\Omega}^2 + 2(\boldsymbol{\theta}^n, \tilde{G}^{n+1}) + 2\delta \sum_{j=1}^{s} b_j (\boldsymbol{\theta}^n, T^{n,j}) \\ + 2\delta \sum_{j=1}^{s} b_j (\tilde{G}^{n+1}, T^{n,j}) + \delta^2 \sum_{i,j=1}^{s} b_i b_j (T^{n,i}, T^{n,j}) \ . \end{aligned} \tag{5.13}$$

On the other hand by (5.2) and (5.3) we have

$$\boldsymbol{\theta}^n = \boldsymbol{\theta}^{n,i} - \tilde{G}^{n,i} - \delta \sum_{j=1}^{s} a_{ij} T^{n,j} \ , \qquad \text{where} \tag{5.14}$$

$$\tilde{G}^{n,i} = G^{n,i} + J[\lambda^n] \ , \qquad \text{with} \tag{5.14a}$$

$$G^{n,i} = \boldsymbol{\rho}^{n,i} - \mathbf{e}^{n,i} + (P_S^n - I)\mathbf{u}(t^n) \ . \tag{5.14b}$$

Substituting this expression of $\boldsymbol{\theta}^n$ in (5.13) and using the fact that by property (S) the array $\{m_{ij}\}$ is positive semidefinite we get

$$\|\boldsymbol{\theta}^{n+1}\|_{0,\Omega}^2 \leq \|\boldsymbol{\theta}^n\|_{0,\Omega}^2 + \|\tilde{G}^{n+1}\|_{0,\Omega}^2 + 2(\boldsymbol{\theta}^n, \tilde{G}^{n+1}) + 2\delta \sum_{i=1}^{s} b_i \left\{ (\tilde{G}^{n+1} - \tilde{G}^{n,i}, T^{n,i}) + (T^{n,i}, \boldsymbol{\theta}^{n,i}) \right\} \tag{5.15}$$

Now we estimate the terms on the right hand side of (5.15). We have

$$G^{n+1} = (P_S^n - I)\mathbf{u}(t^n) + \boldsymbol{\rho}^{n+1} - \mathbf{e}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}(t^{n+1}) + (I - P_S^n)(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) - \mathbf{e}^{n+1}. \tag{5.16}$$

Hence Taylor's theorem, $(4.18), (4.19), (3.5b)$ yield

$$\|G^{n+1}\|_{0,\Omega} \leq c\delta(h^{\mu} + \delta^{\sigma}) \ , \tag{5.17}$$

and by $(5.12a)$ we get that

$$\|\tilde{G}^{n+1}\|_{0,\Omega} \leq c\delta(h^{\mu} + \delta^{\sigma}) + \|J[\lambda^n]\|_{0,\Omega} \ . \tag{5.17a}$$

Now (5.17) and (5.17a) yield

$$|(\boldsymbol{\theta}^n, \tilde{G}^{n+1})| \le \delta\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + c\delta(h^{2\mu} + \delta^{2\sigma}) + \frac{1}{2}\epsilon_0\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + \frac{1}{2}\epsilon_0^{-1}\|J[\lambda^n]\|^2_{0,\Omega}. \tag{5.18}$$

Further from (5.3) we see that

$$(T^{n,i}, \boldsymbol{\theta}^{n,i}) = -\nu a_k^\gamma(\boldsymbol{\theta}^{n,i}, \boldsymbol{\theta}^{n,i}) + b(\mathbf{u}^{n,i}, \mathbf{u}^{n,i}, \boldsymbol{\theta}^{n,i}) - b(\mathbf{v}_k^{n,i}, \mathbf{v}_k^{n,i}, \boldsymbol{\theta}^{n,i}) + (\mathbf{f}(t^{n,j}) - \mathbf{f}^{n,j}, \boldsymbol{\theta}^{n,m}).$$

