Interface crack propagation in porous and time-dependent materials analyzed with discrete models

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Abstract A model describing the crack propagation at the interface between a rigid sub-stratum and a beam is considered. The interface is modeled using a fiber bundle model (i.e. using a discrete set of elements having a random strength). The distribution of avalanches, defined as the distance over which the crack is propagated under a fixed force, is studied in order to capture the effect of ageing and time-dependent response of the interface. The avalanches depend not only on the statistical distribution of strength but more importantly on time (or displacement) correlations. Namely, local fiber breakage kinetics is related to a correlation length, which sets the size of the fracture process zone which occurs ahead of the crack due to progressive failure. First, a variation of porosity of the interface is considered. It corresponds for instance to diffusion controlled dissolution processes. Interpreting the results in Delaplace et al. [Delaplace A, Roux S, Pijaudier-Cabot G (2001) J Eng Mech 127:646–652], it is shown that the size of the fracture process zone increases with increasing porosity in accordance with experimental observations [Haidar K, Pijaudier-Cabot G, Dubé J-F, Loukili A (2005) Mater Struct 38:201–210]. The creep–fracture interaction is analyzed in the second part of the paper. It is found based on a Maxwell model that the size of the process zone depends on the fracture propagating velocity and on the distribution of forces in the interface due to the interaction between the interface and the rest of the specimen. The observed decrease of the size of the process zone, in creep experiments, compared to the size of the process zone in a time-independent process, is justified by the proposed model for an interface that is less viscous than the rest of the material.

Keywords Zip model · FPZ size · Size effects · Creep · Ageing · Fracture · Viscoelasticity · Time effect · Concrete failure · Discrete approach

1 Introduction

Progressive failure of quasi-brittle heterogeneous materials is a succession of micro-cracks nucleation, propagation and arrest. First, the material response is elastic, then microcracking appears and eventually these microcracks coalesce in order to form a macro-crack which propagates suddenly. The above rupture events are controlled by the randomness
of the distribution of the material properties and also by correlation lengths, due to the interaction between microcracks, that are in general unknown. Such correlation lengths have a strong influence on the failure of quasi-brittle materials. Instead of a perfect crack, with a very small process zone of non-linear response at its tip, a rather large fracture process zone (FPZ) develops in the material ahead of the tip of the macrocrack Bažant and Planas (1998). This process zone induces a size effect, which is typical of quasi-brittle heterogeneous structures.

Among several possible explanations for such a size effect on the structural strength, a simple one is that it is a purely deterministic effect, resulting from the energetic interaction between the elastic part of the structure and the FPZ. From the point of view of design of structures, e.g. reinforced concrete structures, this size effect is important. According to the size effect law proposed by Bažant, the nominal strength of a structural component can be decreased by as much as 50% if the real size of the structure is five times larger than the laboratory specimen size. This size effect law incorporates also the definition of a fracture energy, seen as the energy dissipation in the asymptotic limit of a structure of infinite size with a very large initial notch, which would fail at crack initiation.

A relevant continuum model has to take into account this size effect, and at the same time to capture strain softening due to progressive cracking in a way that is physically and mathematically sound. This is usually performed by the introduction of an internal length in the continuum model (see e.g. the review in Bažant and Jirasek (2002)). The internal length serves two purposes at the same time: (1) it is a way of constraining the energy dissipation to occur in a region of finite size upon strain localization due to softening and to keep the governing equations well posed (Pijaudier-Cabot and Benallal 1993), (2) the occurrence of the FPZ upon cracking, whose size is a function of the internal length, and the inherited size effect are captured at the same time. As a matter of fact, size effect, e.g. of geometrically similar notched three point bending beams, is an indirect technique for the calibration of the internal length and other model parameters for isotropic non-local damage models for concrete (Le Bellégo et al. 2003a). Theoretical strain localization analyses show that the width of the fracture process zone ought to be proportional to the internal length. Such a result was already observed numerically (Bažant and Pijaudier-Cabot 1988) and was used in order to design an approximate method for the determination of the internal length (Bažant and Pijaudier-Cabot 1989). Experiments and finite element analyses on several model materials made with mixes of mortar and polystyrene beads (Haidar et al. 2005) have shown that the width of the FPZ and Irwin’s length (which can be seen as the length of the FPZ ahead of the macrocrack tip) are also correlated to the variation of the internal length in the non-local continuum. Irwin’s length is defined as

\[ l_{cl} = \frac{E G_f}{\sigma^2} \]  

where \( E \) is the Young’s modulus, \( G_f \) is the fracture energy and \( \sigma \) is the tensile strength. Figure 1 shows that the FPZ measured experimentally from acoustic events analysis during fracture, the internal length obtained after calibration of the non-local damage model on the various model materials (using size effect test data) and Irwin’s length correlate quite well.

