Adiabatic shearing of non-homogeneous thermoviscoplastic materials

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Abstract

In this work we present the role of material non-homogeneities, driven by position-dependent thermomechanical parameters on the emergence, evolution and localization of temperature and strain non-uniformities during the shearing of thermoviscoplastic materials of the rate type. We first present the solution of the quasistatic approximation and show the existence of two stages: the first is characterized by a possible “travelling” of non-uniformities, due to non-homogeneities, while in the second the non-uniformities localize and increase at regions with intense heterogeneities. Moreover, we show that, even under stability conditions, the non-uniformities increase at material regions with inhomogeneities, which affect both their spatial distribution and time evolution. The related strain non-uniformities in the form of shear banding, as well as the comparison with the non-uniformities caused by initial defects or temperature gradient, are presented. Finally, numerical results confirm the analytical findings. © 2003 Elsevier Ltd. All rights reserved.

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1. Introduction and main results

Temperature and/or strain non-uniformities tend to destabilize the shearing of metals under high strain rates. Usually, we consider the process as adiabatic, since heat loss during deformation is small, relative to heat generation. Experimental and numerical investigations suggest that instability occurs in the form of strain localization and formation of shear bands. A strain non-uniformity, initially existing in a zone or initiated by high strain rate, leads to a transverse gradient in the heat generated in...
this zone by the plastic work. This thermal gradient furthers sharpens the strain gradient across the shearing zone, causing, in turn, a steeper temperature gradient thanks to the adiabaticity conditions, and greater thermal softening effects in the center of the zone (Rogers, 1979). During the last years, considerable progress has been made in understanding the physics and mathematics of strain localization. The two dimensional solutions of propagating shear band of Wright and Walter (1996), Chen and Batra (1999), the two dimensional experiments of Kalthoff (2000) and Xue et al. (2001) and the two- and three-dimensional computational studies of Batra and Gummalla (2000) and Batra and Ravinsankar (2000), the three dimensional formulation of Roessig and Mason (1999), the experimental measurement and the theoretical explanation of the strain, strain-rate and temperature dependent plastic work converted into heating (Hodowany et al., 2000; Rosakis et al., 2000), the thermodynamic potential in continuum thermoplasticity (Scheidler and Wright, 2001), all these selected parts of the literature prove the considerable progress in both theoretical, numerical and experimental aspects concerning the formation, characterization and properties of adiabatic shear bands (for an excellent monograph, see the book by Wright, 2002). Within the realm of adiabatic shearing, the main emphasis is usually placed on determining the validity of high-speed modeling by experimental analysis (Gilat and Cheng, 2002; Molinari et al., 2002) or analysis of the deformation process (Burns and Davies, 2002). Moreover, unified theories of rate-dependent deformation in rheological models are discussed (Lubarda et al., 2002, Krempel and Khan, 2003). The mathematical characterization of shear banding is related to the existence theory and large time behavior of solutions of the system of nonlinear partial differential equations describing the shearing, introduced by Tzavaras (1992). The instability criterion was first published by Molinari and Clifton (1983) and subsequently developed in Molinari and Clifton (1987). Simple stability and instability conditions as relationships between the thermomechanical parameters $m$ (strain hardening), $n$ (strain-rate sensitivity) and $\alpha$ (thermal softening), completed the mathematical model of softened materials. It was shown that the maximum time-interval, for which classical solutions exist, is infinite if $p = m + n - \alpha > 0$ or finite, if $p < 0$. However, the time behavior at earlier times is necessary in order to capture possible transient aspects of shear band formation (Walter, 1992), although Walter notices that analysis at long times do bear on what occurs at earlier times. In the above literature, the materials are supposed homogeneous and the strain concentration results from geometrical defects, initial temperature gradient or external loading. In recent papers, much effort has been made to establish stability and instability criteria, as well as to analyze the dynamic response of non-homogeneous materials. Material non-homogeneities occur, for instance, as a result of incomplete production, or phase transition, or the specific material structure (composite materials or materials with welded seams). Composite materials such as functionally graded materials have continuously varying properties. Ozturk and Erdogan (1993, 1995), Wu et al. (1999), Wang et al. (2000) among others have studied the dynamic behavior of such materials containing interfacial imperfections (cracks) in the general three dimensional formulation. Other classes of materials such as laminates possess non-homogeneity of discontinuous nature. Theoretical
and numerical issues concerning the application of effective properties in the analysis of heterogeneous materials have been also examined from the micromechanical point of view (Needleman, 1992; Dolbow and Nadeau, 2002; Aboudi et al., 2003; Majta and Zurek, 2003). Recently, Naboulsi and Palazotto (2001, 2003) proposed a damage model for metal–matrix composite.

