Relaxation models and finite element schemes for the shallow water equations

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1 Introduction

We consider the one-dimensional system of shallow water equations (or Saint-Venant system) with a source term

$$h_t + (hu)_x = 0, \tag{1a}$$

$$(hu)_t + (hu^2 + \frac{g}{2}h^2)_x = -ghZ', \tag{1b}$$

which describes the flow at time $t \geq 0$ at point $x \in \mathbb{R}$, where $h(x,t) \geq 0$ is the height of water, $u(x,t)$ is the velocity, $Z(x)$ is the bottom height and g the gravity constant. In the sequel will denote $Q = hu$ the discharge. System (1) belongs in the more general class of hyperbolic systems with source terms

$$u_t + f(u)_x = q(u), \tag{2}$$

where u is a vector valued function and f, q are the given flux and source functions. In this paper we propose relaxation models and corresponding time discrete and finite element schemes for approximating (1). Our schemes can be formulated for the more general system (2) and special attention is given in the steady state approximations and their relation to the exact steady states especially for (1).

The system (1) was originally derived by A. de Saint–Venant [22], while recently Gerbeau and Perthame [10] derived a more complete system starting from the Navier–Stokes equations. The above system is a hyperbolic system with a source term, due to the topography of the bottom, which includes the unknown quantity h. In recent years many methods were proposed for the numerical approximation of solutions of hyperbolic conservations laws with source terms. The main difficulty being here, obtaining a scheme that is able to resolve shocks accurately and at the same time to have a reasonable steady state behavior. This is not a trivial matter since straight forward adaptations of upwind schemes have bad steady state performance. Hence particular attention has to be given in the numerical treatment of the source terms. In this direction finite volumes are widely used, and in particular the well balanced schemes introduced by Greenberg and LeRoux, [11], Gosse and

LeRoux [14] and further developed by Gosse [12],[13]. Another approach is based on wave propagation algorithms, [20], [5] and Godunov type schemes [9],[20]. Recently Jin [18] introduced a new method based on averaging the values at the interfaces. An alternative class of schemes were considered in [3]. These schemes are based on the kinetic interpretation of the system. An entropy satisfying kinetic scheme which also accurately computes the steady states is presented in [21]. Finite difference discretizations of (1) based on relaxation models are consider in [2], [4].

In the sequel we consider relaxation models and finite element relaxation schemes for (2), (1). We discuss first certain relaxation models and their time discrete counterparts. These models include the standard relaxation approximation for problems of balance laws considered e.g. in [6] as well as new relaxation approximations that we believe that are better suited to designing schemes for problems with non zero source terms. In addition we propose new relaxation approximations for the shallow water equations that admit as solutions exactly the steady states of the original system (1). We then apply the ideas of [1] discretizing by finite elements these models. Standard finite element spaces are used for the spatial discretization while implicit-explicit Runge–Kutta methods provide the time stepping mechanisms. A special variant of the resulting schemes which satisfies a discrete version of the steady state equation for (1) is also proposed.

The resulting schemes

- are based on a regularization mechanism with finite speed of propagation (no extra diffusion is added in the discretization of the relaxation model)
- do not need the solution of approximate local Riemann problems
- are formulated as low order, or high order schemes, simply by selecting the discretization space appropriately
- can be extended in multi-dimensions in a straightforward manner by using the finite element framework.

Since this method has no other regularizing mechanism besides that of the relaxation it should be combined with adaptive mesh refinement (close to the shocks) as in [1].

This paper is organized as follows: The needed preliminary material, the relaxation models and the properties of their time discrete counterparts are introduced in Section 2. In Section 3 we present the finite element schemes. Finally in Section 4 a series of experiments displaying the features of the method are presented.

2 Relaxation Models

For simplicity we consider first the scalar 1-D conservation law. The extension to the case of systems, if is done according to [19] is straightforward. Special relaxation models for the shallow water system (1) will be discussed in the sequel.