These experiments on model materials, with various amounts of polystyrene beads, were designed in order to mimic the ageing process in cementitious materials due to calcium leaching, i.e. a progressive dissolution of the material due to contact with water. In this case, the width and the length of the FPZ increase with the amount of material porosity and can be considered as proportional to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Evolution of the width of the FPZ measured experimentally \((l_{FPZ})\), internal length \((l_c)\), and the Irwin’s length \((l_{cl})\) with polystyrene content in model materials}
\end{figure}
the internal length, at least in the three point bending beam tests on notched specimen considered in Haidar et al. (2005). Furthermore, the same experimental study has shown that large variations of the fracture energy occur with the amount of polystyrene content. Such variations were also observed in fracture tests performed after accelerated leaching on mortar beams (Le Bellégo et al. 2003b). Without variations of the internal length, it was not possible to model the consequences of the leaching process on the fracture properties of aged materials, and in particular the variations of the size effect parameters observed experimentally by Le Bellégo et al. (2003b) which indicate variations of the size of the FPZ.

There is another situation of structural response where similar variations of the size effect properties have been observed. It is the case of creep–fracture interaction studied in Bažant and Gettu (1992). The authors conducted size effect tests on geometrically similar specimens loaded at different rates and observed that the slower the loading rate, the more brittle the response. Pijaudier-Cabot et al. (2005) performed a series of three point bending beam tests (notched specimens) with and without applying a sustained load for 3 months up to 85% of the maximum load prior to the fracture test. They arrived to a similar conclusion, namely that creep influences the residual capacity of the beams, but also that upon size effect tests, the specimens which were subjected to creep exhibited a more brittle response. This result yields a shift to the right of the data set on the size effect plot as shown in Fig. 2. The same phenomena has been observed by Bažant and Gettu (1992) and Bažant and Li (1997) for fracture coupled to linear creep.

In the present paper, we use a discrete random modeling based on a fiber bundle model in order to investigate the size of the FPZ. It is the so-called ZIP model developed in Delaplace et al. (2001). We consider the two examples depicted in the above paragraphs, namely the case of a material whose porosity increases and the case of creep–fracture interaction. As we will see in Sect. 3, it is quite straightforward to demonstrate that, according to the discrete model, when the microstructural size increases (in order to capture the heterogeneities in a representative volume) the FPZ length ought to increase too. The case of creep–fracture interaction needs additional mathematical considerations, but upon some simplifying assumptions, it will be shown in Sect. 4 that the FPZ length decreases as creep develops.

2 Review of the Zip model

During the past decades, fiber bundle models have received considerable attention and have been studied extensively. Originally introduced to explain ruptures in heterogeneous materials under tension (Daniels 1945), fiber bundle models have been applied to cracks and fractures, earthquakes, and other related breakdown phenomena (Andersen et al. 1997; Zapperi et al. 1997). They are directed towards the study of the material heterogeneities, i.e. at a scale much smaller than that of the representative volume of the material. These models consist of a set of parallel fibers having statistical distributed strength. The sample is loaded parallel to the fiber direction, and a fiber breaks if its elongation exceeds a threshold value. When a fiber breaks, its load is transferred to other surviving fibers in the bundle, according to a specific transfer rule. Among the possible options of load transfer are the assumption of equal load sharing (global-sharing rule) (Daniels 1945) and the much studied variants—local load-sharing rule—where the load on the failing fiber is distributed equally among the nearest surviving fibers (Harlow and Phoenix 1991). There are also a number of studies that may be placed between the two extremes that global and local load-sharing rules constitute.

![Fig. 2](image.png)  
**Fig. 2** Evolution of size effect test data for 3 point bending tests on notched specimens subjected to creep prior fracture (after Pijaudier-Cabot et al. (2005))
Among them is the study by Delaplace et al. (2001) on the so-called ZIP model. They have worked out analytically and numerically the statistics of avalanches in a system whose geometry mimics the propagation of crack front at the interface between a rigid support and an elastic beam. Other works on hierarchical fiber bundle models can be found in Newman and Phoenix (2001), Delaplace (1999) and Zhang et al. (1996). For a review of the literature in the subject, one may refer for instance to the work of Batrouni et al. (2002).

