A Limiting Viscosity Approach for the Riemann Problem in Isentropic Gas Dynamics

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ABSTRACT. We consider the Riemann problem for the equations of isentropic gas dynamics in Eulerian coordinates. We construct solutions of this problem as limits of solutions of a "viscosity" regularized problem that is rigged so as to preserve the invariance of the original problem under dilatations of the independent variables. The solutions thus constructed may contain vacuum regions. Using the same approach, we also construct solutions of the Riemann problem in case the data contain a vacuum state.

1. Introduction. The equations describing one dimensional isentropic motions of a compressible, inviscid gas, written in spatial (Eulerian) coordinates and in conservative form, are

$$\rho_t + (\rho u)_x = 0 \tag{1.1}$$

$$(\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x = 0 \tag{1.2}$$

$$-\infty < x < \infty\,,\ t > 0$$

Here, $\rho = \rho(x,t)$ and $u = u(x,t)$ stand for the density and the velocity of the gas, respectively, while $m := \rho u$ describes the momentum flux in the direction of the flow. The density ρ is nonnegative; the regions in the physical space where $\rho = 0$ are identified with vacuum regions of the flow. The pressure p is assumed to be a smooth function of ρ satisfying

$$p'(\rho) > 0 \text{ for } \rho > 0\,. \tag{1.3}$$

Under (1.3), Equations (1.1) and (1.2) form a hyperbolic system of conservation laws. The characteristic speeds $\lambda_{1,2} = u \mp \sqrt{p'(\rho)}$ are distinct when $\rho > 0$, but may coalesce when $\rho = 0$. The problem of solving (1.1)–(1.2) subject to initial data

$$\left(\rho(x,0),\ u(x,0)\right) = \begin{cases} (\rho_-, u_-), & x < 0 \\ (\rho_+, u_+), & x > 0 \end{cases} \tag{1.4}$$

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is called the Riemann problem for (1.1), (1.2).

The classical approach for the solution of the Riemann problem relies on introducing an appropriate criterion that singles out the admissible shocks. Then the solution is effected by constructing admissible wave curves and corresponding admissible wave fans. A comprehensive admissibility criterion, motivated by the study of several concrete systems, such as (1.1)–(1.2), has been proposed by Liu and the Riemann problem was solved for general classes of strictly hyperbolic systems (Liu [13, 14]). At the present time, the issue of admissibility remains unresolved for more complicated situations including nonstrictly hyperbolic systems (cf. Glimm [8], Keyfitz and Kranzer [10], Schaeffer and Shearer [17]), systems that change type (cf. Shearer [19], Slemrod [20]), or even strictly hyperbolic systems when the Riemann data are large (cf. Sever [18]). The reader is referred to Dafermos [4, 5] for a discussion and comparisons of different admissibility criteria.

Here, we test an alternative approach which is pursued in [1, 3, 9, 11, 20, 22] and bypasses the need for a-priori knowledge of an appropriate admissibility criterion. Namely, we construct solutions of (1.1)–(1.4) as $\varepsilon \searrow 0$ limits of the "viscosity" regularized problem

$$\begin{aligned} &\rho_t + (\rho u)_x = \varepsilon t \rho_{xx} \\ &(\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = \varepsilon t (\rho u)_{xx} \end{aligned} \qquad -\infty < x < \infty\,,\ t > 0 \tag{1.5}$$

with initial data (1.4). The artificial regularization employed in (1.5) preserves the invariance property of (1.1)–(1.2) under dilatation of coordinates $(x,t) \to (\alpha x, \alpha t)$, $\alpha > 0$, as well as the entropy structure of the system (1.5) as compared with usual viscous regularizations. The system (1.5)–(1.4) admits self-similar solutions of the form $\big(\rho_\varepsilon(\frac{x}{t}), u_\varepsilon(\frac{x}{t})\big)$, functions of the single variable $\xi = \frac{x}{t}$. The functions $\big(\rho_\varepsilon(\xi), m_\varepsilon(\xi)\big)$, with $m_\varepsilon(\xi) = \rho_\varepsilon(\xi) u_\varepsilon(\xi)$, are generated by solving the boundary–value problem

$$\begin{aligned} &\varepsilon \rho_\varepsilon'' = -\xi \rho_\varepsilon' + m_\varepsilon' \\ &\varepsilon m_\varepsilon'' = -\xi m_\varepsilon' + \left(\frac{m_\varepsilon^2}{\rho_\varepsilon} + p(\rho_\varepsilon) \right)' \\ &\big(\rho_\varepsilon(-\infty), m_\varepsilon(-\infty)\big) = (\rho_-, m_-)\,, \\ &\big(\rho_\varepsilon(+\infty), m_\varepsilon(+\infty)\big) = (\rho_+, m_+)\,, \end{aligned} \qquad -\infty < \xi < \infty \tag{$\mathcal{P}_\varepsilon$}$$

where $m_- = \rho_- u_-$, $m_+ = \rho_+ u_+$.

In this paper we study the boundary–value problem $(\mathcal{P}_\varepsilon)$ with Riemann data (ρ_-, m_-) and (ρ_+, m_+), ρ_- and ρ_+ positive. Our goal is to first construct connecting orbits $\big(\rho_\varepsilon(\xi), m_\varepsilon(\xi)\big)$ for $(\mathcal{P}_\varepsilon)$, with $\rho_\varepsilon(\xi) > 0$, and then pass to the limit $\varepsilon \searrow 0$ to obtain solutions for the Riemann problem $(\mathcal{P})$ consisting of (1.1), (1.2) and (1.4). Of special interest is to show how entropy pairs and variation estimates are relevant in these constructions. In addition, this framework provides a natural set up to construct solutions that include the vacuum state.

As the system (1.1)–(1.2), together with its companion system of isentropic, Lagrangian gas dynamics, have been instrumental in the development of the theory of hyperbolic conservation laws, there is a large literature on these equations. (In fact the two systems are equivalent for weak solutions that are bounded away from the vacuum state, Wagner [23]). The Riemann problem is solved by constructing the shock and rarefaction wave curves (e.g., Smoller and Johnson [21], Liu [13], Liu and Smoller [15] in the presence of vacuum). DiPerna [7] and Ding, Chen and Luo [6] construct solutions of the Cauchy problem for (1.1)–(1.2), when the states at infinity are the same, as limits of viscous regularizations or finite difference schemes.

The scale invariant regularization employed in (1.5) was proposed by Dafermos [1] and used in Dafermos and DiPerna [3] to construct solutions for broad classes of 2×2 systems that include the equations of isentropic, Lagrangian, gas dynamics. In a subsequent work, Dafermos [2] studied the structure of solutions of the Riemann problem constructed by this method and the relationship with other admissibility criteria. Although the basic assumption on the fluxes in [1, 2, 3] does not cover (1.1)–(1.3), this work owes a sizable amount to the ideas and techniques developed in these papers.

In Sections 2 and 3 we show that, under the assumption (1.3), for each fixed $\varepsilon > 0$, the boundary value problem $(\mathcal{P}_\varepsilon)$ admits solutions $\big(\rho_\varepsilon(\xi), m_\varepsilon(\xi)\big)$ defined for $-\infty < \xi < \infty$. The main theorem is stated in Section 2; its proof is based on a Leray–Schauder type fixed point theorem together with a-priori estimates which are derived in Section 3. The resulting solution has the property that $\rho_\varepsilon(\xi) > 0$, $-\infty < \xi < \infty$. In addition, the solutions have specific forms, namely, either the functions $\rho_\varepsilon(\xi)$ and $u_\varepsilon(\xi)$ are both monotone (or constant), or one of them is monotone and the other bell shaped.

In Section 4, we take advantage of the restrictions on the shapes of solutions to show that the family $\{(\rho_\varepsilon(\cdot), u_\varepsilon(\cdot)); \ 0 < \varepsilon \leq 1\}$ is of uniformly bounded variation and thus precompact in the topology of pointwise convergence. The limit points generate bounded solutions $\big(\rho(\frac{x}{t}), u(\frac{x}{t})\big)$ of the Riemann problem $(\mathcal{P})$, with $\rho(\xi) \geq 0$ for $-\infty < \xi < \infty$. An additional hypothesis is now required, stating that the stored enthalpy $\int^\rho \frac{p'(s)}{s}\, ds$ diverges as $\rho \to \infty$. The final result is stated in Theorem 4.5. We emphasize that this existence theorem does not require any size assumptions on the Riemann data and, therefore, the resulting solution is expected to attain the vacuum state for some data.