In the case of a conservation law with a source term

$$\begin{aligned} u_t + f(u)_x &= q(u), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3}$$

a relaxation system that corresponds to the one of [19] for $q = 0$, considered e.g. in [6, 18], takes the form

$$\begin{aligned} u_t + v_x &= q(u), \\ v_t + c^2 u_x &= -\frac{1}{\epsilon}(v - f(u)), \end{aligned} \tag{4}$$

yielding the following regularization of (3),

$$u_t + f(u)_x = q(u) + \epsilon q(u)_t - \epsilon(u_{tt} - c^2 u_{xx}). \tag{5}$$

Hence (4) introduces a wave type regularization of (3) and the additional term $\epsilon q(u)_t$. We do not expect that (4) will preserve the general steady states of (3) unless they (or q) have a special structure that, allows that as we will see in the sequel for the shallow water system. Nevertheless the error terms in the steady state regime are $O(\epsilon)$, and since computations with relaxation schemes are done with very small ϵ (or even with $\epsilon = 0$ in the relaxed case) (4) will yield a reasonable model for computations in this regime. Another important feature of the relaxation model (4) in the homogeneous case is the implicit-explicit time discretization: Consider time discrete approximations $U^n \cong u(t^n), V^n \cong V(t^n)$ where $t^n = n\delta$, δ being the temporal stepsize. It is known that the stiff source term in (4) should be descritized implicitly,

$$\begin{aligned} \frac{U^{n+1} - U^n}{\delta} + V_x^n &= 0, \\ \frac{V^{n+1} - V^n}{\delta} + c^2 U_x^n &= -\frac{1}{\epsilon}(V^{n+1} - f(U^{n+1})). \end{aligned} \tag{6}$$

Obviously the system (6) can be decoupled, and since U^{n+1} is known from the first equation, the solution of a nonlinear system is not needed. However for (4) this is not the case. Indeed, discretizing the source terms implicitly we get

$$\begin{aligned} \frac{U^{n+1} - U^n}{\delta} + V_x^n &= q(U^{n+1}), \\ \frac{V^{n+1} - V^n}{\delta} + c^2 U_x^n &= -\frac{1}{\epsilon}(V^{n+1} - f(U^{n+1})), \end{aligned} \tag{7}$$

which is fully coupled.

As an alternative approach we consider the following relaxation system

$$\begin{aligned} &u_t + v_x = 0, \\ &v_t + c^2 u_x = -\frac{1}{\epsilon}(v - f(u)) - \frac{1}{\epsilon}R(u), \end{aligned} \tag{8}$$

where $R(u)$ is an antiderivative of $q(u)$,

$$R(u(x,t)) = \int_a^x q(u(s,t))ds \,.$$

In this case (8) provides exactly a wave-type regularization of (3), indeed we have

$$u_t + f(u)_x = q(u) - \epsilon(u_{tt} - c^2 u_{xx}). \tag{9}$$

An implicit-explicit time discretization is now possible when we treat the source terms implicitly:

$$\begin{aligned} &\frac{U^{n+1} - U^n}{\delta} + V_x^n = 0, \\ &\frac{V^{n+1} - V^n}{\delta} + c^2 U_x^n = -\frac{1}{\epsilon}(V^{n+1} - f(U^{n+1})) - \frac{1}{\epsilon}R(U^{n+1}). \end{aligned} \tag{10}$$

This idea is applied successfully in the sequel in the case of shallow water system, yielding efficient schemes with nice steady-state behavior.

2.1 Relaxation models for the shallow water system

Next we discuss the properties of certain relaxation models for for the Saint-Venant equations corresponding to (4) and (8) respectively. In addition we introduce a relaxation system that preserves the steady states of (1). The steady state solution of (1) satisfies

$$(hu)_x = 0 \quad (hu^2 + \frac{g}{2}h^2)_x = -ghZ', \tag{11}$$

i.e.,

$$Q = hu = \text{constant}, \tag{12}$$

$$\frac{u^2}{2} + g(h + Z) = \text{constant}. \tag{13}$$

A particular case that provides a benchmark for many approximating schemes is the flow at rest, i.e., when $u = 0$, and $h + Z = \text{constant}$.