In this section, we focus on a system of elastic-perfectly brittle fibers loaded in parallel between a rigid substratum and a semi-infinite elastic beam as depicted in Fig. 3. It can be considered as a schematic model for mode I crack propagation (the rigid side of the interface representing an axis of symmetry). A normal displacement is imposed at one point of the elastic beam that can move along the interface as if a wedge was pushed between the substratum and the beam (Fig. 3). The fiber strength is randomly distributed with a uniform probability of critical fiber extension between 0 and 1. As the wedge is moving along the x-axis, the fibers are elongated and break. Fiber breakage on the interface results into three different parts due to the randomness of fiber strength. Namely, under the wedge and in a small area ahead all fibers are broken. This region constitutes the crack itself. Further ahead there is an active area with surviving and broken fibers. Moving along this zone, broken fibers become more and more scarce. In the safe area, there are no broken fibers.

In the spirit of continuum modeling (i.e. for length scales much larger than the fiber separation) and using beam theory, one can write the governing equation for the mean deflection of the beam $y(x)$. Over the infinitesimal length $dx$ there is a very large number of fibers of resulting stiffness $k$.

Upon loading, these fibers break successively and, according to the classical bundle model (Daniels 1945), the bundle response to an elongation $y(x) = ky(x)$ for a uniform distribution of critical fiber extension between 0 and 1. $(1 - y(x))$ actually represents the probability of survival of fibers. It follows that the governing equation for the beam deflection is:

$$ EI \frac{d^4 y(x)}{dx^4} = -k y(x)(1 - Y(x)), \quad (2) $$

where

$$ Y(x) = \max_{\text{loading history}} (y(x)). \quad (3) $$

The distinction between $y$ and $Y$ is made because fiber breakage is an irreversible process. Upon unloading, the stiffness of the surviving fibers over length $dx$ is recovered, and not the original stiffness that includes broken fibers. Equation 2 holds for $y < 1$, whereas $d^4 y(x)/dx^4 = 0$ for larger $y$. The boundary conditions are $y(\infty) = dy(\infty)/dx = 0$, $y(0) = 1$ and $d^2 y(0)/dx^2 = 0$ (no bending moment applied at the loading point).

There is no analytical solution to this problem. However, since the quadratic non-linear term becomes unimportant at a large distance from the origin, the asymptotic shape will have the form

$$ y'(x) = A e^{-x/\xi'} \cos \left( x/\xi' + \phi \right), \quad (4) $$

where $\xi'$ is defined in (7), $\phi$ is such that $\cos(\phi) = 1/A > 0$ in order for $y'$ to satisfy the boundary conditions, while $A$ is chosen such that

$$ \frac{dy'(x)}{dx} \bigg|_{x=0} = -\frac{1}{\xi'} \left( 1 + A \sqrt{1 - (1/A)^2} \right) \quad (5) $$

is equal to the corresponding value of the exact solution. The oscillatory component is the one that makes the deflection non-monotonous and requires the distinction between $y$ and $Y$.

Omitting this oscillatory term and the non-linearity, one arrives to a simplified model which is found to yield a correct approximation compared to full numerical calculations with a deformable beam in the zip model (Delaplace et al. 2001). In this simplified model, the displacement profile along the upper beam is imposed. The interface opening has an exponential shape: for any abscissa $x$, the profile $y$ is given by

$$ y(x) = \exp \left( \frac{U(t) - x}{\xi'} \right) \quad (6) $$

![Fig. 3 Schematic representation of “zip” model. The wedge moves along the x-axis](image)
where the length scale
\[ \xi' = \sqrt{2} (EI/k)^{1/4}, \]
is fixed \((E\) is the Young modulus of the beam, \(I\) its transverse geometrical inertia and \(k\) the stiffness of the fibers divided by the spacing in between the fibers), and \(U(t)\) is the time-dependent horizontal displacement of the wedge.

The zip model mimics the propagation of a crack in a stationary regime. The average total force acting on the beam due to the fibers is constant, but this force fluctuates due to the random distribution of the fiber strengths. As shown by Delaplace, these fluctuations contain useful information on the FPZ and may be analyzed with the help of avalanche statistics. There exists various definitions of avalanches. The one used in Delaplace et al. (2001) consists in fixing a level of force, and computing the distance \(\Delta U\) over which the crack can propagate until the retrieval of the same force level. The smallest avalanche is 1 (fiber spacing) and the largest one can be in theory infinite when computed for a level of force corresponding to the maximum of the fluctuation of the force but with an infinitely small probability.

