A Discontinuous Galerkin Method For The Incompressible Navier-Stokes Equations

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Abstract. Approximations to solutions of the inhomogeneous boundary value problem for the Navier-Stokes equations are constructed via the discontinuous Galerkin method. The velocity field is approximated using piecewise polynomial functions that are totally discontinuous across interelement boundaries and which are pointwise divergence-free on each element (locally solenoidal). The pressure is approximated by standard continuous piecewise polynomial functions.

1 Introduction

We consider the stationary Navier-Stokes equations for viscous incompressible flow as given in the primitive variable formulation

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial\Omega \,. \tag{3}$$

Here $\mathbf{u} = (u_1, \cdots, u_N) : \Omega \to R^N$ represents the velocity field, and $p : \Omega \to R$, the pressure; the function $\mathbf{f} = (f_1, \cdots, f_N) : \Omega \to R^N$ denotes the prescribed external body forces, $\mathbf{g} = (g_1, \cdots, g_n) : \partial\Omega \to R^N$, the admitted flux across the boundary $\partial\Omega$ and $\nu > 0$ is a constant measuring viscosity. Note that $\mathbf{g}$ must satisfy the compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, d\sigma = 0$.

One of the attributes of the numerical method presented herein is the use of totally discontinuous piecewise polynomial vector functions to approximate the velocity field $\mathbf{u}$. These functions satisfy the incompressibility condition (2) pointwise on each element of a partition of Ω. A special weak formulation is designed to account for the interelement jumps that ensue. For the pressure, standard continuous piecewise polynomial functions are used. For the method that we shall describe below, a range of theoretical issues including the stability and convergence of the approximations at the optimal rates were presented in [11]. These results extended those obtained in [5] for the Stokes problem. Also, in [13], the application of implicit Runge-Kutta methods to the corresponding time dependent problem were analyzed.

In this paper, we outline the method for the stationary problem (1)–(3) and give a summary of these results. In addition, a small number of numerical experiments displaying optimal convergence rates for the errors and other features are presented. A more substantial array of numerical experiments will be provided in a forthcoming work [12].

2 The energy spaces

We shall next construct appropriate settings for the velocity, the pressure and their approximations as well as the Galerkin formulation. To begin, we consider partitions $\mathcal{T}_k = \{\Omega_1, \ldots, \Omega_{d_k}\}$ of Ω parametrized by $k > 0$. For simplicity, we shall use the generic name of element to denote Ω_i, which will be typically a triangle in 2-D or a tetrahedron in 3-D. We note that our formulation allows for more general shapes. In particular, the outlying elements may have a curved edge or face if Ω is not polygonal.

The formal setting for the velocity will be provided by the (mesh-dependent) "energy" space $\mathbf{E}_k = \mathbf{H}^2(\Omega_1) \times \cdots \times \mathbf{H}^2(\Omega_{d_k})$, where $\mathbf{H}^2$ is the Sobolev space of index 2 (cf. [1]). We may view $\mathbf{E}_k$ as a subspace of $\mathbf{L}^2(\Omega)$. In addition to the L^2 norm, we equip $\mathbf{E}_k$ with the mesh-dependent H^1-like norm

$$\|\mathbf{v}\|_{1,k} = \left\{ \sum_{i=1}^{d_k} \left(\|\nabla \mathbf{v}^{(i)}\|^2_{\partial\Omega_i} + \sum_{j\in\mathcal{N}_i} \tau_{ij} \left[k_i \left| \frac{\partial \mathbf{v}^{(i)}}{\partial n} \right|^2_{\partial\Omega_{i,j}} + k_i^{-1} \left| \mathbf{v}^{(i)} - \mathbf{v}^{(j)} \right|^2_{\partial\Omega_{i,j}} \right] + k_i \left| \frac{\partial \mathbf{v}^{(i)}}{\partial n} \right|^2_{\partial\Omega_i^e} + k_i^{-1} |\mathbf{v}^{(i)}|^2_{\partial\Omega_i^e} \right) \right\}^{1/2}$$