Also we have, using (2.7), (2.8), (3.5b)

$$b(\mathbf{u}^{n,i}, \mathbf{u}^{n,i}, \boldsymbol{\theta}^{n,i}) - b(\mathbf{v}_k^{n,i}, \mathbf{v}_k^{n,i}, \boldsymbol{\theta}^{n,i}) \le a_k^\gamma(\boldsymbol{\theta}^{n,i}, \boldsymbol{\theta}^{n,i})c\{\epsilon + \epsilon_0 + h\} + \epsilon_0^{-1}c\|\boldsymbol{\theta}^{n,i}\|_{0,\Omega} + \epsilon^{-1}ch^{2\mu}.$$

Hence, taking $\epsilon + \epsilon_0 + h \le \frac{\nu}{2c}$ it follows, using (5.8) and (5.9),

$$(T^{n,i}, \boldsymbol{\theta}^{n,i}) \le c^*\|\boldsymbol{\theta}^n\|^2_{0,\Omega} - \frac{\nu}{2}a_k^\gamma(\boldsymbol{\theta}^{n,i}, \boldsymbol{\theta}^{n,i}) + c^*\left\{h^{2\mu} + \delta^{2\sigma+1}\right\}. \tag{5.19}$$

Now from (5.14) we have

$$\delta\sum_{i=1}^{s} b_i(T^{n,i}, \tilde{G}^{n+1} - \tilde{G}^{n,i}) = \sum_{i,j=1}^{s} a_{ij}^{-1} b_i(\tilde{G}^{n+1} - \tilde{G}^{n,i}, \boldsymbol{\theta}^{n,i} - \boldsymbol{\theta}^n - \tilde{G}^{n,i}). \tag{5.20}$$

Using $(5.9), (5.17)$ and $(5.17a)$ we see that

$$|(\tilde{G}^{n+1}, \boldsymbol{\theta}^{n,i})| \le c\delta\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + c\delta(h^{2\mu} + \delta^{2\sigma}) + \frac{1}{2}\epsilon_1\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + \frac{1}{2}\epsilon_1^{-1}\|J[\lambda^n]\|^2_{0,\Omega}. \tag{5.21}$$

Also $G^{n,i} = (P_S - I)(\mathbf{u}(t^n) - \mathbf{u}^{n,i}) - \mathbf{e}^{n,i}$ so (4.17) implies

$$\|G^{n,i}\|_{0,\Omega} \le c\delta(h^\mu + \delta^\sigma). \tag{5.22}$$

Thus form $(5.9), (5.22)$ we get

$$|(\tilde{G}^{n,i}, \boldsymbol{\theta}^{n,i})| \le c\delta\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + c\delta(h^{2\mu} + \delta^{2\sigma}) + \frac{1}{2}\epsilon_2\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + \frac{1}{2}\epsilon_2^{-1}\|J[\lambda^n]\|^2_{0,\Omega}. \tag{5.23}$$

Similarly we get

$$|(\tilde{G}^{n,i}, \boldsymbol{\theta}^n)| \le c\delta\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + c\delta(h^{2\mu} + \delta^{2\sigma}) + \frac{1}{2}\epsilon_3\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + \frac{1}{2}\epsilon_3^{-1}\|J[\lambda^n]\|^2_{0,\Omega}, \tag{5.24}$$

$$|(\tilde{G}^{n+1}, \tilde{G}^{n,i})| \le c\delta^2(h^{2\mu} + \delta^{2\sigma}) + (1+\delta)\|J[\lambda^n]\|^2_{0,\Omega}. \tag{5.25}$$

Then taking $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{\beta_n}{2}$ and setting $M_n = 4\beta_n^{-1} + 1$ we get in view of (5.1), using (5.17) – (5.25) in (5.15)

$$\|\boldsymbol{\theta}^{n+1}\|^2_{0,\Omega} + c\delta\nu\sum_{i=1}^{s} a_k^\gamma(\boldsymbol{\theta}^{n,i}, \boldsymbol{\theta}^{n,i}) \le (1 + c^*\delta + \beta_n)\|\boldsymbol{\theta}^n\|^2_{0,\Omega} + c^*\delta(h^{2\mu} + \delta^{2\sigma}) + (\delta + M_n)\|J[\lambda^n]\|^2_{0,\Omega}. \tag{5.26}$$