The avalanches are characterized by their statistical distribution, \(p(\Delta U, F)\). In order to give a global characterization of the signal without considering a specific value of the force, one may consider the above avalanches for any crack length \(\Delta U\) such that a fiber would break, and then average over all breaking events. These avalanches are the “forward” avalanches.

Figure 4 shows the distribution of these avalanches \(N(\Delta)\) (number of avalanches of size \(\Delta \equiv \Delta U\)) for two values of \(\xi'\). The avalanche distributions follow two regimes with respect to a crossover value \(\Delta^*\) of the avalanche size \(\Delta\). In the first regime the distribution of avalanches is a power law \(N(\Delta) \propto \Delta^{\tau_1}\) with an exponent \(\tau_1\). In this regime the force versus crack length displays correlations similar to a random walk \((\tau_1 \simeq -1.5)\). In the second regime, the distribution of avalanches is again a power law but with a different exponent \(\tau_2\). In this regime, the forces are uncorrelated \((\tau_2 \simeq -2)\) (Fig. 4). The cross over value \(\Delta^*\) scales the extent of the FPZ. Same as in Hillerborg’s fictitious crack model (Hillerborg et al. 1976), it is the length of the FPZ, i.e. the size beyond which fiber breakage is not influenced by the crack tip.

In Fig. 5 the scaled variable \(\Delta/\xi'\) and the scaled distribution \(\xi'^{1.5} p(\Delta)\) are used to show that curves corresponding to different values of \(\xi'\) collapse onto one master curve. The size of the fracture process zone scales as \(\xi'\). More specifically the size of the FPZ increases as \(\xi'\) increases. This property has been demonstrated (Delaplace et al. 2001).
3 Zip model—case of interface with variation of porosity

Let us now consider that both the beam and the interface material are made of the same material as in the above example, but which contains voids. Hence, we look now at the fracture of a porous material. The Young modulus $E$ of the beam is a positive, monotonically decreasing function $F$ of porosity $g$: $E = E_0 F(g)$, where $E_0$ is a reference value (material without voids). Porosity $g$ here is defined as the ratio of the voids area to the total area of the beam. The influence of the porosity on the Young’s modulus is defined generically by the function $F$. One could use any homogenization technique in order to model the specific variation of the void distribution on the elastic property of the material, without any consequence on the present analysis. It is straightforward to show that the ratio $k$ of the average stiffness of the interface is the same function: $k = k_0 F(g)$, where $k_0$ is a reference value. In the fracture process zone, however, it is not the average porosity of the interface that is of importance. In the 2D material that the zip model aims at describing, fracture follows the weakest possible path. Consequently, the porosity of the interface should be less than the average interface porosity as the crack connects the pores in the material in such a way that the work of fracture is minimum. In the case of a regular array of voids for instance, the interface porosity $g_l$ along the dotted line in Fig. 6 is $g_l = n2r/R \propto \sqrt{g}$. One can generalize this remark and set that $g_l$ scales as $g^b$, where $b$ will be in principle a real number smaller than 1. One could argue that in the porous material the roughness of the crack will increase, thus increasing the apparent fracture energy. In the zip model, it would be equivalent to change the fiber spacing (or the average threshold of failure) which is not sensitive on the avalanche statistics as seen in Fig. 5.

Replacing the stiffnesses accounting for porosity into (7), one gets a modified value of the length scale $\xi^*$ in the porous material, compared to the same material without voids:

$$\xi^* = \sqrt{2 \left( \frac{E_0 F(g)I}{k_0 F(g^b)} \right)^{1/4}}.$$  \hspace{1cm} (8)

Since $g < 1$ and $F$ is a positive, monotonically decreasing function of $g$, one can conclude that $F(g)/F(g^b)$ increases with increasing $g$. It is then obvious from (8) that increasing porosity yields $\xi^* > \xi'$.

Upon a growth of porosity, the internal length scale grows. Since the size of the FPZ scales as the internal length, there is an increase of the FPZ upon a growth of porosity. So the present theoretical model is consistent with experimental observations on the dependence of the size of FPZ on porosity, as well as with numerical computations with a non-local damage model.

4 Zip model—case of viscoelastic fibers loaded between a rigid substratum and a viscoelastic beam

The model described below is a generalization to viscoelasticity of the ZIP model (Fig. 3). We consider the case of a crack propagating in a linear, non-ageing, viscoelastic material following Maxwell model.

The fibers are viscoelastic and fail suddenly at a given total elongation. Their stiffness is $k$ and their viscous coefficient $\mu$. The critical elongation at fiber breakage is a random uncorrelated variable in order to incorporate inhomogeneity into the model, same as in the elastic case. The fibers are loaded in parallel between a rigid support and a semi-infinite viscoelastic beam of Young modulus $E$, transverse geometrical inertia $I$ and viscous coefficient $M$.