$$\begin{aligned}
k_i &= \text{diameter of } \partial\Omega_i, \\
\partial\Omega_{i,j} &= \partial\Omega_i \cap \partial\Omega_j \quad \text{ if } \partial\Omega_i \text{ and } \partial\Omega_j \text{ are adjacent },\\
\partial\Omega_i^e &= \partial\Omega_i \cap \partial\Omega \\
\mathbf{v}^{(i)} & \quad \text{restriction of } \mathbf{v} \text{ to } \partial\Omega_i,\\
\mathbf{v}^{(i)} - \mathbf{v}^{(j)} & \quad \text{jump in } \mathbf{v} \text{ across } \partial\Omega_{i,j},\\
\frac{\partial \mathbf{v}^{(i)}}{\partial n} & \quad \text{normal derivative of } \mathbf{v}^{(i)} \text{ with respect to}\\
& \quad \text{the unit outward normal to } \partial\Omega_i\\
\mathcal{N}_i &= \{j : \partial\Omega_j \text{ is adjacent to } \partial\Omega_i\}\\
\tau_{ij} &= 1 \text{ if } i > j \text{ and } \tau_{ij} = 0 \text{ if } i \le j,\\
(\mathbf{u},\mathbf{v})_D &= \int_D \mathbf{u}\cdot\mathbf{v}\,dx = \sum_{i=1}^N \int_D u_i v_i dx, \quad \|\mathbf{v}\|_D = (\mathbf{v},\mathbf{v})_D^{1/2},\\
(\mathbf{u},\mathbf{v}) &= (\mathbf{u},\mathbf{v})_\Omega,\\
<\mathbf{u},\mathbf{v}>_\Gamma &= \int_\Gamma \mathbf{u}\cdot\mathbf{v}\,ds = \sum_{i=1}^N \int_\Gamma u_i v_i ds, \text{ edge or surface integrals },\\
|\mathbf{v}|_\Gamma &= <\mathbf{v},\mathbf{v}>_\Gamma^{1/2} .
\end{aligned}$$

To approximate the pressure, we use a partition $\mathcal{T}_h = \{\Omega_1^h, \ldots, \Omega_{d_h}^h\}$ of Ω possibly different from $\mathcal{T}_k$. In order to satisfy the Babuska-Brezzi stability condition establishing the compatibility of the velocity and pressure finite element spaces, we shall assume that $\mathcal{T}_k$ is possibly finer than $\mathcal{T}_h$ in the sense that every element Ω_ℓ^h is a union of members of $\mathcal{T}_k$.

Since the pressure is determined up to an additive constant only, it is convenient to work with quotient spaces X/R obtained by identifying all functions in the space X that differ by constants. Such a space is $L^2(\Omega)/R$. (Note that equivalently one could work with the space $L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}$). $L^2(\Omega)/R$ is a Banach space when equipped with the quotient norm $\|q\|_{L^2(\Omega)/R} = \inf_{c\in R}\|q-c\|_{L^2(\Omega)}$. We shall also use the following mesh-dependent and L^2-like norm on the quotient space $H^1(\Omega)/R$

$$\|q\|_{0,h} = \left\{ \|q\|^2_{L^2(\Omega)/R} + \sum_{\ell=1}^{d_h} h_\ell^2 \|\nabla q^{(\ell)}\|^2_{\Omega_\ell^h} \right\}^{1/2},$$

where h_ℓ is the diameter of Ω_ℓ^h.

3 The finite element spaces

The set of vector functions

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}y\\0\end{pmatrix}, \begin{pmatrix}0\\x\end{pmatrix}, \begin{pmatrix}x\\-y\end{pmatrix} \right\}$$

forms a basis for the space of linear solenoidal functions in R^2. Augmenting it by

$$\left\{ \begin{pmatrix}y^2\\0\end{pmatrix}, \begin{pmatrix}0\\x^2\end{pmatrix}, \begin{pmatrix}x^2\\-2xy\end{pmatrix}, \begin{pmatrix}-2xy\\y^2\end{pmatrix} \right\},$$

gives a basis for quadratics. In R^3, for $r_1 = 2$, a basis is given by the set

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\x\\0\end{pmatrix}, \begin{pmatrix}0\\0\\x\end{pmatrix}, \begin{pmatrix}y\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\y\end{pmatrix}, \right.$$

$$\left. \begin{pmatrix}z\\0\\0\end{pmatrix}, \begin{pmatrix}0\\z\\0\end{pmatrix}, \begin{pmatrix}x\\-y\\0\end{pmatrix}, \begin{pmatrix}x\\0\\-z\end{pmatrix} \right\}.$$