An alternative approach requiring genuine nonlinearity, but in turn, yielding explicit uniform bounds is outlined at the end of Section 4. As a byproduct of this analysis, we prove in Section 5 an existence theorem for the Riemann problem $(\mathcal{P})$, when the Riemann data include the vacuum state.

2. Existence of Connecting Orbits for $(\mathcal{P}_\varepsilon)$. This Section deals with the construction of connecting orbits for the boundary-value problem $(\mathcal{P}_\varepsilon)$, when $\varepsilon > 0$ is fixed. The construction is performed in two steps. First, we consider the two parameter boundary-value problem

$$\varepsilon\rho'' = \mu m' - \xi\rho', \tag{2.1}$$

$$\varepsilon m'' = \mu\left(\frac{m^2}{\rho} + p(\rho)\right)' - \xi m', \qquad -L < \xi < L \tag{2.2}$$

$$\rho(-L) = \rho^* + \mu(\rho_- - \rho^*) \qquad \rho(L) = \rho^* + \mu(\rho_+ - \rho^*) \tag{2.3}$$

$$m(-L) = \mu\rho_- u_- = \mu m_- \qquad m(L) = \mu\rho_+ u_+ = \mu m_+ \tag{2.4}$$

with parameters $0 \le \mu \le 1$, $L \ge 1$. Here $\rho^* = \min\{\rho_-, \rho_+\}$, ρ_- and $\rho_+ > 0$, $m(\xi) = \rho(\xi)u(\xi)$, $m_- = \rho_- u_-$ and $m_+ = \rho_+ u_+$. With the help of a Leray–Schauder type fixed point theorem and a–priori estimates for solutions of (2.1)–(2.4) (which are established in Section 3) we show that, for each fixed $0 \le \mu \le 1$ and $L \ge 1$, the boundary-value problem (2.1)–(2.4) has solutions $(\rho(\xi), m(\xi))$ with $\rho(\xi) > 0$. Then, the solutions of $(\mathcal{P}_\varepsilon)$ are constructed as $L \to \infty$ limits of solutions of (2.1)–(2.4) for $\mu = 1$.

More precisely, we prove:

Theorem 2.1. *Assume there are positive constants M and δ depending on u_-, u_+, ρ_-, ρ_+, $p(\rho)$ and ε, but independent of μ and L, such that every solution $(\rho(\xi), m(\xi))$ of (2.1)–(2.4) with $\rho(\xi) > 0$, corresponding to any $0 \le \mu \le 1$, $L \ge 1$, satisfies*

$$\begin{aligned} &\sup_{-L \le \xi \le L} \big(|m(\xi)| + \rho(\xi)\big) \le M, \\ &\inf_{-L \le \xi \le L} \rho(\xi) \ge \delta. \end{aligned} \tag{2.5}$$

Then there exists a solution of $(\mathcal{P}_\varepsilon)$, denoted again by $(\rho(\xi), m(\xi))$, such that $\rho(\xi) > 0$ for $-\infty < \xi < \infty$.

Proof. Set

$$y(\xi) = \begin{pmatrix} \rho(\xi) \\ m(\xi) \end{pmatrix}, \qquad f(y(\xi)) = \begin{pmatrix} m(\xi) \\ \dfrac{m^2(\xi)}{\rho(\xi)} + p(\rho(\xi)) \end{pmatrix}.$$

Then (2.1)–(2.4) is rewritten

$$\begin{aligned} &\varepsilon y''(\xi) = \mu f(y(\xi))' - \xi y'(\xi), \\ &y(-L) = \mu y_- + (1-\mu)y^*, \quad y(L) = \mu y_+ + (1-\mu) y^*, \end{aligned} \qquad -L < \xi < L, \tag{2.6}$$

where

$$y_- = \begin{pmatrix} \rho_- \\ \rho_- u_- \end{pmatrix}, \quad y_+ = \begin{pmatrix} \rho_+ \\ \rho_+ u_+ \end{pmatrix}, \quad y^* = \begin{pmatrix} \rho^* \\ 0 \end{pmatrix}.$$

The problem of solving (2.6) is transformed into a fixed point problem as follows:

Let $X = C^0([-L,L]; R^2)$ equipped with the sup–norm and define Ω by

$$\Omega = \Big\{ (\rho(\cdot), m(\cdot)) \in X : \inf_{-L \le \xi \le L} \rho(\xi) > \frac{\delta}{2} > 0, \\ \sup_{-L\le \xi \le L} \big(|m(\xi)| + \rho(\xi)\big) < M+1 \Big\}.$$

Ω is a bounded, open subset of X and the constant function $(\rho^*, 0)$ belongs to Ω.

Given a smooth function $x(\xi)$, the solution $z(\xi)$ of the boundary–value problem

$$\varepsilon z''(\xi) = f\big(x(\xi)\big)' - \xi z'(\xi), \; z(-L) = y_-, \; z(L) = y_+ \tag{2.7}$$

is computed, using the variation of parameters formula, by

$$\begin{aligned} z(\xi) = z_0 \int_{-L}^{\xi} e^{-\zeta^2/(2\varepsilon)}\, d\zeta + \frac{1}{\varepsilon}\int_{-L}^{\xi} f\big(x(\zeta)\big)\, d\zeta \\ - \frac{1}{\varepsilon^2} \int_{-L}^{\xi}\int_0^{\zeta} \tau f\big(x(\tau)\big) e^{(\tau^2-\zeta^2)/(2\varepsilon)} d\tau\, d\zeta + y_-, \end{aligned} \tag{2.8}$$

where z_0 is calculated by

$$\begin{aligned} z_0 \int_{-L}^{L} e^{-\zeta^2/(2\varepsilon)}\, d\zeta = (y_+ - y_-) - \frac{1}{\varepsilon}\int_{-L}^{L} f\big(x(\zeta)\big)\, d\zeta \\ + \frac{1}{\varepsilon^2}\int_{-L}^{L}\int_0^{\zeta} \tau f\big(x(\tau)\big) e^{(\tau^2-\zeta^2)/(2\varepsilon)}\, d\tau\, d\zeta. \end{aligned} \tag{2.9}$$

Relations (2.8) and (2.9) define a map $T : \overline{\Omega} \to X$ carrying $x(\xi)$ to $z(\xi)$. The range of T is contained in $C^1([-L,L];R^2)$ and

$$z'(\xi) = z_0 e^{-\xi^2/(2\varepsilon)} + \frac{1}{\varepsilon} f\big(x(\xi)\big) - \frac{1}{\varepsilon^2}\int_0^\xi \tau f\big(x(\tau)\big) e^{(\tau^2-\xi^2)/(2\varepsilon)}\, d\tau . \tag{2.10}$$

Hence, T maps bounded subsets of $\overline{\Omega}$ into equicontinuous subsets of X. In addition T is continuous and thus $T : \overline{\Omega} \to X$ is a compact map.

By means of the map T, solutions of (2.6) are identified with fixed points of the map $\mu T(\cdot) + (1-\mu)y^*$. Since we look for solutions with $\rho(\xi) > O$ and in view of the a–priori estimate (2.5), these fixed points are sought for $\big(\rho(\cdot), m(\cdot)\big)$ in Ω. We recall the following fixed point theorem (J. Mawhin [16], Thm IV.1).

Proposition 2.2. *Let X be a real normed vector space and Ω a bounded open subset of X. Let $F = I - T$ with $T : \overline{\Omega} \to X$ compact and $\mathcal{F} : \overline{\Omega} \times [0,1] \to X$ be of the form*

$$\mathcal{F}(y,\mu) = y + G(y,\mu),$$

where $G : \overline{\Omega} \times [0,1] \to X$ is compact and $\mathcal{F}(\cdot,1) = F$. If the following conditions are satisfied:

(i) $\qquad \mathcal{F}(y,\mu) \neq 0 \qquad$ *for $y \in \partial\Omega$, $0 \le \mu \le 1$,*

(ii) $\qquad \mathcal{F}(y,0) = 0 \qquad$ *for some $y \in \Omega$,*

then

$$Fy = 0$$

has at least one solution in Ω.