The position of the fiber $i$ is denoted as $x_i$ and its deformation as $y_i$. The critical elongation of each

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Fig. 6 Schematic representation of a material as a square of edge $R$. The circles of radius $r$ in within represent $n^2$ uniformly distributed voids
fiber denoted as $y_i^{\alpha r}$ is, as mentioned above, a random uncorrelated variable having a statistical distribution in the interval $[0, 1]$. The spacing between the fibers is set to 1 (i.e. $x_{i+1} - x_i = 1$), thus defining a fixed microstructural size. A normal displacement of 1 is imposed at one point of the beam, which may move along the interface, as if a wedge is pushed in a double cantilever geometry (Fig. 3).

The fiber deformation $y_i$ is related to force $F_i$ in the $i$th fiber as:

$$\frac{\partial F_i(t)}{\partial t} + \frac{k}{\mu} F_i(t) = k \frac{\partial y_i(t)}{\partial t}$$  \hspace{1cm} \text{(Maxwell model),} \hspace{1cm} \text{(9)}$$

where $t$ denotes the time and $\partial$ denotes partial differentiation with respect to the indicated variable. It can also be shown without altering the assumptions of beam theory that the deformation of the beam $y(x,t)$ satisfies a similar equation, if again the Maxwell model is used,

$$\frac{\partial F(x,t)}{\partial t} + \frac{E}{M} F(x,t) = EI \frac{\partial^5 y(x,t)}{\partial t \partial x^4}$$  \hspace{1cm} \text{(10)}$$

where $F(x,t)$ denotes the force acting on the beam and the other parameters are already defined.

In the spirit of continuum modeling, and following the same reasoning as in Sect. 2 (i.e. for length scales much larger than the fiber separation such that over an infinitesimal length $dx$ we have a large bundle of viscoelastic fibers) the system defining the mean deflection of the beam $y(x,t)$ reads now:

$$\begin{aligned}
\frac{\partial F(x,t)}{\partial t} + \frac{E}{M} F(x,t) &= EI \frac{\partial^5 y(x,t)}{\partial t \partial x^4} \\
\frac{\partial F(x,t)}{\partial t} + \frac{k}{\mu} F(x,t) &= -k \partial \left( y(x,t)(1 - Y(x,t)) \right)/\partial t,
\end{aligned}$$

where

$$Y(x,t) = \max_{t' \leq t} \left( y(x,t') \right).$$

The above system holds for a uniform distribution of fiber extension between 0 and 1, and for $y < 1$, whereas $\partial^4 y/\partial x^4 = 0$ for larger $y$. The boundary conditions are such that

$$\forall t, \ y(\infty,t) = \partial y(\infty,t)/\partial x = 0,$$

$$\partial^2 y (U(t), t) / \partial x^2 = 0, \hspace{1cm} \text{(13a)}$$

and for $t \geq t_0, \ y(U(t), t) = 1, \hspace{1cm} \text{(13b)}$$

where $U(t)$ denotes the time-dependent horizontal displacement of the wedge along the $x$-axis, $t_0$ is the time at which the wedge starts moving, and the relation $\partial^2 y (U(t), t) / \partial x^2 = 0$ implies that no bending moment is being applied at the loading point, while the initial conditions are

$$y(x,0) = y_0(x) \text{ and } F(x,0) = F_0(x). \hspace{1cm} \text{(14)}$$

4.1 Simplified version of the model

Same as in Delaplace et al. (2001) (Eqs. 2–5), a simplified model in which the profile of the upper beam is derived, but now for the case of a viscoelastic beam connected to a viscoelastic interface. For this purpose, we omit the quadratic non-linear term in the equation governing the fiber elongation since it becomes unimportant at large distances from the origin (left hand side of the beam). The set of governing equations of the problem reads now:

$$\begin{aligned}
\partial F(x,t)/\partial t + \frac{E}{M} F(x,t) &= EI \partial^5 y(x,t)/\partial t \partial x^4 \\
\partial F(x,t)/\partial t + \frac{k}{\mu} F(x,t) &= -k \partial y(x,t)/\partial t,
\end{aligned}$$

In notation, we denote the system (15) with boundary conditions (13) and initial conditions (14) as $(\Sigma^*)$.