In Baxevanis and Charalambakis (2002), it was shown that the same, with the homogeneous case (Tzavaras, 1992), stability conditions suffice to ensure the large time behavior of strain. Moreover the isothermal case was treated.

In this paper, we study the anisothermal shearing. We first consider the quasistatic approximation and study the behavior of temperature and strain for both intermediate and long times. We show that, at the early stages of the process, non-uniformities may “travel” from one material zone to another, driven by the thermomechanical non-homogeneities. After a “small” time $t_0$, non-uniformities localize and persist until the end of the process. A second conclusion is that, even though stability conditions are valid, the temperature and strain non-uniformities never tend towards zero at long time. This is due primarily (and, in the case of strong hardening, exclusively) to the heterogeneities, expressed by the function

$$\varphi(x) = \frac{[m(x) + n(x) - \alpha(x)]}{n(x)} = \frac{p(x)}{n(x)}$$

We show that $\varphi(x)$ and its gradient control the time evolution and the spatial distribution of non-uniformities. We notice that linearized analysis techniques for composites (Georgievskii, 1997, 2000) do not take into account the evolution of the strain gradient.

Next, we present the numerical solution of the dynamic problem and compare with the above findings. Moreover we compare numerically the dynamic and the quasi-static problem.

2. Analysis and numerical simulation

The simple shearing of a plate made by a non-homogeneous thermoviscoelastic material is described by the following system of nonlinear partial differential equations in terms of the velocity $v(x,t)$, the strain $\gamma(x,t)$ and the temperature $\theta(x,t)$ (subscript indicates differentiation with respect to the variable indicated),

$$\rho(x)v_t(x,t) = \sigma_x(x,t),$$

$$c(x)\rho(x)\theta_t(x,t) = \beta(x)\sigma(x,t)\gamma_t(x,t),$$

$$v_x(x,t) = \gamma_t(x,t),$$

where the stress $\sigma(x,t)$ is given by the empirical law

$$\sigma(x,t) = G(x)\theta(x,t)^{-\alpha(x)}\gamma(x,t)^m(x)\gamma_t(x,t)^n(x),$$

$$\alpha(x) < 1, \ m(x) < 1, \ n(x) < 1.$$
Eq. (2) is the equation of motion, (3) is the balance of energy and (4) is the geometrical compatibility condition. In (2)–(4), $\rho(x)$, $c(x)$ and $\beta(x)$ denote the density, specific heat and portion of plastic work converted into heating, respectively, while in (5), $G(x)$ is a material function and $\alpha(x)$, $m(x)$ and $n(x)$ are the smooth approximations of thermal softening, strain hardening and strain-rate sensitivity, respectively. We assume that the shearing is caused by steady shear forces at the boundaries,

$$\sigma(0, t) = \sigma(h, t) = S,$$

where $h$ is the thickness of the plate. Finally, we assume the following initial conditions for temperature and strain

$$\theta(x, 0) = \theta_0(x) > 0, \quad \gamma(x, 0) = \gamma_0(x) > 0, \quad 0 \leq x \leq h,$$

which are compatible with the boundary conditions. It is possible to show by using maximum principles, that the positiveness of initial data ensures the positiveness of temperature and strain rate. This is necessary since (5) is meaningful only if $\gamma_1 > 0$, $\gamma > 0$, $\theta > 0$ and $\sigma > 0$. Finally, we need the following technical assumption,

$$\theta_0(x) \geq \gamma_0(x)\beta(x)S/(c(x)\rho(x)), \quad 0 \leq x \leq h,$$

which will be explained later. In any case, this inequality is completely justified by the physical values of all functions in (8).