Hence it follows that

$$\begin{aligned}
\|\boldsymbol{\theta}^{n+1}\|_{0,\Omega}^2 &\le (1+c^*\delta+\beta_n)\|\boldsymbol{\theta}^n\|_{0,\Omega}^2 + c^*\delta\{h^{2\mu}+\delta^{2\sigma}\} + (\delta+M_n)\|J[\lambda^n]\|_{0,\Omega}^2 \\
&\le \prod_{j=0}^{n}(1+c^*\delta+\beta_j)\|\boldsymbol{\theta}^0\|_{0,\Omega}^2 \\
&\quad + c\sum_{m=0}^{n}\left(\prod_{j=m+1}^{n}(1+c^*\delta+\beta_j)\right)\left(c^*\delta(h^{2\mu}+\delta^{2\sigma}) + (\delta+M_m)\|J[\lambda^m]\|_{0,\Omega}^2\right).
\end{aligned}$$

Now, fix n and choose $M_m = M = \mathcal{N}_C(n)$, $m = 1,\dots,n$ where $\mathcal{N}_C(n)$ is the number of times where $X_{k,h}^j \ne X_{k,h}^{j-1}, j = 1,\dots,n$. Then $\beta_j = \beta = \frac{4}{M-1}+1$, for $X_{k,h}^j \ne X_{k,h}^{j-1}$,

$$\prod_{j=0}^{n}(1+c^*\delta+\beta_j) \le \prod_{\substack{j=0\\ \beta<c^*\delta}}^{n}(1+2c^*\delta)\prod_{\substack{j=0\\ \beta\ge c^*\delta}}^{n}(1+2\beta) \le \prod_{j=0}^{n}(1+2c^*\delta)(1+2\beta)^M \le 3e^{2c^*t_{n+1}+8}.$$

Now $\boldsymbol{\theta}^0 = \mathbf{v}_k^0 - \mathbf{w}_k^0 = \mathbf{v}_k^0 - P_S^0\mathbf{u}(0) = \mathbf{v}_k^0 - \mathbf{u}^0$ so by (1.13) we have

$$\|\boldsymbol{\theta}^0\|_{0,\Omega} \le ch^\mu\ . \tag{5.27}$$

Set now $C_n = (3e^{2c^*t_n+8})^{1/2}$. Then , using (5.27)

$$\|\boldsymbol{\theta}^{n+1}\|_{0,\Omega} \le C_{n+1}h^\mu + cC_{n+1}\delta^{\frac{1}{2}}\{h^\mu+\delta^\sigma\} + C_{n+1}\sqrt{\mathcal{N}_C(n)}\left(\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\right)^{\frac{1}{2}}, \tag{5.28}$$

which is (5.10). $\square$

Remark 5.1. Tracing back the constants in the estimates leading to (5.26) we see that $c^* = \frac{c}{\nu}$ and thus the constant C_n in (5.10) is of order $\frac{1}{\nu}e^{c\frac{1}{\nu}}$. If we assume that, see also [2], [28],

$$\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))} \le C\nu \tag{A}$$

then c^* can be taken independent of ν and thus in the final estimate (5.29) the constant C_n is of order $\frac{1}{\nu}$. Indeed we have, using (2.6), (2.7), (2.8),

$$b(\mathbf{u}^{n,i},\mathbf{u}^{n,i},\boldsymbol{\theta}^{n,i}) - b(\mathbf{v}_k^{n,i},\mathbf{v}_k^{n,i},\boldsymbol{\theta}^{n,i}) \le c\|\mathbf{u}\|_{L^\infty(\mathbf{H}^2(\Omega))}a_k^\gamma(\boldsymbol{\theta}^{n,i},\boldsymbol{\theta}^{n,i}) + ch^{2\mu},$$

thus using assumption (A) we get

$$(T^{n,i},\boldsymbol{\theta}^{n,i}) \le -\frac{\nu}{2}a_k^\gamma(\boldsymbol{\theta}^{n,i},\boldsymbol{\theta}^{n,i}) + c\left\{h^{2\mu}+\delta^{2\sigma+1}\right\}.$$

The result then follows in view of Lemma 5.1 and the estimates of the other terms of (5.13).