Let X, Ω and T be as defined above and set $Fy = y - Ty$, $G(y,\mu) = -\mu Ty - (1-\mu)y^*$ and $\mathcal{F}(y,\mu) = y - \mu Ty - (1-\mu)y^*$. Since $T : \overline{\Omega} \to X$ is compact, $G : \overline{\Omega} \times [0,1] \to X$ is compact. Moreover, the hypotheses (2.5) imply that (i) is satisfied, while, since $y^* \in \Omega$ is a solution of $\mathcal{F}(y,0) = 0$, condition (ii) is also satisfied. Therefore, Proposition 2.2 yields existence of solutions of (2.6) in Ω, for any $\varepsilon > 0$ fixed, $0 \le \mu \le 1$ and $1 \le L < \infty$.

To conclude the proof we extend the solutions $y(\xi;L)$ of (2.6) with $\mu = 1$ onto the entire real line by setting $y(\xi;L) = y_-$ for $\xi < -L$ and $y(\xi;L) = y_+$ for $\xi > L$. Following Dafermos [1, p. 4] and using (2.5) we establish the estimate

$$|y'(\xi;L)| \le C e^{(2\alpha|\xi| - \xi^2)/(2\varepsilon)}, \qquad -L \le \xi \le L, \tag{2.11}$$

where the constants C and α depend on M, δ and ε only. On account of (2.5) and (2.11) the set $\{y(\cdot;L) : L \ge 1\}$ is precompact in $C^o((-\infty,\infty);R^2)$. Thus, there exists a sequence $\{L_n\}$ tending to infinity as $n \to \infty$ and a function $y(\xi)$, such that $y(\xi;L_n) \to y(\xi)$ uniformly on $(-\infty,\infty)$. It is a matter of routine analysis to show that $y(\xi) = \big(\rho(\xi), m(\xi)\big)$ is a solution of $(\mathcal{P}_\varepsilon)$ and the details are omitted. Moreover, by virtue of $(2.5)_2$, $\rho(\xi) > 0$ for $-\infty < \xi < \infty$.

3. The a–priori estimates. The scope of this section is twofold: to derive the a–priori estimates required to apply Theorem 2.1, and to prepare the ground for studying the $\varepsilon \searrow 0$ limit in the following section. Regarding the first goal, we show:

Theorem 3.1. *Let* $(\rho(\xi), m(\xi))$ *be a solution of* (2.1)–(2.4)*, with* $\rho(\xi) > 0$*, corresponding to some* $0 \le \mu \le 1$*,* $L \ge 1$*. If* $p(\rho)$ *satisfies* (1.3)*, then* $(\rho(\xi), m(\xi))$ *satisfies the a–priori estimates* (2.5).

Theorem 3.1 will be established in a series of lemmas. In preparation, we note that if $(\rho(\xi), m(\xi))$ is a solution of (2.1)–(2.4) with $\rho(\xi) > 0$, then $\rho(\xi)$ and $u(\xi) = m(\xi)/\rho(\xi)$ satisfy the differential equations

$$\varepsilon \rho'' = \mu(u\rho' + \rho u') - \xi \rho', \tag{3.1}$$

$$\varepsilon(\rho u'' + 2\rho' u') = \mu(\rho u u' + p'(\rho)\rho') - \xi \rho u', \tag{3.2}$$

$$-L < \xi < L$$

with boundary conditions

$$\begin{aligned} \rho(-L) &= \rho^* + \mu(\rho_- - \rho^*), \\ \rho(L) &= \rho^* + \mu(\rho_+ - \rho^*), \\ u(-L) &= \frac{m(-L)}{\rho(-L)} = \mu \frac{\rho_- u_-}{\rho^* + \mu(\rho_- - \rho^*)}, \\ u(L) &= \frac{m(L)}{\rho(L)} = \mu \frac{\rho_+ u_+}{\rho^* + \mu(\rho_+ - \rho^*)}. \end{aligned} \tag{3.3}$$

The first lemma provides a classification of the possible shapes of (ρ, u) satisfying (3.1)–(3.2). The lemma remains valid in case $L = \infty$, in particular for (ρ, m) solutions of $(\mathcal{P}_\varepsilon)$. This result should be contrasted to Theorem 4.1 of Dafermos [1] and is in correspondence to the form of the shock and wave curves for the equations of isentropic gas dynamics in Eulerian coordinates (1.1)–(1.4), (e.g. Liu and Smoller [15]).

Lemma 3.2. *Let* $(\rho(\xi), u(\xi))$ *with* $\rho(\xi) > 0$ *satisfy* (3.1)–(3.2) *in* $(-L, L)$ *for some* $0 < \mu \le 1$ *and* $1 \le L \le \infty$*. If* $p(\rho)$ *satisfies* (1.3)*, then one of the following holds:*

(i) *ρ and u are constant functions,*
(ii) *ρ and u are monotone functions,*
(iii) *ρ is monotone increasing (decreasing); u has precisely one critical point which is a minimum (maximum),*
(iv) *u is monotone increasing (decreasing); ρ has precisely one critical point which is a minimum (maximum).*

Proof. We begin with the following elementary observations:

(a) If $\rho'(\tau) = u'(\tau) = 0$ for some $\tau \in (-L, L)$, then ρ and u are constant functions.
(b) The only solutions with degenerate critical points are the constant functions.
(c) If τ is a point of minimum (maximum) of ρ, then $\rho'(\tau) = 0$, $\rho''(\tau) > 0$ $(\rho''(\tau) < 0)$ and $u'(\tau) > 0$, $(u'(\tau) < 0)$; if τ is a point of minimum (maximum) of u, then $u'(\tau) = 0$, $u''(\tau) > 0$ $(u''(\tau) < 0)$ and $\rho'(\tau) > 0$ $(\rho'(\tau) < 0)$.

Statement (a) follows from the standard uniqueness theorem for ordinary differential equations applied to (3.1), (3.2). Statement (b) is a consequence of (a) upon noting that, if $\rho'(\tau) = \rho''(\tau) = 0$, then (3.1) implies $u'(\tau) = 0$, while, if $u'(\tau) = u''(\tau) = 0$, then (3.2) implies $\rho'(\tau) = 0$. Finally, statement (c) provides the local behavior of ρ and u at critical points; it follows from (3.1) and (3.2), respectively, upon using (b) and (1.3).

Next, suppose that ρ and u are nonconstant functions and that u attains a minimum at some τ, $-L < \tau < L$. Then, one of the following happens:

(1) either (iii) is true, or
(2) u has two consecutive critical points with a minimum at τ and a maximum at σ, or
(3) u has precisely one critical point which is a minimum at τ, while ρ has at least one critical point in $(-L, L)$.

Assume that (2) holds and, for concreteness, let $\tau < \sigma$. Then $u'(\xi) > 0$ for $\tau < \xi < \sigma$ and, by (c), $\rho'(\tau) > 0$ and $\rho'(\sigma) < 0$. Thus, ρ attains a maximum at some point ϑ, $\tau < \vartheta < \sigma$. Using (c), $u'(\vartheta) < 0$ which is a contradiction. Therefore, case (2) is impossible.

If (3) holds, $u'(\xi) < 0$ for $\xi \in [-L, \tau)$, while $u'(\xi) > 0$ for $\xi \in (\tau, L]$. Moreover, $\rho'(\tau) > 0$ and either ρ attains a minimum at some $\vartheta \in (-L, \tau)$, or ρ attains a maximum at some $\sigma \in (\tau, L)$. On account of (c), in the former case $u'(\vartheta) > 0$, while, in the latter case, $u'(\sigma) < 0$. Thus, (3) is also impossible.

We conclude that if u has a minimum at some $\tau \in (-L, L)$, then ρ and u have the shapes dictated in (iii). The rest of the proof follows a similar pattern and it is omitted. □

In the sequel, we follow Dafermos and DiPerna [3] and use the concept of entropy-entropy flux pairs (Lax [12]) to obtain (ε-independent) bounds for solutions of (2.1)–(2.4). A function $\eta(\rho, u)$ is called entropy for (1.1)–(1.2) with corresponding entropy flux $q(\rho, u)$ if

$$q_\rho = u\eta_\rho + \frac{p'(\rho)}{\rho}\eta_u\,, \qquad q_u = \rho\eta_\rho + u\eta_u\,. \tag{3.4}$$

Such pairs are generated by first solving

$$\eta_{\rho\rho} = \frac{p'(\rho)}{\rho^2}\eta_{uu} \tag{3.5}$$

for $\eta(\rho,u)$ and then integrating (3.4) to obtain $q(\rho,u)$. A comprehensive analysis of solutions of (3.5) is conducted by DiPerna [7].