The relaxation models for (1) corresponding to (4) and (8) respectively are

$$h_t + v_x = 0, \tag{14a}$$
$$Q_t + w_x = -ghZ', \tag{14b}$$
$$v_t + c_1^2 h_x = -\frac{1}{\epsilon}(v - Q), \tag{14c}$$
$$w_t + c_2^2 Q_x = -\frac{1}{\epsilon}(w - (\frac{Q^2}{h} + \frac{g}{2}h^2)), \tag{14d}$$

and

$$h_t + v_x = 0, \tag{15a}$$
$$Q_t + w_x = 0, \tag{15b}$$
$$v_t + c_1^2 h_x = -\frac{1}{\epsilon}(v - Q), \tag{15c}$$
$$w_t + c_2^2 Q_x = -\frac{1}{\epsilon}(w - (\frac{Q^2}{h} + \frac{g}{2}h^2)) + \frac{1}{\epsilon}R(Z;h), \tag{15d}$$

where

$$R(Z;h)(x,t) = \int_a^x gh(y,t)Z'(y)dy\,.$$

The constants c_1, c_2 should be chosen appropriately, so that certain subcharacteristic conditions hold true. Formally in the relaxation limit $\epsilon \to 0$ we recover (1), for both relaxation systems (14) and (15). Notice that in the case where $Z \equiv 0$ then the two systems are identical. The above systems are combined in the sequel with implicit-explicit finite element discretizations yielding a reliable class of schemes for the approximation of (1). Note that due to the special form of the source term in the shallow water system, cf. (14a), even the first system, (14), admits decoupled implicit-explicit discretizations. Although the resulting schemes perform in a nice way in all the test problems, the exact steady states of (1) are not obtained as solutions of (14) and (15) with the exception of the flow at rest with $Z'' = 0$. Overall the qualitative behavior of (15) is better, cf. Section 4.

3 Finite Element Schemes

We consider now the finite element discretizations of (14), and (15). We assume for simplicity that the data our problem and its solution in $[0,T]$ has compact support in a set $\Omega_1 \subset \mathbb{R}$. Let Ω be an appropriate set such that $\Omega_1 \subset \Omega$ where our numerical solution will be sought. We consider a classical one dimensional C^0 finite element space defined on an element decomposition $\mathcal{T}_\kappa$ of Ω:

$$\mathcal{S}_\kappa = \mathcal{S}_\kappa^1 = \{\phi \in C(\mathbb{R}) : \phi|_T \in \mathbb{P}_1(T), T \in \mathcal{T}_\kappa, \phi|_{\Omega^c} \equiv 0\}\,.$$

The schemes are extended in a straightforward manner using finite element spaces of arbitrary degree p:

$$\mathcal{S}_\kappa = \mathcal{S}_\kappa^p = \{\phi \in C(\mathbb{R}) : \phi|_T \in \mathbb{P}_p(T), T \in \mathcal{T}_\kappa, \phi|_{\Omega^c} \equiv 0\}\,.$$

We denote by T the elements, intervals in the one dimensional case, of the partition $\mathcal{T}_\kappa$, and κ_T the length of the interval T. Then the finite element discretization of (15) is : we seek functions $h_\kappa, Q_\kappa, v_\kappa, w_\kappa \in \mathcal{S}_\kappa$ so that,

$$(\partial_t h_\kappa, \phi) + (\partial_x v_\kappa, \phi) = 0, \quad \forall \phi \in \mathcal{S}_\kappa, \tag{16a}$$

$$(\partial_t Q_\kappa, \psi) + (\partial_x w_\kappa, \psi) = 0, \quad \forall \psi \in \mathcal{S}_\kappa, \tag{16b}$$

$$(\partial_t v_\kappa, \phi) + c_1^2(\partial_x h_\kappa, \phi) = -\frac{1}{\epsilon}(v_\kappa - Q_\kappa, \phi), \quad \forall \phi \in \mathcal{S}_\kappa, \tag{16c}$$

$$\begin{aligned}(\partial_t w_\kappa, \psi) + c_2^2(\partial_x Q_\kappa, \psi) = &-\frac{1}{\epsilon}(w_\kappa - F(h_\kappa, Q_\kappa), \psi) \\ &+ \frac{1}{\epsilon}(R(Z; h_\kappa), \psi), \quad \forall \psi \in \mathcal{S}_\kappa,\end{aligned} \tag{16d}$$

where

$$F(h, Q) = \frac{Q^2}{h} + \frac{g}{2}h^2.$$

These schemes are high order and are combined with corresponding implicit-explicit Runge-Kutta time discretizations as in [1]. The behavior of the resulting schemes in several test problems is presented in Section 4.