It is typical in constructing finite element spaces to use affine transformations to map "master" basis functions to each element $\partial\Omega_i$. It turns out however that the incompressibility property is not preserved by general affine transformations. Therefeore, the local basis functions are constructed by translations and scaling of the above functions. We denote the finite element space thus obtained by $\mathbf{V}_k^{r_1}$ where $r_1 - 1$ is the degree of the polynomials used. The fact that the spaces $\mathbf{V}_k^{r_1}$ possess optimal approximations properties is established in [5].

To approximate the pressure, we use spaces $P_h^{r_2}$ of continuous piecewise polynomial functions of degree $r_2 - 1, r_2 \geq 2$ defined on the partition $\mathcal{T}_h$. These spaces are quite standard cf. [8].

4 The discontinuous Galerkin formulations

We begin by defining the bilinear form $a_k^\gamma(\cdot,\cdot) : \mathbf{E}_k \times \mathbf{E}_k \to R$

$$\begin{aligned} a_k^\gamma(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{d_k} \Bigg\{ (\nabla \mathbf{u}^{(i)}, \nabla \mathbf{v}^{(i)})_{\partial\Omega_i} + \sum_{j\in\mathcal{N}_i} \tau_{ij} \Bigg[-\left\langle \frac{\partial \mathbf{u}^{(i)}}{\partial n}, \mathbf{v}^{(i)} - \mathbf{v}^{(j)} \right\rangle_{\partial\Omega_{i,j}} \\ - \left\langle \frac{\partial \mathbf{v}^{(i)}}{\partial n}, \mathbf{u}^{(i)} - \mathbf{u}^{(j)} \right\rangle_{\partial\Omega_{i,j}} \\ + \gamma k_i^{-1} \left\langle \mathbf{u}^{(i)} - \mathbf{u}^{(j)}, \mathbf{v}^{(i)} - \mathbf{v}^{(j)} \right\rangle_{\partial\Omega_{i,j}} \Bigg] - \left\langle \frac{\partial \mathbf{u}^{(i)}}{\partial n}, \mathbf{v}^{(i)} \right\rangle_{\partial\Omega_i^e} \\ - \left\langle \frac{\partial \mathbf{v}^{(i)}}{\partial n}, \mathbf{u}^{(i)} \right\rangle_{\partial\Omega_i^e} + \gamma k_i^{-1} \left\langle \mathbf{u}^{(i)}, \mathbf{v}^{(i)} \right\rangle_{\partial\Omega_i^e} \Bigg\}, \end{aligned} \tag{4}$$

which constitutes a weak formulation for the Dirichlet integral $(\nabla \mathbf{u}, \nabla \mathbf{v})$. Indeed, if $\mathbf{u} \in \mathbf{H}^2(\Omega)$, then $\forall \mathbf{v} \in \mathbf{E}_k$

$$a_k^\gamma(\mathbf{u},\mathbf{v}) = -(\Delta \mathbf{u}, \mathbf{v}) - \sum_{i=1}^{d_k} \left\langle \mathbf{u}^{(i)}, \frac{\partial \mathbf{v}^{(i)}}{\partial n} - \gamma k_i^{-1} \mathbf{v}^{(i)} \right\rangle_{\partial\Omega_i^e} . \tag{5}$$