We consider the quasi-static approximation of the above problem,

$$\sigma_x(x, t) = 0,$$

from which, by using (6),

$$\sigma(x, t) = S.$$

Combining (3) and (10), integrating over time and using (7)

$$\gamma(x, t) = \gamma_0(x) + A(x)[\theta(x, t) - \theta_0(x)],$$

where

$$A(x) = c(x)\rho(x)/(S\beta(x))(> 0).$$

Eq. (11) shows that strain and temperature evolve simultaneously. Substituting $\gamma(x, t)$ and $\theta(x, t)$ from (11) into the stress, we deduce respectively

$$S^{1/n(x)} = A(x)G^{1/n(x)}$$

$$\times \left\{ \gamma_0(x) + A(x)[\theta(x, t) - \theta_0(x)] \right\}^{m(x)/n(x)} \theta(x, t)^{-\alpha(x)/n(x)} \theta_0(x, t),$$

and

$$S^{1/n(x)} = G^{1/n(x)}$$

$$\times \left\{ \theta_0(x) - [\gamma_0(x)/A(x)] + \gamma(x, t)/A(x) \right\}^{-\alpha(x)/n(x)} \gamma(x, t)^{m(x)/n(x)}$$

$$\times \gamma_t(x, t).$$
Eqs. (18) and (19) admit analytical solutions under special assumptions on the exponents $m(x)/n(x)$. We have, from Fig. 3, that

$$C_{\text{fig.3}}(x) = \frac{\gamma_0(x)}{\lambda(x) - \theta_0(x)} \leq 0, \text{ due to (8) and (13)}.$$

We write (13) and (14) in the form of two evolution equations, for every $0 \leq x \leq h$,

$$\begin{align*}
\theta(x, t) + \Delta(x) \gamma_{\text{fig.3}}(x, t) - \phi(x, t) = \Gamma(x), \\
\gamma(x, t) - \phi_0(x) + A(x) \theta_0(x) = \gamma_{\text{fig.3}}(x, t) = A(x)^{\phi(x)/n(x)} S/G^{1/n(x)}.
\end{align*}$$

Eqs. (18) and (19) admit analytical solutions under special assumptions on the exponents $m(x)/n(x)$ and $\alpha(x)/n(x)$. We first consider the case of instability, $\rho(x) < 0$, $\forall x$. As an example, we take $\alpha(x)/n(x) = 3$, $m(x) = n(x)$, $\forall x$, and find that the solution of (19) is given by [see (A5) and (A4) in Appendix A]

$$\begin{align*}
\gamma(x, t) = \gamma_0(x) - A(x) \theta_0(x) \\
+ \left[ A(x) \theta_0(x) - \gamma_0(x) \right] - \left[ 1 - 1 - 2C(x) + 2t/D(x) \right]^{1/2},
\end{align*}$$

where $C(x)$ is a function of initial non-uniformities,

$$C(x) = 1 - \gamma_0(x)/(A(x) \theta_0(x)) - (1/2)[1 - \gamma_0(x)/(A(x) \theta_0(x))]^2$$

and $D(x)$ a non-homogeneity function

$$D(x) = A(x)^{\alpha(x)/n(x)} (G(x)/S)^{1/n(x)} \left[ A(x) \theta_0(x) - \gamma_0(x) \right]^{\rho(x)/n(x)}.$$

The time at which the strain “blows-up” is given by $t_\infty = \min_{x \in [0, h]} [C(x) D(x)]$, larger than the critical time of the process, since only a finite value of the strain is needed to create shear band. For instance, for a material zone with $\rho = 7800 \text{ kg/m}^3$, $\beta = 0.9$, $c = 500 \text{ J/(kg k)}$, $G = 436 \text{ Mpa}$, $\gamma_0 = 0.01$, $\theta_0 = 27 \text{ °C}$ and $\gamma_0 = 10^3$, we have $t_\infty = 4.666 \text{ ms}$, while the time needed to obtain a critical strain 0.5 is only $t_\infty = 1.7 \text{ ms}$ (see Fig. 1). However, as we will see in the sequel, the critical time is even smaller, due to the fact that the material zones exhibit high values of strain gradient, as a result of the non-homogeneities.

In Fig. 2, we see the time-evolution of the strain at two material points $x_1$ and $x_2$. The strain at the point $x_1$ is equal to $\gamma_1$ and the strain at the point $x_2$ is equal to $\gamma_2$. In Fig. 3, we see the “small” time-evolution of the same quantities. In Fig. 4a we see the strain distribution at time $t_1$. We have, from Fig. 3, that $\gamma_1 > \gamma_2$. In Fig. 4b, we see the strain distribution at time $t_2$, after the intersection of the two curves at $t_0$ (Fig. 3). We have, from Fig. 3, that $\gamma_1 < \gamma_2$. Thus, the non-uniformities move from $x_1$ to $x_2$. Moreover, we observe in Fig. 2 that non-uniformities localize after $t_0$ and increase until collapse. We next turn to the time evolution of non-uniformities, expressed by the strain gradient (A6) (see Appendix A). By ordering the different functions in accord with their rates, we obtain
The thermomechanical non-homogeneities participate with the larger rate in the evolution of the strain gradient, followed by the influence of initial non-uniformities.