The qualitative features of our finite element schemes as well as their relation to the classical finite difference schemes of [19] were discussed in detail in [1]. We briefly recall here that the effective equation corresponding to these schemes has a dispersion term coming from the finite element discretization. If this term is dominant, the approximate solution will have oscillatory character close to the shocks. Hence in order to exclude this possibility, as it was observed in [1], a condition of the form

$$\kappa < \gamma\epsilon$$

should be respected in a neighborhood of the shock, where γ is a constant. Hence, if the the local mesh size κ close to the shock is small enough the schemes are free of oscillations, cf. [1] for details.

In order to locate the shock and to use finer mesh there we use the *estimator function* g introduced in [1]. This is a piecewise linear function with point values

$$\kappa_j|u(x_j, t^m) - u(x_{j-1}, t^m)|$$

where $\kappa_j = x_j - x_{j-1}$ and has the property to *locate accurately the (approximate) shocks.* Its construction was motivated by the a posteriori estimates of [15, 16]. This function is used in the mesh refinement shock computations in the next section. We note though that the proposed schemes are not necessarily linked to a particular method of the mesh selection. This can be done in many alternative ways. The only requirement of our method is to have fine enough mesh size close to the shocks. (Note that in practice γ ranges from $10^2 - 10^3$).

4 Numerical experiments

We present the results of a series of numerical experiments illustrating the various features of the schemes. We have tested the fully discrete analogs, of the finite element method for (14), (15) in the piecewise linear case. We still refer these methods as (14), (15). Overall the qualitative behavior of (15) is better than of (14). So we display results of (14) only in the first example—the flow at rest. A general observation is that, although our schemes do not satisfy exactly the steady state equation (12) but rather approximations of it, the results in the standard steady state benchmarks presented in the sequel are very promising. Issues related to fine $h-p$ adaptive computations as well as computations in more than one dimension will be the subject of a forthcoming work.

In all the experiments reported here the time step δ is computed according to the *CFL*-condition

$$\max\{c_1, c_2\}\frac{\delta}{\kappa} \leq \frac{1}{2}.$$

The adaptive strategy followed in these experiments is very similar to the one reported in [1] : while the number of nodes is kept fixed throughout the computation, their locations are changing accordingly.

4.1 Flow at rest

We consider the system (1) with initial conditions

$$\begin{aligned} u(x,0) &= 0 \quad \forall x \in \mathbb{R}, \\ h(x,0) + Z(x) &= H \quad \forall x \in \mathbb{R}, \end{aligned}$$

then clearly

$$\begin{aligned} u(x,t) &= 0 \quad \forall x \in \mathbb{R}, t \geq 0, \\ h(x,t) + Z(x) &= H \quad \forall x \in \mathbb{R}, t \geq 0, \end{aligned}$$

is a solution to (1).

We test our scheme to this steady state flow where the bathymetry is non-trivial and is given by

$$Z(x) = \begin{cases} 0.2 - 0.05(x-10)^2, & 8 \leq x \leq 12, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

in channel of length $L = 20m$ and $H = 2m,\ \epsilon = 5.E-4,\ c_1 = c_2 = 5$.

Figures 1, and 2 show a magnified view of the results obtained by these schemes. The variance of the values of the water level $H = 2m$, are of $O(\epsilon)$. The differences are become more clear in the comparison of discharge in figure 2. There clearly (15) is better than (14), $(Q = 0)$.