Let $(\eta(\rho,m),q(\rho,m))$ be an entropy–entropy flux pair, expressed in terms of the variables ρ and m. Using (3.4) we deduce that solutions of (2.1)–(2.2) satisfy the identity

$$-\xi\eta' + \mu q' = \varepsilon\eta'' - \varepsilon(\rho',m').\nabla^2\eta(\rho',m')^T \tag{3.6}$$

where $\eta = \eta(\rho(\xi),m(\xi))$, $q = q(\rho(\xi),m(\xi))$. In exploiting (3.6), it is helpful to use entropy functions $\eta(\rho,m)$ that are convex (or linear) with respect to the variables ρ and m. Classical examples of such pairs correspond to entropies associated with the mass, momentum and mechanical energy, namely

$$\begin{gathered} \pm(\rho,\rho u), \quad \pm(\rho u, \rho u^2 + p(\rho)), \\ \left(\frac{1}{2}\rho u^2 + \rho e(\rho),\ \frac{1}{2}\rho u^3 + \rho e(\rho)u + p(\rho)u\right), \end{gathered} \tag{3.7}$$

where $e(\rho) = \int^\rho p(s)/s^2\,ds$.

We now present a lemma indicating how to use (3.6) to bound the total entropy production. It is expedient to state the lemma for subclasses $\mathcal{F}$ of solutions $(\rho(\cdot),m(\cdot))$ of (2.1)–(2.4), $0 \le \mu \le 1$, $1 \le L < \infty$, $\varepsilon > 0$; in applications of the lemma, $\mathcal{F}$ incorporates the monotonicity properties of solutions dictated by Lemma 3.2. Given any constant entropy level $\overline{\eta}$, consider the level set $\mathcal{C}_{\overline{\eta}} = \{(\rho,m) \in R^+ \times R : \eta(\rho,m) = \overline{\eta}\}$, as well as the set of values in $\mathcal{C}_{\overline{\eta}}$ that are attained along solutions of the class $\mathcal{F}$

$$\mathcal{C}_{\mathcal{F},\overline{\eta}} = \mathcal{C}_{\overline{\eta}} \cap \{(\rho(\xi),m(\xi)),\ -L \le \xi \le L\}_{(\rho(\cdot),m(\cdot))\in\mathcal{F}}. \tag{3.8}$$

If $\mathcal{C}_{\mathcal{F},\overline{\eta}}$ is nonempty, let

$$Q := \sup_{(\rho_1,m_1),(\rho_2,m_2)\in\mathcal{C}_{\mathcal{F},\overline{\eta}}} |q(\rho_1,m_1) - q(\rho_2,m_2)| \tag{3.9}$$

be the oscillation of $q(\rho,m)$ on the set $\mathcal{C}_{\mathcal{F},\overline{\eta}}$. We prove:

Lemma 3.3. *Let $\mathcal{F}$ be a class of solutions $\big(\rho(\cdot),m(\cdot)\big)$ of* (2.1)–(2.4) *with $\rho(\cdot) > 0$ and assume that $\eta(\rho,m)$ is a convex entropy for* (1.1)–(1.2) *with corresponding entropy flux $q(\rho,m)$. If $\overline{\eta}$ is any constant such that*

$$\overline{\eta} \geq \max\{\max_{0\leq\mu\leq 1} \eta\big(\rho(-L),m(-L)\big),\ \max_{0\leq\mu\leq 1} \eta\big(\rho(L),m(L)\big)\} \tag{3.10}$$

then for any $(\rho(\cdot),m(\cdot)) \in \mathcal{F}$ and $(\alpha,\beta) \subset (-L,L)$

$$\int_\alpha^\beta \big(\eta\big(\rho(\xi),m(\xi)\big) - \overline{\eta}\big)\, d\xi \leq N, \tag{3.11}$$

where $N = 0$ in case $\mathcal{C}_{\mathcal{F},\overline{\eta}}$ is empty and $N = Q$ in case $\mathcal{C}_{\mathcal{F},\overline{\eta}}$ is nonempty.

Proof. Let $\big(\rho(\cdot),m(\cdot)\big) \in \mathcal{F}$ and fix $\overline{\eta}$ so that (3.10) holds. If $\eta\big(\rho(\xi),m(\xi)\big) \leq \overline{\eta}$ for $-L \leq \xi \leq L$, then (3.11) is trivially true with $N = 0$. So, assume that the set $\big\{\xi \in [-L,L] : \eta\big(\rho(\xi),m(\xi)\big) > \overline{\eta}\big\}$ is nonempty and let α_0, β_0 be any two points in $[-L,L]$ such that $\alpha_0 < \beta_0$, $\eta\big(\rho(\alpha_0),\, m(\alpha_0)\big) = \eta\big(\rho(\beta_0),\, m(\beta_0)\big) = \overline{\eta}$ with $\eta'\big(\rho(\alpha_0),m(\alpha_0)\big) \geq 0$ and $\eta'\big(\rho(\beta_0),m(\beta_0)\big) \leq 0$. Integrating (3.6) over $[\alpha_0,\beta_0]$ and using the convexity of $\eta(\rho,m)$, we obtain

$$\begin{aligned} -\int_{\alpha_0}^{\beta_0} \xi\big(\eta(\rho(\xi),m(\xi)) - \overline{\eta}\big)'\, d\xi \qquad & \\ + \mu\big[q\big(\rho(\beta_0),m(\beta_0)\big) - q\big(\rho(\alpha_0),m(\alpha_0)\big)\big] &\leq 0, \end{aligned} \tag{3.12}$$

which, on account of (3.9), yields

$$\int_{\alpha_0}^{\beta_0} \big(\eta\big(\rho(\xi),m(\xi)\big) - \overline{\eta}\big)\, d\xi \leq Q. \tag{3.13}$$

Given any α, β in $(-L,L)$ with $\alpha < \beta$ we choose α_0, β_0 as follows: If $\eta\big(\rho(\alpha),m(\alpha)\big) > \overline{\eta}$, then $\alpha_0 = \sup\{\xi \in [-L,\alpha) : \eta\big(\rho(\xi),m(\xi)\big) \leq \overline{\eta}\}$, while if $\eta\big(\rho(\alpha),m(\alpha)\big) \leq \overline{\eta}$, then $\alpha_0 = \inf\{\xi \in (\alpha,L] : \eta\big(\rho(\xi),m(\xi)\big) > \overline{\eta}\}$. If $\eta\big(\rho(\beta),m(\beta)\big) > \overline{\eta}$, then $\beta_0 = \inf\{\xi \in (\beta,L] : \eta\big(\rho(\xi),m(\xi)\big) \leq \overline{\eta}\}$, while if $\eta\big(\rho(\beta),m(\beta)\big) \leq \overline{\eta}$ then $\beta_0 = \sup\{\xi \in [-L,\beta) : \eta\big(\rho(\xi),m(\xi)\big) > \overline{\eta}\}$. Either $\eta\big(\rho(\xi),m(\xi)\big) \leq \overline{\eta}$ for $\alpha \leq \xi \leq \beta$ and (3.11) is true with $N = 0$, or α_0, β_0 are well defined, $\alpha_0 < \beta_0$ and

$$\int_\alpha^\beta \big(\eta\big(\rho(\xi),m(\xi)\big) - \overline{\eta}\big)\, d\xi \leq \int_{\alpha_0}^{\beta_0} \big(\eta\big(\rho(\xi),m(\xi)\big) - \overline{\eta}\big)\, d\xi \leq Q \qquad \square \tag{3.14}$$

In general the quantity Q (and thus also N in (3.11)) depends on $\overline{\eta}$, $\eta(\rho,m)$, $q(\rho,m)$ and the class $\mathcal{F}$ and may be infinite. However, it often turns out that Q is finite and independent of μ, L and ε. For instance, Q is finite if the level set $\mathcal{C}_{\overline{\eta}}$ or its restriction $\mathcal{C}_{\mathcal{F},\overline{\eta}}$ is contained in a compact subset of $R^+ \times R$. As an application of Lemma 3.3 we obtain the following:

Corollary 3.4.