Next, we consider the case of stability $p(x) > 0$. We differentiate (18) with respect to $x$ and use again (18) to obtain

$$
\gamma_2(x, t) \sim [\varphi_2(x)]O\left(t^{1/2}/(1 - t^{1/2})\right) + [\gamma_{02}(x) + \theta_{02}(x)]O\left(1/(1 - t^{1/2})\right).
$$

The thermomechanical non-homogeneities participate with the larger rate in the evolution of the strain gradient, followed by the influence of initial non-uniformities.

Fig. 1. Solution of the quasistatic problem.

Fig. 2. “Large” time behavior of the strain at two material zones. The non-uniformities localize and increase.
\begin{equation}
\theta_{tx}(x, t) + \left\{ \left[ \frac{m(x)}{n(x)} \right] \theta_t(x, t) / [\theta(x, t) + \Delta(x)] \right\} + \\
\left[ \alpha(x) / n(x) \right] \ln [\theta(x, t) + \Delta(x)] \theta_t(x, t) - [\alpha(x) / n(x)] \ln \theta(x, t) \theta_t(x, t) \\
+ \left[ \frac{m(x)}{n(x)} \right] \Delta x \theta_t(x, t) / [\theta(x, t) + \Delta(x)] - \left[ \Gamma_s(x) / \Gamma(x) \right] \theta_t(x, t) = 0.
\end{equation}

Fig. 3. “Small” time behavior. Time-evolution of the strain at two material zones.

Fig. 4. Strain distribution at two different “small” times. The non-uniformities travel to the right.
For each \( x, 0 \leq x \leq h \), (24) is a non-linear ordinary differential equation with respect to \( t, 0 \leq t < \infty \). In Appendix C we prove that the time-behavior of the temperature gradient is given by

\[
\theta_x(x, t) \sim \left[ \phi_x(x) \right] O(t^{2/\phi(x)}) + \left[ \Gamma_x(x) \right] O(t^{1/\phi(x)}) + \left[ \theta_0(x) \right] O(t\alpha(x)) + [\Delta_x(x)] O(t^0).
\]  

(25)

In (25), only the non-homogeneities (for instance, \( \phi_x(x) \)), and the initial non-uniformities \( \theta_0(x) \), are presented, together with their order of time rate (for instance, \( O(t^{2/\phi(x)}) \)).

Using (11), (II.9) and (25), we find a similar expression for the strain,

\[
\gamma_x(x, t) \sim A(x) \theta_x(x, t) + A_x(x) t^{1/\phi(x)} + [\Delta_x(x)] O(t^0).
\]  

(26)

We see in (25) and (26) that the non-uniformities are a combination of the time-evolution of the following functions: the three non-homogeneity functions \( \phi_x(x), \Gamma_x(x), \Delta_x(x) \) and the initial non-uniformities \( \theta_0(x), \gamma_0(x) \), which act with different rates. The fist function reflects the non-homogeneity of the thermomechanical parameters (strain hardening, strain-rate sensitivity and thermal softening). This non-homogeneity causes the “fastest” amplification of non-uniformities. Additionally, the thermomechanical parameters enter into all rates of convergence. Regarding \( \Gamma_x(x) \) and \( \Delta_x(x) \), they reflect the non-homogeneity of the density, the specific heat, the portion of plastic work converted into heat, and the shear modulus. We verify that their contribution to the evolution of non-uniformities becomes less important as the time increases. Finally, the contribution of the initial non-uniformity, due to initial temperature gradients \( \theta_0(x) \) or initial geometrical defects \( \gamma_0(x) \), is also less important and may be negligible for \( \phi(x) < 1 \). Concerning the interplay between non-homogeneity and non-uniformity, we note that, in the above case \( [\phi(x) < 1, \alpha(x) > 0] \), the absence of non-homogeneities \( [\Gamma_x(x) = \Delta_x(x) = 0] \) prevents the non-uniformities from growing-up and forming shear bands.