We can now prove our main convergence Theorem for the velocity and pressure approximations.

Theorem 5.1. *Assume that* $\mathbf{v}_k^0$ *is chosen according to* (1.13). *and that the IRK method satisfies* (P). *Then if* $\mathbf{v}_k^n$ *and* $\mathbf{u}$ *are the solutions of* (1.14) – (1.15) *and* (1.1) – (1.4) *respectively, it follows that*

$$\max_{0\le n\le N}\|\mathbf{v}_k^n-\mathbf{u}(t^n)\|_{0,\Omega}\le c(h^\mu+\delta^\sigma)+c\sqrt{\mathcal{N}_C(N-1)}\Big(\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\Big)^{1/2}\;, \tag{5.29}$$

where $\mathcal{N}_C(n)$ *denotes the number of times where* $X_{k,h}^j\neq X_{k,h}^{j-1}, j=1,\dots,n$, *and* $J[\lambda^n]=\mathbf{w}_k^n-\mathbf{w}_k^{n+}=(P_S^{n-1}-P_S^n)\mathbf{u}(t^n)$ *denotes the jump of the Stokes projection at* t^n. *Further, if* ψ_h^n *is as defined in* (1.17) *and* p *is the solution of* (1.1) – (1.4) *then*

$$\begin{aligned}\max_{0\le n\le N}\|\psi_h^n-p(t^n)\|_{0,h}\le & c\{h^{\mu-1}+\delta^{-1}h^\mu+\delta^{\sigma-1}\}\\ &+C\sqrt{\mathcal{N}_C(N-1)}\Big(\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\Big)^{1/2}+C\mathcal{N}_C(N-1)\delta^{-1}\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\;.\end{aligned} \tag{5.30}$$

Proof. We write

$$\mathbf{v}_k^n-\mathbf{u}(t^n)=\mathbf{v}_k^n-\mathbf{w}_k^n+\mathbf{w}_k^n-\mathbf{u}(t^n)=\boldsymbol{\theta}^n+\mathbf{w}_k^n-\mathbf{u}(t^n)\;. \tag{5.31}$$

Also

$$\|\mathbf{w}_k^n-\mathbf{u}(t^n)\|_{0,\Omega}\le ch^\mu\;. \tag{5.32}$$

Using this and $(5.10), (5.31)$ we get (5.29). For the pressure we write

$$\psi_h^{n+1}-p(t^{n+1})=\psi_h^{n+1}-\pi_h^{n+1}+\pi_h^{n+1}-p(t^{n+1}). \tag{5.33}$$

By (5.11) we have $\forall[\mathbf{v},q]\in X_{k,h}^n$,

$$\begin{aligned}\delta\sum_{j=1}^{s}b_j\left\{-\mathcal{B}([\boldsymbol{\theta}^{n,j},\zeta^{n,j}],[\mathbf{v},q])\right\}&=(\boldsymbol{\theta}^n,\mathbf{v})-(\boldsymbol{\theta}^{n+1},\mathbf{v})+(\tilde{G}^{n+1},\mathbf{v})\\ &+\delta\sum_{j=1}^{s}b_j\left\{b(\mathbf{u}^{n,j},\mathbf{u}^{n,j},\mathbf{v})-b(\mathbf{v}_k^{n,j},\mathbf{v}_k^{n,j},\mathbf{v})+F^{n,j}([\mathbf{v},q])-\tilde{F}^{n,j}([\mathbf{v},q])\right\}.\end{aligned} \tag{5.34}$$

Now from $(5.6), (2.7), (2.8), (5.9)$ and (5.32) we get

$$\begin{aligned}&|b(\mathbf{u}^{n,j},\mathbf{u}^{n,j},\mathbf{v})-b(\mathbf{v}_k^{n,j},\mathbf{v}_k^{n,j},\mathbf{v})|\\ &\le\Bigg\{cC_nh^\mu+cC_n\delta^{\frac12}(h^\mu+\delta^\sigma)+cC_n\sqrt{\mathcal{N}_C(n)}\Big(\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\Big)^{1/2}\\ &+cC_n^2\delta^{-1}h^{2\mu}+cC_n^2(h^{2\mu}+\delta^{2\sigma})+cC_n^2\mathcal{N}_C(n)\delta^{-1}\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\Bigg\}\|\mathbf{v}\|_{1,k}\;.\end{aligned} \tag{5.35}$$