(a) *If u is monotone increasing and ρ has a minimum at $\tau \in (-L, L)$,*

$$\rho(\xi) \geq \min\{\rho_-, \rho_+\} - \frac{N}{|\xi - \tau|}, \qquad \xi \in [-L, L] \backslash \{\tau\}. \tag{3.15}$$

(b) *If u is monotone decreasing and ρ has a maximum at $\tau \in (-L, L)$,*

$$\rho(\xi) \leq \max\{\rho_-, \rho_+\} + \frac{N}{|\xi - \tau|}, \qquad \xi \in [-L, L] \backslash \{\tau\}. \tag{3.16}$$

(c) *If ρ is monotone increasing and u has a minimum at $\tau \in (-L, L)$, then for any α, β in $(-L, L)$ with $\alpha < \beta$,*

$$\int_\alpha^\beta (m(\xi) - \underline{m})\, d\xi \geq -N, \tag{3.17}$$

where $\underline{m} = \min\{0, m_-, m_+\}$, *and*

$$u(\xi) \geq \frac{\underline{m}}{\rho_-} - \frac{N}{|\xi - \tau|}, \qquad \xi \in [-L, L] \backslash \{\tau\}. \tag{3.18}$$

(d) *If ρ is monotone decreasing and u has a maximum at $\tau \in (-L, L)$, then for any α, $\beta \in (-L, L)$ with $\alpha < \beta$,*

$$\int_\alpha^\beta (m(\xi) - \overline{m})\, d\xi \leq N, \tag{3.19}$$

where $\overline{m} = \max\{0, m_-, m_+\}$, *and*

$$u(\xi) \leq \frac{\overline{m}}{\rho_+} + \frac{N}{|\xi - \tau|}, \qquad \xi \in [-L, L] \backslash \{\tau\}. \tag{3.20}$$

The constants N in (3.15)–(3.20) *are all independent of μ, L and ε.*

Proof. We prove (a) and (d). The proofs of (b) and (c) are similar.

To show (a) we consider the class $\mathcal{F}$ consisting of solutions of (2.1)–(2.4) with u monotone increasing and ρ having exactly one minimum and apply Lemma 3.3 for the entropy–entropy flux pair $(-\rho, -m)$ and the entropy level $\overline{\eta} = -\underline{\rho} = -\min\{\rho_-, \rho_+\}$. In this case $Q \le \max_{0\le\mu\le1}\{\underline{\rho}\big(u(L) - u(-L)\big)\}$ is bounded independently of μ, L and ε. Thus, (3.11) implies that

$$\int_\alpha^\beta (\rho(\zeta) - \underline{\rho}) d\zeta \ge -N \tag{3.21}$$

for any $(\alpha, \beta) \subset (-L, L)$. The choice $\alpha = \xi$, $\beta = \tau$ together with the monotonicity properties of $\rho(\xi)$ gives

$$-N \le \int_\xi^\tau (\rho(\zeta) - \underline{\rho})\, d\zeta \le (\rho(\xi) - \underline{\rho})(\tau - \xi), \tag{3.22}$$

whence (3.15) follows for $\xi \in [-L, \tau)$. Similarly, the choice $\alpha = \tau$, $\beta = \xi$ yields (3.15) for $\xi \in (\tau, L]$.

Next we establish (d). Let $\mathcal{F}$ be the class of solutions with ρ monotone decreasing and u having exactly one maximum. We apply Lemma 3.3 for the pair $\big(m, \frac{m^2}{\rho} + p(\rho)\big)$ with $\overline{\eta} = \overline{m} = \max\{0, m_-, m_+\}$. In this case $\mathcal{C}_{\mathcal{F},\overline{\eta}}$ is contained in $[\rho_+, \rho_-] \times \{\overline{m}\}$ and thus Q is bounded independently of μ, L and ε. As a consequence (3.11) implies (3.19). Finally, (3.20) follows from (3.19) by using as argument similar to the one leading to (3.15). □

Next, we establish the a–priori bounds (2.5) for $\rho(\xi)$.

Lemma 3.5. *Let $\big(\rho(\xi), m(\xi)\big)$ be as in Theorem 2.1. There are positive constants M and δ, independent of μ and L, such that*

$$0 < \delta \le \rho(\xi) \le M, \qquad -L \le \xi \le L. \tag{3.23}$$

Proof. We only need to analyze solutions with $\rho(\xi)$ admitting a critical point. We show (3.23) in the representative case that $\rho(\xi)$ has a positive minimum at $\tau \in (-L, L)$ and $u(\xi)$ is monotone increasing. It suffices to show that $\rho(\tau)$ is bounded from below by a constant independent of μ and L.

Set $\underline{\rho} = \min\{\rho(-L), \rho(L)\}$ and let α_0, β_0 be such that $-L \le \alpha_0 < \tau < \beta_0 \le L$ and $\rho(\alpha_0) = \rho(\beta_0) = \underline{\rho}$. Given any $\xi \in [\alpha_0, \tau)$, let ξ' be such that $\rho(\xi') = \rho(\xi)$. Then $\xi' \in (\tau, \beta_0]$ and $\rho'(\xi') > 0$. Integrating (2.1) over $[\xi, \xi']$, we arrive at

$$\varepsilon\rho'(\xi') - \varepsilon\rho'(\xi) = \mu\rho(\xi)\big(u(\xi') - u(\xi)\big) - \int_\xi^{\xi'} \zeta\rho'(\zeta)\, d\zeta. \tag{3.24}$$

Using the monotonicity properties of u together with

$$-\int_{\xi}^{\xi'} \zeta \rho'(\zeta)\, d\zeta = \int_{\xi}^{\xi'} \big(\rho(\zeta) - \rho(\xi)\big)\, d\zeta < 0 \tag{3.25}$$

and (3.24), we conclude that ρ satisfies the differential inequality

$$0 < \varepsilon \rho'(\xi) + A\rho(\xi), \qquad \alpha_0 \leq \xi < \tau \tag{3.26}$$

with $A = \max_{0 \leq \mu \leq 1}\big(u(L) - u(-L)\big)$ positive and independent of μ, L and ε.

Integrating (3.26), we obtain

$$\rho(\tau) \geq \rho(\zeta) e^{-(A/\varepsilon)(\tau - \zeta)}, \qquad \alpha_0 \leq \zeta \leq \tau. \tag{3.27}$$

In addition (3.15) reads

$$\rho(\zeta) \geq \underline{\rho} - \frac{N}{\tau - \zeta}, \qquad \alpha_0 \leq \zeta \leq \tau. \tag{3.28}$$

Relations (3.27) and (3.28) together imply that $\rho(\tau)$ is bounded from below as follows:

(a) If $\tau - \alpha_0 \leq \frac{2N}{\underline{\rho}}$, (3.27) gives for $\zeta = \alpha_0$

$$\rho(\tau) \geq \underline{\rho} e^{-2AN/(\varepsilon \underline{\rho})}$$

(b) If $\tau - \alpha_0 > \frac{2N}{\underline{\rho}}$, let $\zeta^* = \tau - \frac{2N}{\underline{\rho}}$ and use (3.27) and (3.28) for $\zeta = \zeta^*$ to deduce

$$\rho(\tau) \geq \rho(\zeta^*) e^{-(A/\varepsilon)(\tau - \zeta^*)} \geqq \frac{1}{2} \underline{\rho} e^{-2AN/(\varepsilon \underline{\rho})}. \qquad \square$$

The last goal is to show the bound (2.5) for $m(\xi)$. A difficulty arises in this case from the lack of a–priori control on the oscillations of the function $m(\xi)$. We overcome this difficulty using the L^1-type estimates (3.17), (3.19) in conjunction with the special structure of the differential equation (2.2).

Lemma 3.6. *Let $\big(\rho(\xi), m(\xi)\big)$ be as in Theorem 2.1. There is a positive constant M, independent of μ and L, such that*

$$|m(\xi)| < M, \qquad -L \leq \xi \leq L. \tag{3.29}$$

Proof. The bound (3.29) is obviously true for cases (i), (ii) and, by virtue of (3.23), case (iv) of Lemma 3.2. We now consider case (iii).

Suppose first that ρ is monotone decreasing and u has precisely one maximum at τ, $-L < \tau < L$. Clearly $m(\xi) \geq \rho(\xi)\min\{u(-L),u(L)\}$ and thus $m(\xi)$ is bounded from below. It remains to show that $m(\xi)$ is bounded from above independently of μ and L.