We close this analysis by comparing the non-uniformities with the corresponding temperature or strain: Using \((B8), (B9), (C8)\) and \((26)\), we conclude that,

\[
\theta_x(x, t) \sim K \theta(x, t)^2, \quad \gamma_x(x, t) \sim K \gamma(x, t)^2.
\]  

(27)

This comparison shows that the non-uniformity of temperature or strain evolve much faster than the temperature or strain respectively. This is a qualitative description of shear banding, which is developed under “stability” conditions! Similar conclusions, but for instability conditions, permitted to Fressengeas and Molinari (1987) to elaborate more “engineering” stability criteria, by controlling the evolution of the relative perturbation in the deformation process of homogeneous materials.

We now turn to the dynamic shearing, in which the stress is not uniform in \( x \) and its gradient makes more difficult the analytical study of the non-uniformities, in the above context of nonlinear analysis. We continue this investigation by the numerical
simulation of the dynamical problem, using the commercial program PDECOL
(Madsen and Sincovec, 1979). We first compare the dynamical problem (2)–(7) with the
quasi-static approximation (18), under the same initial conditions for temperature
\( \theta_0(x) \) and strain \( \gamma_0(x) \) and the additional assumption
\[
\sigma_0(x) = S
\]
in the dynamical problem. This condition, together with (4) and (5), gives the initial
temperature \( \theta_0(x) \) as a function of the initial strain \( \gamma_0(x) \) and the initial strain
rate \( \gamma_0(x) \). This is necessary in order to compare the spatial distribution and time-
evolution of two similar situations, starting from the same temperature and deformation
distribution and the same uniform stress. We first study a slab of thickness
\( h = 2.5 \text{ mm} \), with thermomechanical parameters \( m(x), n(x), \alpha(x) \) satisfying the stability condition
\( p(x) > 0, \forall x \in [0, h] \). These parameters oscillate between the values
\( \alpha(x): 0.017 \text{ and } 0.03, m(x): 0.011 \text{ and } 0.01, n(x): 0.01446 \text{ and } 0.02 \). The oscillation is
achieved using appropriately the function \( x - \sin(\pi x) \). These parameters give a meaningful fluctuation to the initial temperature for uniform initial strain and strain rate.
All the other material parameters are selected on the basis of hot-rolled steel (Clifton
et al., 1984). We note that the initial strain-rate is uniform and equal to \( 10^3 \text{ s}^{-1} \). In
Fig. 5 we see the temperature patterns corresponding to different values of time, for
the dynamical and the quasi-static problem. We first conclude that the quasi-static
approximation is very satisfactory. Moreover, we see that, even under stability
conditions, the non-uniformity increases continuously with time. In Fig. 6 we see the
time evolution of temperature non-uniformity, measured by its gradient, for both
the dynamical and the quasi-static problem. The maximum value of the ratio of the absolute value of their difference over the smallest of them at 0.03 ms is approximately 0.047. The gradient \( \phi_s(x) \) is responsible for the numerical “blow up” of the

Fig. 5. Temperature distribution for various values of time. Case of stability with weak hardening
\( [m(x) < \alpha(x)] \).
temperature gradient in such a short time interval \((t = 0.03 \text{ ms})\). The above case is characterized by weak hardening \([m(x) < \alpha(x), \ 0 \leq x \leq 1]\). We next consider a material with thermomechanical parameters oscillating between \(\alpha(x): 0.017 \text{ and } 0.03, \ m(x): 0.03 \text{ and } 0.031, \ n(x): 0.01446 \text{ and } 0.02\). Now \(m(x) > \alpha(x) (\phi(x) > 1), \ 0 \leq x \leq 1\). In Fig. 7 we see again the comparison between the two approaches. We note that the evolution of the temperature non-uniformity is now convex with respect to \(t\)-axis, as it is predicted by (25). Moreover, we see that the process takes 0.07 ms longer, for the strain to take the same, with the previous case, value at the boundary, since the non
homogeneity function $\phi(x)$ inherent in (25) is now greater than 1. Finally, we consider the dynamical case of instability $p < 0$. In Fig. 8 we see the time evolution of strain patterns, for a plate, made by hot-rolled steel with parameters used in Clifton et al. (1984), reinforced by two thin slabs. The thermomechanical parameters are respectively $\alpha_p = 0.38$, $m_p = 0.015$, $n_p = 0.019$, $\alpha_s = 0.375$, $m_s = 0.0151$, $n_s = 0.0191$. We note that the fluctuation of the above thermomechanical parameters is relatively small compared with those used in the case of stability. The process exhibits numerical instability due to the strain gradient at the interfaces, which “blows-up”