Some further comments on the nature of the form a_k^γ are in order

1. The first, second and fifth terms on the right side of (4) are byproducts of integration by parts and range over the interior and boundary edges of $\mathcal{T}_k$ respectively. The array τ_{ij} is used to ensure that each interior edge is visited only once. This device is also convenient as a method for relating the ordering of the edges in a natural way to the ordering of the elements $\partial\Omega_i$.
2. The third and sixth terms have been added to ensure symmetry of the form a_k^γ. Note that the third term is zero for smooth $\mathbf{u}$, while the sixth is a known quantity since $\mathbf{u}\,\big|_{\partial\Omega}$ is given. We note that the theoretical results remain valid if these terms are removed.
3. The fourth and seventh are so-called "penalty" terms which, upon choosing γ sufficiently large, induce coercivity of the form a_k^γ. The choice of γ is independent of the partition $\mathcal{T}_k$.

The next few results highlight the analysis presented in [5] and [11]. In particular, the role of the penalty parameter γ is exhibited.

Proposition 1. (i)

$$|a_k^\gamma(\mathbf{u},\mathbf{v})| \leq (1+\gamma)\,\|\mathbf{u}\|_{1,k}\ \|\mathbf{v}\|_{1,k}\,, \qquad \forall \mathbf{u},\mathbf{v} \in \mathbf{E}_k.$$

(ii) *There exist positive constants γ_0 and c_a such that for all $\gamma \geq \gamma_0$,*

$$a_k^\gamma(\mathbf{v},\mathbf{v}) \geq c_a \|\mathbf{v}\|_{1,k}^2\,, \qquad \forall \mathbf{v} \in \mathbf{V}_k^{r_1}.$$

The value of γ_0 depends on r_1 but is independent of the meshsize k. Indeed, the bilinear form a_k^γ is singular if γ is small. It is interesting to note that recently a class of related of methods which discard the penalty terms have been proposed cf. [6]. In these methods the bilinear form is made nonsingular by what essentially amounts to changing the sign of the 3rd term on the right side of (4).

Proposition 2. *There exists a constant $c > 0$ such that*

$$(\mathbf{v},\nabla q) \leq c\,\|\mathbf{v}\|_{1,k}\|q\|_{0,h}\,, \quad \forall \mathbf{v} \in \mathbf{E}_k,\ \forall q \in H^1(\Omega).$$

Proposition 3. *Let $r_1 \geq 1$ and $r_2 \geq 2$ be given. Suppose $\mathcal{T}_k$ is sufficiently fine with respect to $\mathcal{T}_h$. Then, there exists a positive constant c, independent of k and h, such that*

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_k^{r_1}} \frac{(\mathbf{v},\nabla q)}{\|\mathbf{v}\|_{1,k}} \geq c\|q\|_{0,h}\,, \quad \forall q \in P_h^{r_2}. \tag{6}$$

This is the crucial Babuska-Brezzi (inf-sup) condition. Existence and convergence of the numerical approximations depend on it in an essential manner. It is a simple exercise in Linear Algebra to show that if (6) holds then we must necessarily have that $\dim \mathbf{V}_k^{r_1} \geq \dim P_h^{r_2} - 1$. In this sense, using discontinuous elements for the velocity in conjunction with continuous elements for the pressure constitutes a step in the right direction. Indeed, taking $\mathcal{T}_k$ fine with respect to $\mathcal{T}_h$ is a way of increasing the dimension of $\mathbf{V}_k^{r_1}$ with respect to the dimension of $P_h^{r_2}$. By the same token, (6) cannot hold for arbitrary choices of r_1 and r_2 without taking $\mathcal{T}_k$ finer than $\mathcal{T}_h$. However, our numerical experiments, all conducted with $r_1 = r_2$ or $r_1 = r_2 + 1$, suffered no apparent ill effects from taking $\mathcal{T}_k = \mathcal{T}_h$. We conjecture that (6) holds under these conditions.

At this point, we draw attention to similarities between our method and others in the literature. Indeed, in view of the fourth term in (4), our method can be termed as an "interior penalty" formulation. Such methods have been extensively studied in the context of elliptic and other types of problems [3],

[9], [4], [15], [2]. In addition, since the inf-sup property (6) holds for arbitrary choices of r_1 and r_2, provided of course we choose $\mathcal{T}_k$ finer than $\mathcal{T}_h$, relates our method in spirit to so-called stabilized methods cf. [7], [10].