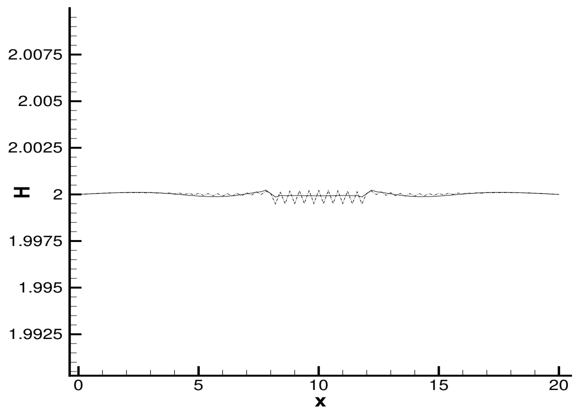

Fig. 1. *Magnified view.* Water level: (14)(thicker solid line), (15)(fine solid line)

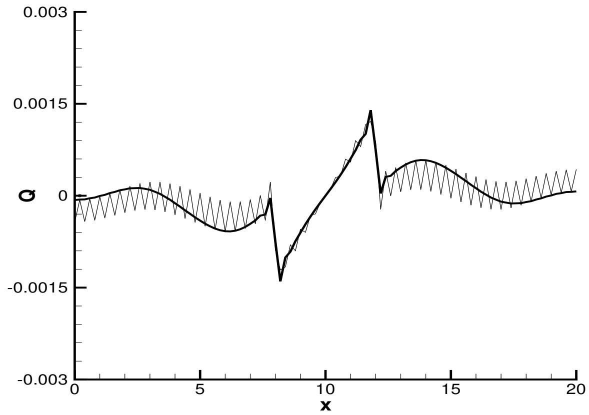

Fig. 2. *Magnified view.* Discharge: (14)(fine solid line),(15)(thicker solid line)

4.2 Transcritical flow with shock

In this test case the upstream boundary condition for the discharge is $Q = 0.18m^2/s$ and the downstream boundary condition for the water level is $h = 0.33m$. Local refinement was applied to obtain the results. The final water level of (14), (15) is displayed in Figure 3. Figure 4 displays the discharge given by the scheme (15).

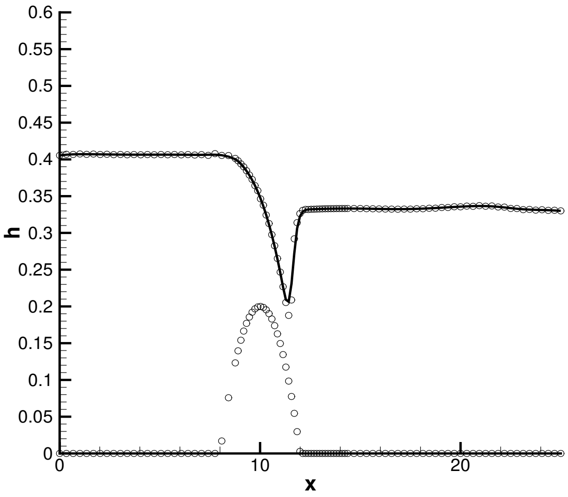

Fig. 3. Transcritical flow with shock over a hump : Water level. (15)(circles), (14)(solid line).

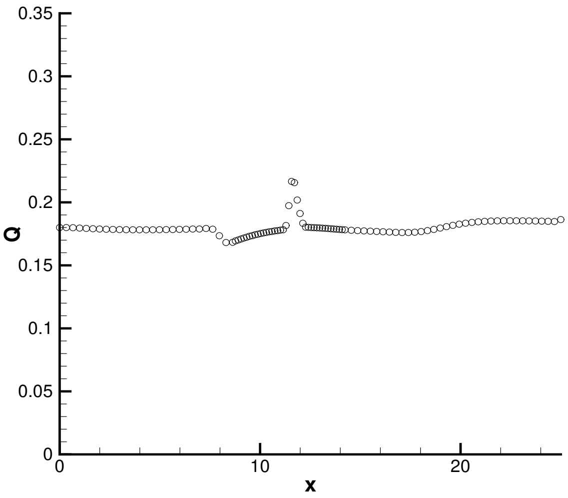

Fig. 4. Transcritical flow with shock over a hump : Discharge,(15).