We eliminate $F$ in (15) in order to obtain

$$\frac{\partial}{\partial t} \left( EI \frac{\partial^5 y(x,t)}{\partial t \partial x^4} + k \frac{\partial y(x,t)}{\partial t} \right)$$

$$= -\frac{k}{\mu} EI \frac{\partial^5 y(x,t)}{\partial t \partial x^4} - \frac{E}{M} k \frac{\partial y(x,t)}{\partial t}.$$  \hspace{1cm} \text{(16)}$$

By separation of variables, i.e. by assuming that

$$\partial y(x,t)/\partial t = u(t)v(x),$$

we take from (16)

$$\frac{\partial u(t)/\partial t}{u(t)} = -\frac{k}{\mu} EI \left( \frac{\partial^4 v(x)/\partial x^4}{\partial t} \right) - \frac{E}{M} k v(x) \hspace{1cm} = \alpha,$$$$

where $\alpha$ is a positive constant. Indeed, the displacement $y(x,t)$ is a monotonically increasing function, $u(t)$ and $\partial u(t)/\partial t$ are both positives. Solving the above equation for $u$ and $v$, and then integrating (17), we find a solution of system (15)

$$y(x,t) = \frac{1}{\alpha} e^{-x/\xi} \cos(x/\xi)e^{\alpha t},$$

$$F(x,t) = \frac{k}{\alpha + \alpha} e^{-x/\xi} \cos(x/\xi)e^{\alpha t},$$

$$\begin{aligned}
\frac{\partial^2 y (U(t), t) / \partial x^2 = 0 \implies 
\partial^2 y (U(t), t) / \partial x^2 = 0 \\
\partial^2 y (U(t), t) / \partial x^2 = 0,
\end{aligned}$$

(14)
where
\[ \xi = \sqrt{2} \left( \frac{EI}{k \mu + \alpha} \right)^{1/4}. \]  

(20)

We have here the definition of a new internal length, similar to the one in the time independent zip model \( \xi' \).

By multiplying in (19) with \( \alpha \), taking into account the linearity of (15) (i.e. any linear combination of solutions is a solution itself), and omitting the oscillatory term \( \cos \left( \frac{x}{\xi} \right) \), we end up to another, approximate, solution of (15)

\[
\begin{align*}
    y(x,t) &= e^{(\alpha t t-x)/\xi} \\
    F(x,t) &= \frac{ak}{\mu+\alpha} e^{(\alpha t t-x)/\xi}.
\end{align*}
\]

(21)

Further, for initial conditions

\[
\begin{align*}
    y(x,0) &= e^{-x/\xi}, \quad F(x,0) = \frac{ak}{\mu+\alpha} e^{-x/\xi},
\end{align*}
\]

(22)

(21) is a solution of system \( (\Sigma^*) \) for \( t_0 = 0 \). It represents the problem of the wedge moving along \( x \)-axis under constant velocity \( v = \alpha \xi \). Indeed, for the deformation \( y(x,t) \) at position \( x \) to become equal to 1,

\[ y(x,t) = e^{(\alpha t t-x)/\xi} = 1, \]

time \( t = x/(\alpha \xi) \) is needed.

It should be noted here that \( \alpha \) is a parameter that characterizes the deformation rate. Although the deformation rate is not constant (\( \partial y(x,t)/\partial t = e^{(\alpha t t-x)/\xi} \)), the process can be viewed as quasi-static since for constant \( \xi \), the total force exerted on the wedge, which is the sum of all fiber contributions, remains fixed within the statistical randomness of the fiber strength.

Suppose now that the initial conditions are

\[ (y_0(x) = 0, \quad F_0(x) = 0), \]

then

\[
\begin{align*}
    y(x,t) &= \frac{1}{\alpha} e^{-x/\xi} (e^{\alpha t} - 1) \\
    F(x,t) &= \frac{k}{\mu+\alpha} e^{-x/\xi} (e^{\alpha t} - 1),
\end{align*}
\]

is one solution of system (15). For \( t_0 \), such that \( e^{\alpha t_0} - 1 = \alpha \), the above equation (25) gives

\[
\begin{align*}
    y(x,t_0) &= e^{-x/\xi}, \quad F(x,t_0) = \frac{ak}{\mu+\alpha} e^{-x/\xi}.
\end{align*}
\]

(26)

**Fig. 7** At time \( t_0 \), such that \( e^{\alpha t_0} - 1 = \alpha \), the vertical displacement of the loading point is equal to 1. The wedge now starts moving along the \( x \)-axis.