![Fig. 8. Reinforced plate. Strain distribution at different values of time.](image1)

![Fig. 9. Comparison of critical times between the homogeneous and the non-homogeneous material.](image2)
at \( t_c \approx 0.155 \) ms, for a boundary strain \( \gamma_B \approx 0.166 \) and a maximum strain \( \gamma_{\text{max}} \approx 0.265 \) only, overcoming the action of boundary forces and geometrical defects. The duration of the process is almost 9.4% less of the corresponding of an homogeneous slab made entirely by the same hot-rolled steel, starting from the same initial non-uniformity with the inhomogeneous case (Fig. 9). This could explain the fact that analytical methods, using constitutive laws that do not take into account the phase transformation, predict smaller values for the critical time than the ones observed in experiments.

3. Conclusions

The evolution of deformation non-uniformities are highly influenced by the presence of material non-homogeneities, especially in strain hardening \( [m(x)] \), strain-rate sensitivity \( [n(x)] \) and thermal softening \( [\alpha(x)] \) parameters. More specifically, we conclude that the role of the non-homogeneity function \( [m(x) + n(x) - \alpha(x)]/n(x) \), in both the spatial distribution and the time evolution of deformation patterns, is crucial, even under stability conditions. Under instability conditions, at the early stages of the process the non-uniformities move from one material zone to another, driven by the non-homogeneities. After a “small” time, they localize and increase continuously. Under stability conditions, in material zones with large strain-rate sensitivity and/or small difference between strain hardening and thermal softening, the non-uniformities due to non-homogeneities evolve faster. Moreover, we see that, in the case of important hardening \( [m(x) > \alpha(x)] \), the only possible cause of shear banding is the non-homogeneous nature of the material. It seems that this “weak” stability regime needs to be characterized by new non-uniformity criteria, based on the control of the temperature or strain gradient.

Appendix A. Case of instability

Eq. (19) gives after integration over time

\[
\left[(G/S)^{1/n} / A^{-\alpha/n}\right][A\theta_0 - \gamma_0]^{p/n} \int z^{m/n}(z + 1)^{-\alpha/n} \, dz = t, \tag{A1}
\]

where

\[
z = \gamma/(A\theta_0 - \gamma_0). \tag{A2}
\]

Putting \( \alpha/n = 3 \) and \( m = n \), we obtain, after integration

\[
D\left[1/(2\omega^2) - 1/\omega + C\right] = t, \tag{A3}
\]

where

\[
\omega = \gamma/(A\theta_0 - \gamma_0) + 1, \tag{A4}
\]
and $C$, $D$ are given by (21) and (22). Since $\omega > 1$, the solution of (A3) gives
\[
\omega(x, t) = 1/[1 - [1 - 2C(x) + 2t/D(x)]]
\]
which is equivalent to (20). Differentiating (20) with respect to $x$ we deduce
\[
\gamma_x = (\gamma_0x - A\theta_0x) - \theta_0A_x + (A\theta_0 - \gamma_0)x/[1 - (1 - C + 2t/D)^{1/2}] +
(\gamma_0x - \gamma_0)\left[ (C_x + D_xt/D^2) \right]/(1 - 2C + 2t/D)^{1/2}[1 - (1 - C + 2t/D)^{1/2}].
\]

Appendix B. Case of stability

Eq. (6) and inequality (8) give a lower bound for the strain. Using (6) and (8),
\[
[S/(G(x)A(x)\gamma(x))]^{1/[n(x)]} \leq \gamma(x, t)^{(m(x) - \alpha(x))/n(x)}\gamma_t(x, t),
\]
from which, by integration over $(0, t)$ and recalling that $p(x) = m(x) + n(x) - \alpha(x) > 0$,
\[
\gamma(x, t) \geq K t^{p(x)/p(x)} = K t^{1/\gamma(x)},
\]
where $K$ is a constant which can be estimated in terms of $S$, $G(x)$, $A(x)$, $\gamma_0(x)$, $\theta_0(x)$ and $n(x)$. We will use the same symbol $K$ for every constant, which can be estimated from the data of the problem. Using (11) and (B2),
\[
\theta(x, t) \geq K t^{1/\gamma(x)}.
\]