Set $\overline{m} := \max\{0,m_-,m_+\} \geq \max\{m(-L),m(L)\}$ and consider any point $\xi \in [-L,L]$ such that $m(\xi) > \overline{m}$ and $m'(\xi) > 0$. Let $\xi' = \inf\{\zeta \in (\xi,L] : m(\zeta) < m(\xi)\}$. Since $m(\xi) > \overline{m} \geq m(L)$, ξ' is well defined, $\xi' \geq \xi$, $m(\xi') = m(\xi)$, and $m'(\xi') \leq 0$. Moreover, $m(\zeta) \geq m(\xi)$ for $\xi \leq \zeta \leq \xi'$. Integrating (2.2) over $[\xi,\xi']$, we obtain

$$\begin{aligned} &\varepsilon m'(\xi') - \varepsilon m'(\xi) \\ &= \mu m^2(\xi)\left[\frac{1}{\rho(\xi')} - \frac{1}{\rho(\xi)}\right] + \mu\big[p\big(\rho(\xi')\big) - p\big(\rho(\xi)\big)\big] \\ &\quad - \int_\xi^{\xi'} \zeta m'(\zeta)\, d\zeta. \end{aligned} \tag{3.30}$$

Since ρ is decreasing and

$$-\int_\xi^{\xi'} \zeta m'(\zeta)\, d\zeta = \int_\xi^{\xi'} \big(m(\zeta) - m(\xi)\big)\, d\zeta \geq 0, \tag{3.31}$$

(3.30) together with (1.3) yield

$$\varepsilon m'(\xi) \leq A, \tag{3.32}$$

where $A = \max_{0\leq\mu\leq1}\{p\big(\rho(-L)\big) - p\big(\rho(L)\big)\}$ is positive. Note that the bound (3.32) holds for any $\xi \in [-L,L]$ such that $m(\xi) > \overline{m}$.

To conclude the proof observe that the set $\{\xi \in [-L,L] : m(\xi) > \overline{m}\}$ can be decomposed into a countable union of disjoint subintervals (α_k,β_k), $k = 1,2,\ldots$, such that $m(\alpha_k) = m(\beta_k) = \overline{m}$ and $m(\xi) > \overline{m}$ for $\alpha_k < \xi < \beta_k$. Fix a k and let $m_k = \max_{\alpha_k\leq\xi\leq\beta_k} m(\xi)$ and τ_k be such that $m(\tau_k) = m_k$. If $\tau_k - \alpha_k \leq 1$, then, using (3.32),

$$m_k = m(\tau_k) \leq \overline{m} + \frac{A}{\varepsilon}. \tag{3.33}$$

On the other hand, if $\tau_k - \alpha_k > 1$, we use the identity

$$(\tau_k - \vartheta)(m(\tau_k) - \overline{m}) = \int_\vartheta^{\tau_k} (m(\zeta) - \overline{m})\, d\zeta + \int_\vartheta^{\tau_k}\int_\zeta^{\tau_k} m'(\xi)\, d\xi\, d\zeta \tag{3.34}$$

for $\alpha_k < \vartheta < \tau_k < \beta_k$, together with (3.19) and (3.32), to obtain

$$(\tau_k - \vartheta)(m(\tau_k) - \overline{m}) \leq N + \frac{1}{2}\frac{A}{\varepsilon}(\tau_k - \vartheta)^2. \tag{3.35}$$

For $\vartheta = \tau_k - 1$, (3.35) gives

$$m_k = m(\tau_k) \leq \overline{m} + N + \frac{1}{2}\frac{A}{\varepsilon}. \tag{3.36}$$

Finally, on account of (3.33) and (3.36)

$$m(\xi) \leq \overline{m} + \max\left\{\frac{A}{2\varepsilon} + N, \frac{A}{\varepsilon}\right\}, \qquad -L \leq \xi \leq L. \tag{3.37}$$

A similar argument provides (3.29) in case ρ is increasing and u attains a minimum at τ. □

4. Existence of solutions of the Riemann problem. The intent of this Section is to construct solutions of the Riemann problem $(\mathcal{P})$ as $\varepsilon \searrow 0$ limits of solutions of $(\mathcal{P}_\varepsilon)$. We recall that for fixed $\varepsilon > 0$ and under the hypothesis $p'(\rho) > 0$ for $\rho > 0$, the boundary–value problem $(\mathcal{P}_\varepsilon)$:

$$\varepsilon\rho'' = (\rho u)' - \xi\rho', \tag{4.1}$$

$$\varepsilon(\rho u)'' = \left(\rho u^2 + p(\rho)\right)' - \xi(\rho u)', \qquad -\infty < \xi < \infty \tag{4.2}$$

$$\begin{aligned} \rho(-\infty) &= \rho_-, \qquad \rho(+\infty) = \rho_+, \\ u(-\infty) &= u_-, \qquad u(+\infty) = u_+, \end{aligned} \tag{4.3}$$

admits classical solutions $\left(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\right)$ with $\rho_\varepsilon(\xi) > 0$. Note that in (4.3), ρ_-, $\rho_+ > 0$, also $m_\varepsilon(\xi) = \rho_\varepsilon(\xi)u_\varepsilon(\xi)$.

The analysis in this Section is centered around justifying the hypotheses of the following existence theorem for the Riemann problem $(\mathcal{P})$.

Theorem 4.1. *Let $\left(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\right)$ denote a solution of $(\mathcal{P}_\varepsilon)$ with $\rho_\varepsilon(\xi) > 0$. Suppose the functions $\{\rho_\varepsilon(\cdot), u_\varepsilon(\cdot)\ ;\ 0 < \varepsilon \leq 1\}$ are uniformly bounded and of uniformly bounded variation. Then, there exists a pair of functions of bounded variation $\left(\rho(\xi), u(\xi)\right)$, defined for $-\infty < \xi < \infty$ with $\rho(\xi) \geq 0$, such that $\left(\rho(\frac{x}{t}), u(\frac{x}{t})\right)$ is a weak solution of the Riemann problem $(\mathcal{P})$.*

Proof. By Helly's theorem there is a sequence $\{\varepsilon_n\}$, $\varepsilon_n \searrow 0$ as $n \to \infty$ and functions $\rho(\xi)$, $u(\xi)$ of bounded variation, with $\rho(\xi) \geq 0$, such that $\rho_{\varepsilon_n}(\xi) \to \rho(\xi)$, $u_{\varepsilon_n}(\xi) \to u(\xi)$ for a.e. $\xi \in (-\infty,\infty)$. Accordingly, $\rho_{\varepsilon_n}\left(\frac{x}{t}\right) \to \rho\left(\frac{x}{t}\right)$, $u_{\varepsilon_n}\left(\frac{x}{t}\right) \to u\left(\frac{x}{t}\right)$ for a.e. $(x,t) \in (-\infty,\infty) \times (0,\infty)$. The functions $\left(\rho_{\varepsilon_n}\left(\frac{x}{t}\right), u_{\varepsilon_n}\left(\frac{x}{t}\right)\right)$ satisfy (1.5) with $\varepsilon = \varepsilon_n$. Passing to the limit $\varepsilon_n \searrow 0$ in (1.5), we conclude that $\left(\rho\left(\frac{x}{t}\right), u\left(\frac{x}{t}\right)\right)$ is a weak solution of (1.1)–(1.2).

It remains to verify the initial conditions (1.4). Set

$$y_\varepsilon(\xi) = \begin{pmatrix} \rho_\varepsilon(\xi) \\ m_\varepsilon(\xi) \end{pmatrix}, \quad z_\varepsilon(\xi) = \begin{pmatrix} \rho_\varepsilon(\xi) \\ u_\varepsilon(\xi) \end{pmatrix},$$

$$f\big(z_\varepsilon(\xi)\big) = \begin{pmatrix} \rho_\varepsilon(\xi) u_\varepsilon(\xi) \\ \rho_\varepsilon(\xi) u_\varepsilon^2(\xi) + p\big(\rho_\varepsilon(\xi)\big) \end{pmatrix}.$$

By virtue of (4.1), (4.2), $y_\varepsilon(\xi)$ satisfies the equation

$$\frac{d}{d\xi}\left[y_\varepsilon'(\xi) e^{\xi^2/(2\varepsilon)}\right] = \frac{1}{\varepsilon} A\big(z_\varepsilon(\xi)\big) y_\varepsilon'(\xi) e^{\xi^2/(2\varepsilon)}, \tag{4.4}$$

where the matrix $A\big(z_\varepsilon(\xi)\big)$ is given by

$$A\big(z_\varepsilon(\xi)\big) = \begin{pmatrix} 0 & 1 \\ -u_\varepsilon^2(\xi) + p'\big(\rho_\varepsilon(\xi)\big) & 2u_\varepsilon(\xi) \end{pmatrix}.$$

Relation (4.4) yields the bound

$$|y_\varepsilon'(\xi)| \leq |y_\varepsilon'(0)| e^{(2\alpha|\xi| - \xi^2)/(2\varepsilon)}, \qquad -\infty < \xi < \infty, \tag{4.5}$$

with α any constant such that $\alpha \geq \sup_{-\infty<\xi<\infty} |A(z_\varepsilon(\xi)|$. In view of (4.5), $|y_\varepsilon'(\xi)|$ decays rapidly to zero as $|\xi| \to \infty$ for $\varepsilon > 0$ fixed. Moreover, since $z_\varepsilon(\xi)$ is uniformly bounded, the constant α can be chosen to be independent of ε.