Now using $(1.8b), (5.34), (5.17a), (5.8)$ and (5.35) we obtain

$$\begin{aligned}\sum_{i=1}^{s}\|\zeta^{n,i}\|_{0,h}\le & cC_{n+1}h^\mu+cC_{n+1}\delta^{\frac12}(h^\mu+\delta^\sigma)+cC_{n+1}\sqrt{\mathcal{N}_C(n)}\Big(\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\Big)^{1/2}\\ &+cC_n^2\delta^{-1}h^{2\mu}+cC_n^2(h^{2\mu}+\delta^{2\sigma})+cC_n^2\mathcal{N}_C(n)\delta^{-1}\sum_{m=1}^{n}\|J[\lambda^m]\|_{0,\Omega}^2\;.\end{aligned} \tag{5.36}$$

We write

$$\begin{aligned}(5.37) \qquad \zeta^{n+1} &= \psi_h^{n+1} - \pi_h^{n+1} \\ &= \sum_{j=1}^{s} \xi_j \zeta^{n,j} + \sum_{j=1}^{s} \xi_j (\pi_h^{n,j} - p^{n,j}) + p(t^{n+1}) - \pi_h^{n+1} + \sum_{j=1}^{s} \xi_j p^{n,j} - p(t^{n+1}).\end{aligned}$$

From $(4.2), (4.14)$ and (1.17) now follows that

$$(5.38) \qquad p(t^{n+1}) = \sum_{j=1}^{s} \xi_j p^{n,j} + O(\delta^s).$$

Thus, it follows from $(5.37), (3.5a), (5.38), (5.36)$ that

$$\begin{aligned}(5.39) \qquad \|\zeta^{n+1}\|_{0,h} \leq{}& cC_{n+1}h^{\mu} + cC_{n+1}\delta^{\frac{1}{2}}(h^{\mu} + \delta^{\sigma}) + cC_{n+1}\sqrt{\mathcal{N}_C(n)}\Big(\sum_{m=1}^{n} \|J[\lambda^m]\|_{0,\Omega}^2\Big)^{1/2} \\ &+ cC_n^2\delta^{-1}h^{2\mu} + cC_n^2(h^{2\mu} + \delta^{2\sigma}) + cC_n^2\mathcal{N}_C(n)\delta^{-1}\sum_{m=1}^{n} \|J[\lambda^m]\|_{0,\Omega}^2 \\ &+ ch^{\mu-1} + c\delta^{\sigma-1}\ .\end{aligned}$$

Further using $(3.5a), (5.39)$ in (5.33) we get (5.30). $\square$

ACKNOWLEDGEMENTS

Parts of this work is from the author's Ph.D Thesis [20]. The author would like to express his graditude to Prof. Ohannes Karakashian for his valuable help in completing this work. The author also would like to thank Prof. Charalambos Makridakis for his helpful comments and suggestions.

REFERENCES

[1] G. Akrivis, V. Dougalis, O. Karakashian, *Solving the systems of equations arising in the discretization of some nonlinear p.d.e.'s by implicit Runge-Kutta methods.*, Math. Modell. Numer. Anal. **31** (1997), 251-288.

[2] G. Baker, V. Dougalis, O. Karakashian, *On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations*, Math. Comp. **39** (1982), 339-375.

[3] G. Baker, W. Jureidini, O. Karakashian, *Piecewise solenoidal vector fields and the Stokes problem*, SIAM J.Numerical Analysis **27** (1990), 1466-1485.

[4] F.E. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, In Applications of Partial Differential Equations Proc.Symp.App.Math., Amer.Math.Society, Providence, 1965, pp. 24–29.