Recalling (11), and using that, on account of $\gamma_t(x, t) > 0$, we have $\gamma(x, t) > \gamma_0(x)$,
\[
\gamma(x, t) + [A(x)\theta_0(x) - \gamma_0(x)]\gamma_t(x, t)/\gamma_0(x) \geq A(x)\theta(x, t),
\]
or
\[
\theta(x, t) \leq (1/K)\gamma(x, t).
\]

Using (4) and (B5),
\[
\gamma(x, t)^{(m(x)/n(x)}\gamma_t(x, t) \leq (G(x)^{1/[n(x)]}S^{1/[n(x)]/K})\gamma(x, t)^{\alpha(x)/n(x)},
\]
from which, by integration over time,
\[
\gamma(x, t) \leq K t^{1/\gamma(x)}.
\]

Therefore, combining with (B2) and (11),
\[
(1/K)^{1/\gamma(x)} \leq \gamma(x, t) \leq K t^{1/\gamma(x)},
\]
\[
(1/K)^{1/\gamma(x)} \leq \theta(x, t) \leq K t^{1/\gamma(x)}.
\]
Moreover, using (4), (B8), (B9) and (3),

\[
\frac{1}{K} I_{1-\varphi(x)/\psi(x)}(t) \leq \theta_1(x, t) \leq K I_{1-\varphi(x)/\psi(x)},
\]

(B10)

\[
\frac{1}{K} I_{1-\varphi(x)/\psi(x)}(t) \leq \gamma_1(x, t) \leq K I_{1-\varphi(x)/\psi(x)}.
\]

(B11)

**Appendix C**

We write (24) in the form

\[
dy(., t)/dt + P(., t)y(., t) + Q(., t) = 0.
\]

(C1)

where

\[
y(., t) = \theta_x(., t),
\]

(C2)

\[
P(., t) = \left\{ \left[ \frac{m(x)}{n(x)} \right] \theta(., t) + \Delta(.) \theta(., t) - \left[ \frac{\alpha(.)}{n(.)} \right] \theta(., t) / \theta(., t) \right\},
\]

(C3)

and

\[
Q(., t) = \left\{ \left[ \frac{m(.)}{n(.)} \right] \ln\left[ \theta(., t) + \Delta(.) \theta(., t) - \left[ \frac{\alpha(.)}{n(.)} \right] \theta(., t) / \theta(., t) \right] + \left[ \frac{m(.)}{n(.)} \right] \Delta_x(.) \theta(., t) / \theta(., t) + \Delta(.) - \left[ \frac{\Gamma(.)}{\Gamma(.)} \right] \theta(., t) \right\}.
\]

(C4)

The solution of (C1) is represented in the form

\[
y(., t) = e^{-\int P(., t)dt} \left\{ -\int Q(., t) e^{\int P(., t)dt} dt + C \right\}.
\]

(C5)

Using (B8)–(B11), we obtain the following time behavior of \( P(., t) \) and \( Q(., t) \):

\[
P(., t) \sim \left\{ \left[ \frac{m(.) - \alpha(.)}{n(.)} \right] \right\}^{-1},
\]

(C6)

\[
Q(., t) \sim \left\{ \left[ \frac{\varphi(.)}{\psi(.)} \right] t^{-1} \Delta_x(.) \left[ \frac{\Gamma(.)}{\Gamma(.)} \right] I_{1-\varphi(.)/\psi(.)} + \left[ \frac{m(.)}{n(.)} \right] \Delta_x(.) t^{-1},
\]

(C7)

from which, returning to (C5) and (C2), we conclude that

\[
\theta_x(x, t) \sim \left\{ \left[ \frac{-\varphi(x)}{\psi(x)} \right] t^{2/\psi(x)} - \left[ \frac{\Gamma_x(x)}{\Gamma(x)} \right] I_{1-\varphi(x)}^1 \right\} + \left[ \frac{\theta_0(x)}{t^{1-\varphi(x)}} + \left[ \frac{m(.)}{n(.)} \right] \Delta_x(.) \right\}.
\]

(C8)

By presenting only the non-homogeneity functions we obtain (25).

**References**


Baxevanis, Th., Charalambakis, N., 2002. The role of material non-homogeneities on the formation