Integrating (4.1), (4.2), we obtain

$$\begin{aligned} y_\varepsilon'(0) \int_{-1}^{1} e^{-\xi^2/(2\varepsilon)}\, d\xi = y_\varepsilon(1) - y_\varepsilon(-1) &- \frac{1}{\varepsilon} \int_{-1}^{1} f\big(z_\varepsilon(\xi)\big)\, d\xi \\ &+ \frac{1}{\varepsilon} f\big(z_\varepsilon(0)\big) \int_{-1}^{1} e^{-\xi^2/(2\varepsilon)}\, d\xi \\ &+ \frac{1}{\varepsilon^2} \int_{-1}^{1} \int_{0}^{\xi} \zeta f\big(z_\varepsilon(\zeta)\big) e^{(\zeta^2 - \xi^2)/(2\varepsilon)}\, d\zeta\, d\xi. \end{aligned} \tag{4.6}$$

Since $\rho_\varepsilon(\xi)$ and $u_\varepsilon(\xi)$ are uniformly bounded, taking into account the way each term of (4.6) depends on ε, we deduce from (4.5)

$$|y'_\varepsilon(\xi)| \le \frac{K}{\varepsilon^{3/2}} e^{(2\alpha|\xi|-|\xi|^2)/(2\varepsilon)}, \qquad -\infty < \xi < \infty, \tag{4.7}$$

where K and α are independent of ε.

Recall that $\rho_{\varepsilon_n}(\xi) \to \rho(\xi)$, $u_{\varepsilon_n}(\xi) \to u(\xi)$ and thus $m_{\varepsilon_n}(\xi) = \rho_{\varepsilon_n}(\xi) u_{\varepsilon_n}(\xi) \to m(\xi) = \rho(\xi)u(\xi)$ for a.e. $\xi \in (-\infty,\infty)$. On account of (4.7), $\rho(\xi) = \rho_-$ and $m(\xi) = m_- = \rho_- u_-$ for $\xi < -2\alpha$, while $\rho(\xi) = \rho_+$ and $m(\xi) = m_+ = \rho_+ u_+$ for $\xi > 2\alpha$. Moreover, since ρ_- and ρ_+ are positive, $u(\xi) = u_-$ for $\xi < -2\alpha$ and $u(\xi) = u_+$ for $\xi > 2\alpha$. In particular $\big(\rho(\frac{x}{t}), u(\frac{x}{t})\big)$ satisfies (1.4). □

For the remainder of the section, we let $\big(\rho_\varepsilon(\xi),\, u_\varepsilon(\xi)\big)$ be a solution of $(\mathcal{P}_\varepsilon)$ with $\rho_\varepsilon(\xi) > 0$, $-\infty < \xi < \infty$, and proceed to provide sufficient conditions on $p(\rho)$ that guarantee the uniform (in ε) variation bounds required to apply Theorem 4.1. In this effort we are helped by the special form of $\big(\rho_\varepsilon(\xi),\, u_\varepsilon(\xi)\big)$. For, as noted in Lemma 3.2, either

(i) ρ_ε and u_ε are constant functions, or
(ii) ρ_ε and u_ε are strictly monotone, or
(iii) ρ_ε and u_ε belong to one of the following four disjoint subclasses $F_1 - F_4$:

- F_1 : ρ_ε has precisely one critical point which is a minimum at τ_ε, u_ε is strictly increasing;
- F_2 : ρ_ε is strictly decreasing, u_ε has precisely one critical point which is a maximum at τ_ε;
- F_3 : ρ_ε has precisely one critical point which is a maximum at τ_ε, u_ε is strictly decreasing;
- F_4 : ρ_ε is strictly increasing, u_ε has precisely one critical point which is a minimum at τ_ε.

Due to the restrictions on the shapes of ρ_ε and u_ε, it is sufficient to show that $u_\varepsilon(\xi)$ is uniformly bounded for solutions of class F_2 or F_4 and that $\rho_\varepsilon(\xi)$ is uniformly bounded for solutions of class F_3. Note that, since $\rho(\xi) > 0$, solutions of class F_1 are trivially uniformly bounded.

In the following lemma we record the analogs of (3.15), (3.16), (3.18) and (3.20). The proofs are similar and are omitted.

Lemma 4.2. *If $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ is of class F_1,*

$$\rho_\varepsilon(\xi) \ge \underline{\rho} - \frac{N}{|\xi - \tau_\varepsilon|}, \qquad \xi \in (-\infty,\infty)\backslash\{\tau_\varepsilon\}; \tag{4.8}$$

if $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ is of class F_2,

(4.9)
$$u_\varepsilon(\xi) \le \overline{u} + \frac{N}{|\xi - \tau_\varepsilon|}, \qquad \xi \in (-\infty,\infty)\backslash\{\tau_\varepsilon\};$$

if $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ *is of class* F_3,

(4.10)
$$\rho_\varepsilon(\xi) \le \overline{\rho} + \frac{N}{|\xi - \tau_\varepsilon|}, \qquad \xi \in (-\infty,\infty)\backslash\{\tau_\varepsilon\};$$

if $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ *is of class* F_4,

(4.11)
$$u_\varepsilon(\xi) \ge \underline{u} - \frac{N}{|\xi - \tau_\varepsilon|}, \qquad \xi \in (-\infty,\infty)\backslash\{\tau_\varepsilon\}.$$

The constants N *are independent of* ε, *while* $\underline{\rho} = \min\{\rho_-, \rho_+\}$, $\overline{\rho} = \max\{\rho_-, \rho_+\}$, $\underline{u} = \frac{1}{\rho_-}\min\{0, m_-, m_+\}$ *and* $\overline{u} = \frac{1}{\rho_+}\max\{0, m_-, m_+\}$.

Estimates (4.8)–(4.11) fail to bound the peaks of $\rho_\varepsilon(\xi)$ and $u_\varepsilon(\xi)$ and do not permit application of Theorem 4.1. Nevertheless, they serve as a starting point to establish the desired uniform bounds in Lemmas 4.3 and 4.4.

Lemma 4.3. *If* $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ *is of class* F_2 *or* F_4, *then*

(4.12)
$$\sup_{-\infty<\xi<\infty} |u_\varepsilon(\xi)| \le N,$$

where N *is a positive constant independent of* ε.

Proof. We consider the case that $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ is of class F_2. It suffices to bound from above the peak $u_\varepsilon(\tau_\varepsilon)$, which, with no loss of generality, is assumed positive.