Hence, for \( t \geq t_0 \), solution (25) reads

\[
\begin{align*}
    y(x,t) &= e^{(U(t)-x)/\xi} \\
    F(x,t) &= \frac{ak}{\mu+\alpha} e^{(U(t)-x)/\xi},
\end{align*}
\]

(27)

where \( U(t) = \xi \ln \left( \frac{e^{\alpha t} - 1}{\alpha} \right) \). \( U(t) \) represents the time-horizontal displacement of the wedge for \( t \geq t_0 \) (Fig. 7). In this form it is apparent that (25) satisfies the boundary conditions (13). Thus (25) is a solution of \( (\Sigma^*) \).

Although the wedge does not move, in this case, under constant velocity, the process is still quasi-static since again the total force exerted on the wedge remains constant within the statistical randomness.

At least for two different initial conditions, namely for (22) and (24), the system \( (\Sigma^*) \) admits an asymptotic solution of the form

\[
\begin{align*}
    y(x,t) &= e^{(U(t)-x)/\xi} \\
    F(x,t) &= \frac{ak}{\mu+\alpha} e^{(U(t)-x)/\xi}, \quad (\text{see (21) and (27)}).
\end{align*}
\]

(28)

These solutions represent the motion of a wedge whose time-dependent horizontal displacement for \( t \geq t_0 \) is \( U(t) \) while before is at rest. \( \alpha \) may be seen as a parameter that scales the fracture propagating velocity. Indeed, the time-horizontal displacement of the wedge \( U(t) \) depends on \( \alpha \) (see (21) and (27)) and in turn sets the fracture propagating velocity. Figure 8 shows that the average distance between the moving wedge and the crack tip remains constant, up to the randomness of the model, throughout the total time of computations.

In order to derive the above solutions (27), we omitted the oscillatory component \( \cos \left( \frac{x}{\xi} \right) \) and made the assumption that a fiber breaks when it is strained up to its critical extension \( y^t \) which is statistically distributed in the interval \([0, 1]\). This critical extension in not time dependent. This may
Fig. 8 The evolution of the averages on the relative displacement of the crack tip, $U'(t)$, minus the relative displacement of the wedge $U(t)$ in time intervals of 10,000 broken fibers, is plotted, against time $t^*$, which is equal to the total time (which corresponds to $10^8$ broken fibers) divided by the length of the aforementioned time intervals. Since these averages remain throughout the process very close to zero, the velocity of the crack tip coincides with the velocity of the wedge.

be seen as a simple, if not simplistic, assumption and restrains the application of the present result to creep–damage interaction. From experimental evidence (see e.g. Mazzotti and Savoia 2001), this critical elongation should be a combination of the creep and elastic deformation of the fiber. Note that according to our derivation, the loading history prior failure is identical for all fibers. Hence, the elastic deformation $y_{el}^i(x,t)$ of every fiber remains throughout the process the same fraction of its total deformation $y_{tot}^i(x,t)$:

$$y_{el}^i(x,t) = \frac{ak}{k/\mu + \alpha} y_{tot}^i(x,t).$$  \hspace{1cm} (29)

Therefore, considering that the critical elongation is a linear combination of the elastic and viscous elongation would not change our results. It should be pointed out, however, that rate dependent damage is outside the scope of our approach. In rate dependent damage, it is the microcracking process that is rate dependent (whereas it is not the case here). This cannot be represented by a simple Maxwell model for the fibers and beam materials.

4.1.1 Avalanche distribution

As it is already pointed out, the model described here is the “viscoelastic” analogue of the model in Delaplace et al. (2001). For matters of convenience, we refer to the model described here as “viscoelastic” model and to the one developed in Delaplace et al. (2001) as “elastic” model. Looking now at the parameters $\xi$ (20) and $\xi'$ (7) that scale the size of the fracture process zone in the two models, respectively, we see that in the “viscoelastic” model $\xi$ depends not only on the stiffness fraction $E/\kappa$ as it is for the “elastic” model ($\xi'$) but on the relaxation times $k/\mu$ and $E/M$ as well, and on the parameter $\alpha$ which is related to the rate of deformation. Further, when both the interface and the rest of the material are elastic in the “limit” ($\mu, M \to +\infty$), the “elastic” case can be reproduced, $\xi \to \xi'$. This is also the case ($\xi \to \xi'$) when the rate of deformation is too large to allow for creep, i.e. when $\alpha \to +\infty$ or when the interaction between the interface and the rest of the material does not depend on their relaxation times, i.e. when $(k/\mu = E/M)$. $\xi$ can be either greater or less than $\xi'$, with the two end members of the spectrum to be the case of an elastic beam and a viscoelastic interface and the case of a viscoelastic beam and an elastic interface (Fig. 9).