We next construct a Galerkin approximation to the Stokes problem which is the system (1)–(3) minus the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Multiplying (1) by $\mathbf{v}\in\mathbf{E}_k$ and integrating one obtains after using (5)

$$\nu a_k^\gamma(\mathbf{u},\mathbf{v}) + (\mathbf{v},\nabla p) = (\mathbf{f},\mathbf{v}) - \nu\sum_{i=1}^{d_k}\left\langle \frac{\partial \mathbf{v}^{(i)}}{\partial n} - \gamma k_i^{-1}\mathbf{v}^{(i)}, \mathbf{g}\right\rangle_{\partial\Omega_i^e}.$$

Now multiplying (2) by $q\in H^1(\Omega)$ and integrating by parts, we see that

$$(\mathbf{u},\nabla q) = \langle \mathbf{g}\cdot\mathbf{n}, q\rangle_{\partial\Omega}.$$

Combining the last two equations, we derive the following weak formulation for the Stokes problem

$$\nu a_k^\gamma(\mathbf{u},\mathbf{v}) + (\mathbf{v},\nabla p) + (\mathbf{u},\nabla q) = F_S([\mathbf{v},q]), \quad \forall [\mathbf{v},q]\in\mathbf{E}_k\times H^1(\Omega),$$

where

$$F_S([\mathbf{v},q]) = (\mathbf{f},\mathbf{v}) - \nu\sum_{i=1}^{d_k}\left\langle \frac{\partial \mathbf{v}^{(i)}}{\partial n} - \gamma k_i^{-1}\mathbf{v}^{(i)}, \mathbf{g}\right\rangle_{\partial\Omega_i^e} + \langle \mathbf{g}\cdot\mathbf{n}, q\rangle_{\partial\Omega}.$$

Hence, we define the Galerkin approximation to the Stokes problem as the unique element $[\mathbf{u}_k, p_h]\in\mathbf{V}_k^{r_1}\times P_h^{r_2}/R$ satisfying

$$\nu a_k^\gamma(\mathbf{u}_k,\mathbf{v}) + (\mathbf{v},\nabla p_h) + (\mathbf{u}_k,\nabla q) = F_S([\mathbf{v},q]), \quad \forall [\mathbf{v},q]\in\mathbf{V}_k^{r_1}\times P_h^{r_2}/R.$$

To handle the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ we define the trilinear form $b_1(\cdot,\cdot,\cdot):\mathbf{E}_k^3\to R$ by

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i=1}^{d_k}\Bigg\{\int_{\partial\Omega_i} u_\ell^{(i)}\frac{\partial v_m^{(i)}}{\partial x_\ell} w_m^{(i)}\,dx - \sum_{j\in\mathcal{N}_i}\tau_{ij}\int_{\partial\Omega_{i,j}} u_\ell^{(i)}(v_m^{(i)} - v_m^{(j)})w_m^{(i)} n_\ell^{(i)}\,d\sigma\Bigg\},$$

where we have adopted Einstein's summation convention for repeated indices for components of vectors. Following a well-known device of Témam, we introduce the skew-symmetric form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\left[b_1(\mathbf{u},\mathbf{v},\mathbf{w}) - b_1(\mathbf{u},\mathbf{w},\mathbf{v})\right].$$

Note that we have $b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{E}_k$. additionally, the following consistency result holds

Proposition 4. *Suppose* $\mathbf{u}$ *is in* $\mathbf{H}^2(\Omega)$ *and satisfies* $\operatorname{div}\mathbf{u} = \mathbf{0}$ *in* Ω. *Then*

$$b(\mathbf{u},\mathbf{u},\mathbf{v}) = \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\mathbf{v}dx - \frac{1}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{v})d\sigma, \quad \forall \mathbf{v}\in\mathbf{E}_k.$$