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References

1. Arvanitis, Ch., Katsaounis, T., Makridakis, Ch. (2001): Adaptive finite element relaxation schemes for hyperbolic conservations laws, Math. Model. Numer. Anal., **35**, 17–33
2. Aral, M., Zhang, Y., Jin, S., (1998): Application of relaxation scheme to wave propagation simulation in open channel networks, J. Hydraulic Engineering, **124**, 1125-1133
3. Audusse, E., Bristeau, M.O., Perthame, B. (2000): Kinetic schemes for Saint–Venant equations with source terms on unstructured grids, INRIA Report RR–3989
4. Delis, A., Katsaounis, Th., (2002): Relaxation schemes for the shallow water equations, Report DMA-02-02, ENS, Paris, France
5. Bale, D.S., LeVeque, R.J., Mitran, S., Rossmanith, J.A. (2001): A wave propagation method for conservation laws and balance laws with spatially varying flux functions, Univ. of Washington, Seattle, Applied Math. Tech. Rep.
6. Chalabi, A. (1999): Convergence of relaxation schemes for hyperbolic conservation laws with stiff source terms, Math. Comp., **68**, pp.955–970.
7. Coquel, F., Perthame, B., (1998): Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, SIAM J. Numer. Anal., **35**, 2223–2249
8. Dafermos, C. (1999): Hyperbolic conservation laws in continuum physics, Springer-Verlag
9. Gallouët, T., Herard, J.H., Seguin, N. (2000): Some approximate Godunov schemes to compute shallow water equations with topography, AIAA-2001
10. Gerbeau, J.F., Perthame, B. (2001): Derivation of viscous Saint–Venant system for laminar shallow water, Discrete Contin. Dynam. Systems, Series B, **1**, 89–102
11. Greenberg, J.M., LeRoux, A.Y. (1996): A well balanced scheme for the numerical processing of source terms in hyperbolic equations, SIAM J. Num. Anal., **33**, 1–16
12. Gosse, L. (2000): A well–balanced flux vector splitting scheme designed for hyperbolic systems of conservation laws with source terms, Comput. Math. Appl., **39**, 135–159
13. Gosse, L. (2001): A well balanced scheme using non-conservative products designed for hyperbolic systems of conservation laws with source terms Math. Mod. Meth. Appl. Sci., **11**, 339–365
14. Gosse, L., LeRoux, A.Y. (1996): A well–balanced scheme designed for inhomogeneous scalar conservation laws C.R. Acad. Sci. Paris Sér. I Math., **323**, 543-546
15. Gosse, L., Makridakis, Ch. (1998): A-posteriori error estimates for numerical approximations to scalar conservations laws: schemes satisfying strong and weak entropy inequalities, IACM-FORTH, Technical Report 98-4
16. Gosse, L., Makridakis, Ch. (2000): Two a-posteriori error estimates for one dimensional scalar conservation laws, SIAM J. Numer. Anal., **38**, 964-988

17. Gosse, L., Tzavaras, A.E. (2001): Convergence of relaxation schemes to the equations of elastodynamics, Math. Comp. **70**, 555–577
18. Jin, S. (2001): A steady-state capturing method for hyperbolic systems with geometrical source terms, Math. Model. Numer. Anal., **35**, 631-645
19. Jin, S., Xin, Z. (1995): The relaxing schemes of conservations laws in arbitrary space dimensions, Comm. Pure Appl. Math, **48**, 235–277
20. LeVeque, R.J. (1998): Balancing source terms and flux gradients in high resolution Godunov methods: the quasi–steady wave propagation algorithm, J. Comput. Phys., **146**, 346-365
21. Perthame, B., Simeoni, C. (2001): A kinetic scheme for the Saint–Venant system with a source term, Calcolo, **38**, 201–231
22. de Saint–Venant, A.J.C. (1871): Théorie du mouvement non–permanent des eaux, avec application aux crues des rivière et à l'introduction des mareé dans leur lit, C.R.Acad. Sci. Paris, **73**, 147-154
23. Shu, C.W. (1988): Total-variation-diminishing time discretizations, SIAM J. Sc. Comp., **9**, 1073–1084
24. Stoker, J.J. (1986): Water waves, Interscience Publishers Inc., New York