The case of a beam with relaxation time $E/M$ greater than that $k/\mu$ of the interface corresponds to the observed decrease of the size of the fracture...
process zone in creep experiments compared to the size of the FPZ in a rate independent process.

4.2 Model for the complete problem

We now come back to the complete problem (Σ), i.e. system (11) with boundary conditions (13) and initial conditions (14). Our aim is rather to compute the approximate average shape of the deformed beam. One can see that in system (Σ) for the damaged interface, the quadratic nonlinear term \( \partial(\phi Y)/\partial t \), which is additional compared to system (Σ\(^*\)) for the undamaged interface, becomes unimportant far away from the boundary \( x = 0 \). Thus the asymptotic shape can have a similar expression as the one used in the previous section, that is

\[
\begin{align*}
\frac{\partial y'(x,t)}{\partial x} \bigg|_{x=0} &= -\frac{1}{\xi} \left( 1 + A(t) \sqrt{1 - (1/A(t))^2} \right) e^{\alpha t}, \quad (31) \\
y'(0,t) &= e^{\alpha t}, \quad (30) \\
y'(x,t) &= A(t) e^{-x/\xi} \cos \left( -\frac{x}{\xi} + \phi(t) \right) e^{\alpha t}, \\
F'(x,t) &= \frac{ak}{\xi + \alpha} A(t) e^{-x/\xi} \cos \left( -\frac{x}{\xi} + \phi(t) \right) e^{\alpha t}, \quad (32)
\end{align*}
\]

where \( \phi \) is such that \( \cos(\phi) = 1/A > 0 \) in order for the solution \( y' \) to satisfy the boundary condition \( y'(0,t) = e^{\alpha t} \), while \( A \) is chosen such that

\[
(31) \quad \frac{\partial y'(x,t)}{\partial x} \bigg|_{x=0} = -\frac{1}{\xi} \left( 1 + A(t) \sqrt{1 - (1/A(t))^2} \right) e^{\alpha t},
\]

is equal to the corresponding values of the exact solution of system (Σ). Omitting the oscillatory component one retrieves

\[
\begin{align*}
y'(x,t) &= A(t) e^{(\alpha \xi - x)/\xi} \\
F'(x,t) &= \frac{ak}{\xi + \alpha} A(t) e^{(\alpha \xi - x)/\xi} \quad (32)
\end{align*}
\]

For initial conditions

\[
\begin{align*}
y(x,0) &= A(0) e^{-x/\xi}, \\
F(x,0) &= \frac{ak}{\xi + \alpha} A(0) e^{-x/\xi}, \quad (33)
\end{align*}
\]

expression (32) solves the problem of a wedge moving along x-axis with a time horizontal displacement of \( U^*(t) = \xi(\alpha t + \ln A(t)) \). This asymptotic solution has the same statistical properties with the simplified version of the model (28) derived in the previous section. Indeed, as it was pointed out in Delaplace et al. (2001), the avalanche distribution of (32) depends solely on \( \xi \) and it is independent of \( U^* \). One may also come to the same conclusions for the case of initial conditions (24).

5 Conclusions

A model describing the crack propagation at the interface between a rigid substratum and a beam has been considered. The interface is modeled using a fiber bundle model (i.e. using a discrete set elements having a random strength). The distribution of avalanches, defined as the distance over which the crack is propagated under a fixed force, has been studied in order to capture correlations of breaking events in the course of fracture. Fiber breakage kinetics is related to a correlation length, which sets the size of the fracture process zone that occurs ahead of the crack due to progressive failure.

Ageing is considered as a variation of porosity of the interface. It corresponds for instance to diffusion controlled dissolution processes in cementitious materials (e.g. calcium leaching). Results obtained in Delaplace et al. (2001) show that the size of the fracture process zone is proportional to a length scale. This length scale increases (so does the size of the FPZ) upon increasing porosity and the discrete model is consistent with experiment data and numerical analysis.

The creep–fracture interaction has been analyzed in the second part of the paper. A simplified model is proposed, which is very similar to the simplified time independent zip model described in the first part, but with a length scale that depends now on the time-dependent characteristics of viscoelastic fibers and viscoelastic beam. It is found that the size of the process zone depends on the fracture propagating velocity and on the distribution of forces in the interface due to the interaction between the interface and the rest of the specimen. The observed decrease of the size of the process zone in creep experiments, compared to the size of the process zone in a time-independent process, is found when the interface is less viscous than the averaged bulk material.

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**References**