The Galerkin approximation of the stationary Navier-Stokes problem (1)–(3) is defined as the unique solution $[\mathbf{u}_k, p_h] \in \mathbf{V}_k^{r_1} \times P_h^{r_2}/R$ of

$$\begin{aligned}\nu a_k^\gamma(\mathbf{u}_k,\mathbf{v}) + (\mathbf{u}_k,\nabla q) + (\mathbf{v},\nabla p_h) + b(\mathbf{u}_k,\mathbf{u}_k,\mathbf{v}) \\ = F_{NS}[\mathbf{v},q], \quad \forall[\mathbf{v},q]\in\mathbf{V}_k^{r_1}\times P_h^{r_2}/R,\end{aligned}$$

with

$$F_{NS}[\mathbf{v},q] = F_S([\mathbf{v},q]) - \frac{1}{2}\langle \mathbf{g}\cdot\mathbf{n}, \mathbf{g}\cdot\mathbf{v}\rangle_{\partial\Omega}.$$

The convergence of the numerical approximations defined above is analyzed in [5] and [11]. Let $[\mathbf{u}, p]$ denote the solution of either the stationary Stokes or Navier-Stokes problems and assume it to be sufficiently smooth. For simplicity, suppose that $k = h$ and that $r_1 = r_2 = r$. Then, under certain conditions, (cf. [5], [11] for details) the Galerkin approximations $[\mathbf{u_k}, p_h]$ converge to $[\mathbf{u}, p]$ and satisfy the optimal rates

$$\|\mathbf{u}-\mathbf{u_k}\|_{L^2} + h\|\mathbf{u}-\mathbf{u_k}\|_{1,k} + h\|p-p_h\|_{0,h} \le ch^r. \tag{7}$$

5 Numerical results

For considerations of space, we restricted ourselves to the Stokes problem and relegated algorithmic and implementational issues to our forthcoming work [12]. Taking Ω to be the unit square, we imposed on it a regular partition consisting of right isoceles triangles, with the equal sides having length h. Again, we would like to stress the fact that the grids for the velocity and the pressure were identical. To study the errors, we used the function

$$\mathbf{u} = \frac{1}{\pi^2}(\ \sin\pi(x+y), -\sin\pi(x+y)\ ), \quad p = \frac{1}{\pi^2}\sin\pi(x+y),$$

adjusting $\mathbf{f}$ and $\mathbf{g}$ so that $[\mathbf{u}, p]$ is a solution to (1) and (3).

In Table 1 and Table 2, we exhibit the errors $\|\mathbf{u}-\mathbf{u}_k\|_{L^2}$, $\|\mathbf{u}-\mathbf{u}_k\|_{1,k}$, $\|p-p_h\|_{0,h}$ respectively and the corresponding convergence rates. The first table corresponds to $r_1 = r_2 = 2$ and the second to $r_1 = 3$, $r_2 = 2$. The prevailing values of the Reynolds number $Re = 1/\nu$ and γ are as shown. The rates are seen to conform to theoretical predictions. Note that the rate for the L^2-error for the velocity in Table 1 is slightly larger than the predicted value of 2. Also, the errors for the pressure are the same for both values of r_1.

	$\|\|E(u)\|\|_{L^2}$		$\|\|E(u)\|\|_{1,k}$		$\|\|E(p)\|\|_{0,h}$	
h^{-1}	Error	Rate	Error	Rate	Error	Rate
6	0.254786E-02		0.218224E+00		0.117596E-01	
8	0.969968E-03	3.357	0.154482E+00	1.201	0.645897E-02	2.083
10	0.488330E-03	3.075	0.121078E+00	1.092	0.410444E-02	2.032
12	0.288787E-03	2.881	0.999404E-01	1.052	0.284273E-02	2.015
14	0.189057E-03	2.748	0.852051E-01	1.035	0.208610E-02	2.008
16	0.132241E-03	2.677	0.743051E-01	1.025	0.159624E-02	2.004
18	0.981661E-04	2.530	0.658993E-01	1.019	0.126083E-02	2.003
20	0.755572E-04	2.485	0.592077E-01	1.016	0.102108E-02	2.002
22	0.595668E-04	2.495	0.537598E-01	1.013	0.843775E-03	2.001
24	0.484666E-04	2.370	0.492338E-01	1.011	0.708952E-03	2.001