[5] K. Boukir, B. Metivet, E. Razafindrakoto, Y. Maday, *Presentation of a second order time scheme using the characteristics method and applied to the Navier-Stokes equations,*, I23CS '91 13th Worlds Congress on Computational and Applied Mathematics Dublin, Ireland, vol. 629, 1991.

[6] J.C. Butcher, *The Numerical Analysis of ordinary differential equations. Runge-Kutta and General Linear Methods*, Wiley–Interscience, New York, 1987.

[7] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amesterdam, 1978.

[8] M. Crouzeix, W.H. Hundsdorfer and M.N. Spijker, *On the existence of solutions to the algebraic equations in implicit Runge-Kutta methods*, BIT **23** (1983), 84-91.

[9] T. Dupont, *Mesh modification for evolution equations*, Math. Comp. **39** (1982), 85-107.

[10] V. Girault, P.A. Raviart, *Finite Element Method for Navier-Stokes Equations. Theory and Numerical Analysis*, Springer–Verlag , Springer series in Comp.Physics, 1981.

[11] M. Gunzburger, *Finite Element method for Viscous Incompressible Flows*, Academic Press, 1989.

[12] M. Gunzburger, J.S. Peterson, *On conforming finite element methods for the inhomogeneous stationary Navier–Stokes equations*, Numer. Math. **42** (1983), 173-194.

[13] J.G. Heywood, R. Rannacher, *Finite element approximation to the nonstationary Navier-Stokes problem IV: Error analysis for second order time discretization*, SIAM J.Numerical Analysis.

[14] P. Jamet, P. Raviart, *Numerical Solution of the stationary Navier-Stokes Equations by finite element methods*, In Inter. Symposium on Computing Methods in Applied Sciences and Engineering. (R.Glowinski and J.Lions, ed.), Springer–Verlag, 1974, pp. 193-223.

[15] C.Johnson, R.Rannacher and M.Boman, *Numerics and hydrodynamic stability : Toward error control in computational fluid dynamics.*, SIAM J. Numer. Anal. **32** (1995), 1058–1079.

[16] O. Karakashian, G.D. Akrivis and V.A. Dougalis, *On optimal order error estimates for the nonlinear Schrödinger equation*, SIAM J. Numer. Anal. **30** (1993), 377–400.

[17] O. Karakashian, Ch. Makridakis, *A space time finite element method for the nonlinear Schrodinger equation: The discontinuous Galerkin method*, Math. Comp. **67** (1998), 479–499.

[18] O. Karakashian, W. Jureidini, *A nonconforming finite element method for the stationary Navier-Stokes equations*, SIAM J.Numer. Anal. **35** (1998), 93–120.

[19] O. Karakashian, Th. Katsaounis, *Numerical simulation of incompressible fluid flow with locally divergence free finite element.*, In preperation.

[20] Th. Katsaounis, *On Fully Discrete Galerkin Approximations for the Incompressible Navier-Stokes Equations*, Ph.D. Thesis, University of Tennessee, Knoxville, TN, 37996, USA, 1994.

[21] O. Karakashian, *On a Galerkin-Langrange Multiplier Method for the Stationary Navier-Stokes Equations*, SIAM J.Numer.Anal. **5** (1982), 909-923.

[22] O. Karakashian, W. McKinney, *On optimal high-order in time approximations for the Korteweg-de Vries equation.*, Math.Comp **55** (1990), 473-496.

[23] O.H. Kreiss, J. Lorenz, *Initial Boundary Value Problems and the Navier-Stokes Equations*, Academic Press, 1989.

[24] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York,, 1963.

[25] M. Marion, A. Ali, *Nonlinear Galerkin methods and mixed finite elements: two-grid algorithms for the Navier-Stokes equations*, Numer. Math. **68** (1994), 189-213.

[26] O. Pironeau, *On the transport diffusion algorithm and its application to the Navier-Stokes equations.*, Numer. Math. **38** (1982), 309-332.

[27] E. Suli, *Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations.*, Numer. Math. **53** (1988), 459-483.

[28] R. Temam, *Navier-Stokes equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.