Table 1. $r_1 = r_2 = 2, Re = 1000, \gamma = 50$

	$\|\|E(u)\|\|_{L^2}$		$\|\|E(u)\|\|_{1,k}$		$\|\|E(p)\|\|_{0,h}$	
h^{-1}	Error	Rate	Error	Rate	Error	Rate
6	0.160837E-02		0.129356E+00		0.117676E-01	
8	0.595667E-03	3.453	0.702981E-01	2.120	0.646089E-02	2.084
10	0.291263E-03	3.206	0.451162E-01	1.988	0.410517E-02	2.032
12	0.165575E-03	3.098	0.315465E-01	1.962	0.284316E-02	2.015
14	0.103424E-03	3.053	0.233018E-01	1.965	0.208636E-02	2.008
16	0.689920E-04	3.032	0.178880E-01	1.980	0.159646E-02	2.004
18	0.490183E-04	2.902	0.140943E-01	2.024	0.126089E-02	2.003
20	0.366955E-04	2.748	0.115363E-01	1.901	0.102111E-02	2.002
22	0.277798E-04	2.920	0.958844E-02	1.940	0.843802E-03	2.001
24	0.214329E-04	2.981	0.807169E-02	1.979	0.709007E-03	2.000

Table 2. $r_1 = 3, r_2 = 2, Re = 1000, \gamma = 50$

A very interesting issue is the dependence of the errors on Re and γ. In Fig. 1 we show the level curves for the errors in terms of these two parameters. In these experiments, we took $h = 1/16$. Figures 1a and 1b correspond to $r_1 = r_2 = 2$ while figures 1c and 1d correspond to $r_1 = 3,\ r_2 = 2$. First, we observe that while the error for the velocity increases with Re, the opposite happens with the pressure. However, the variation in the latter is rather insignificant.

On the other hand, for a fixed value of Re, the shapes of the level curves for the velocity indicate the existence of an optimal value of γ in the sense of minimizing the error. This is in accordance with our experience using the bilinear form (4) in the context of the Dirichlet problem for Laplace's equation. It is also worth noting that, while the bilinear form a_k^γ is no longer positive definite for γ small, the errors increase monotonically to an asymptotic limit as $\gamma \to \infty$.

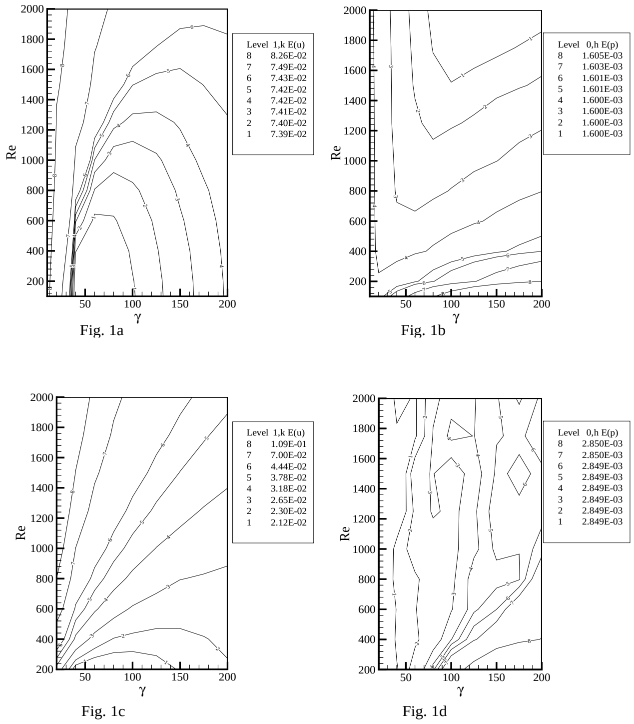

Fig. 1. Level Curves for the errors

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