Choose points $\zeta_\varepsilon \in [\tau_\varepsilon - 2, \tau_\varepsilon - 1]$ and $\xi_\varepsilon \in [\tau_\varepsilon + 1, \tau_\varepsilon + 2]$ such that

$$(\rho_+ - \rho_-) < \rho_\varepsilon'(\zeta_\varepsilon) < 0 \quad \text{and}$$
$$(\rho_+ - \rho_-) < \rho_\varepsilon'(\xi_\varepsilon) < 0.$$

Integrating (4.1) over $[\tau_\varepsilon, \xi_\varepsilon]$, we arrive at

(4.13)
$$\varepsilon\rho_\varepsilon'(\xi_\varepsilon) - \varepsilon\rho_\varepsilon'(\tau_\varepsilon) = \rho_\varepsilon(\xi_\varepsilon)u_\varepsilon(\xi_\varepsilon) - \rho_\varepsilon(\tau_\varepsilon)u_\varepsilon(\tau_\varepsilon) + \tau_\varepsilon\big(\rho_\varepsilon(\tau_\varepsilon) - \rho_\varepsilon(\xi_\varepsilon)\big) + \int_{\tau_\varepsilon}^{\xi_\varepsilon} \big(\rho_\varepsilon(\zeta) - \rho_\varepsilon(\xi_\varepsilon)\big)\, d\zeta.$$

Since $\rho_\varepsilon'(\tau_\varepsilon) < 0$, using (4.9), we deduce from (4.13)

$$\rho_\varepsilon(\tau_\varepsilon)u_\varepsilon(\tau_\varepsilon) \le (2+\varepsilon)(\rho_- - \rho_+) + \rho_-(\overline{u}+N) + |\tau_\varepsilon|\big(\rho_\varepsilon(\tau_\varepsilon) - \rho_\varepsilon(\xi_\varepsilon)\big). \tag{4.14}$$

If $|\tau_\varepsilon|$ were uniformly bounded, then (4.14) would yield (4.12). However, in general, this is not the case. To overcome this difficulty integrate (4.1) over $[\zeta_\varepsilon, \xi_\varepsilon]$ to arrive at

$$\begin{aligned}\varepsilon\rho_\varepsilon'(\xi_\varepsilon) - \varepsilon\rho_\varepsilon'(\zeta_\varepsilon) = \rho_\varepsilon(\xi_\varepsilon)u_\varepsilon(\xi_\varepsilon) - \rho_\varepsilon(\zeta_\varepsilon)u_\varepsilon(\zeta_\varepsilon) + \zeta_\varepsilon\big(\rho_\varepsilon(\zeta_\varepsilon) - \rho_\varepsilon(\xi_\varepsilon)\big)\\ + \int_{\zeta_\varepsilon}^{\xi_\varepsilon} [\rho_\varepsilon(\zeta) - \rho_\varepsilon(\xi_\varepsilon)]\,d\zeta.\end{aligned} \tag{4.15}$$

Combining (4.15) with (4.9), we deduce

$$\begin{aligned}0 &< |\tau_\varepsilon|\big(\rho_\varepsilon(\zeta_\varepsilon) - \rho_\varepsilon(\xi_\varepsilon)\big)\\ &\le (6+2\varepsilon)(\rho_- - \rho_+) + 2\rho_-(\overline{u}+N),\end{aligned} \tag{4.16}$$

which, together with (4.14), implies (4.12) for solutions of class F_2. The proof for solutions of class F_4 is analogous. □

Lemma 4.4. *Let $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ be a solution of class F_3. If*

$$\int_1^\infty \frac{p'(s)}{s}\,ds = \infty, \tag{4.17}$$

then

$$\sup_{-\infty<\xi<\infty} \rho_\varepsilon(\xi) \le N, \tag{4.18}$$

where N is a positive constant independent of ε.

Proof. Let $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ be a solution of class F_3; τ_ε denotes the point where $\rho_\varepsilon(\xi)$ achieves its maximum. It suffices to show that the peak $\rho_\varepsilon(\tau_\varepsilon)$ is uniformly bounded.

Choose $\xi_\varepsilon \in [\tau_\varepsilon + 1, \tau_\varepsilon + 2]$ such that

$$\begin{aligned}u_+ - u_- &< u_\varepsilon(\tau_\varepsilon + 2) - u_\varepsilon(\tau_\varepsilon + 1)\\ &= u_\varepsilon'(\xi_\varepsilon) < 0.\end{aligned} \tag{4.19}$$

Using (4.1) and (4.2), we obtain

$$\varepsilon u'' + 2\varepsilon\frac{\rho' u'}{\rho} = -\xi u' + \left(\frac{u^2}{2}\right)' + \frac{p'(\rho)}{\rho}\rho'$$

which, once integrated over $[\tau_\varepsilon, \xi_\varepsilon]$, yields

$$\varepsilon u_\varepsilon'(\xi_\varepsilon) - \varepsilon u_\varepsilon'(\tau_\varepsilon) + 2\varepsilon \int_{\tau_\varepsilon}^{\xi_\varepsilon} \frac{\rho_\varepsilon'(\zeta) u_\varepsilon'(\zeta)}{\rho_\varepsilon(\zeta)}\, d\zeta = \frac{1}{2}\left(u_\varepsilon^2(\xi_\varepsilon) - u_\varepsilon^2(\tau_\varepsilon)\right) - \int_{\tau_\varepsilon}^{\xi_\varepsilon} \zeta u_\varepsilon'(\zeta)\, d\zeta + \int_{\rho_\varepsilon(\tau_\varepsilon)}^{\rho_\varepsilon(\xi_\varepsilon)} \frac{p'(s)}{s}\, ds. \tag{4.20}$$

Since $u_\varepsilon'(\tau_\varepsilon) < 0$ and $(\rho_\varepsilon' u_\varepsilon')/(\rho_\varepsilon) > 0$ on $(\tau_\varepsilon, \infty)$, we obtain from (4.20)

$$\int_{\rho_+}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{p'(s)}{s} ds \leq -\varepsilon u_\varepsilon'(\xi_\varepsilon) + \frac{1}{2}\left(u_\varepsilon^2(\xi_\varepsilon) - u_\varepsilon^2(\tau_\varepsilon)\right) + \tau_\varepsilon\left(u_\varepsilon(\tau_\varepsilon) - u_\varepsilon(\xi_\varepsilon)\right) + \int_{\tau_\varepsilon}^{\xi_\varepsilon} \left(u_\varepsilon(\zeta) - u_\varepsilon(\xi_\varepsilon)\right) d\zeta . \tag{4.21}$$

Finally, using (4.19) together with the monotonicity properties of u_ε, (4.21) yields the bound

$$\int_{\rho_+}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{p'(s)}{s} ds \leq \left(2 + \frac{1}{2}(|u_-| + |u_+|) + \varepsilon\right)(u_- - u_+) + |\tau_\varepsilon|(u_- - u_+). \tag{4.22}$$

If τ_ε were uniformly bounded, say $|\tau_\varepsilon| \leq \tau_0$, then (4.22) together with (4.17) would provide (4.18). In what follows we complement (4.22) with corresponding bounds for $|\tau_\varepsilon|$ large.

Suppose first that $\tau_\varepsilon > \tau_0 > 0$, with τ_0 a large positive threshold to be determined later. Integration of (4.1) over $[\tau_\varepsilon, \infty)$ gives

$$0 = \rho_+ u_+ - \rho_\varepsilon(\tau_\varepsilon) u_\varepsilon(\tau_\varepsilon) + \tau_\varepsilon(\rho_\varepsilon(\tau_\varepsilon) - \rho_+) + \int_{\tau_\varepsilon}^{\infty} (\rho_\varepsilon(\zeta) - \rho_+)\, d\zeta. \tag{4.23}$$

Since $\rho_\varepsilon(\zeta) > \rho_+$ for $\zeta \in [\tau_\varepsilon, \infty)$, (4.23) leads to

$$\rho_\varepsilon(\tau_\varepsilon)\left(1 - \frac{u_\varepsilon(\tau_\varepsilon)}{\tau_\varepsilon}\right) < \rho_+\left(1 - \frac{u_+}{\tau_\varepsilon}\right). \tag{4.24}$$

Moreover, since $u_+ < u_\varepsilon(\tau_\varepsilon) < u_-$, we can choose τ_0 such that $1 - (u_\varepsilon(\tau_\varepsilon))/\tau_\varepsilon > \frac{1}{2}$ for $\tau_\varepsilon > \tau_0 > 0$. An appropriate choice is $\tau_0 := 2\max\{|u_-|, |u_+|\}$. Then (4.24) implies (4.18) for $\tau_\varepsilon > \tau_0 > 0$.

In case $\tau_\varepsilon < -\tau_0 < 0$, a similar argument gives

$$\rho_\varepsilon(\tau_\varepsilon)\left(1 - \frac{u_\varepsilon(\tau_\varepsilon)}{\tau_\varepsilon}\right) < \rho_-\left(1 - \frac{u_-}{\tau_\varepsilon}\right), \tag{4.25}$$

which, in turn, yields (4.18) for $\tau_\varepsilon < -\tau_0 < 0$. This completes the proof of the lemma. □

We summarize the results of Lemmas 4.3 and 4.4 together with Theorem 4.1 into the following existence theorem for the Riemann problem $(\mathcal{P})$:

Theorem 4.5. *Assume that $p'(\rho) > 0$ for $\rho > 0$ and $\int_1^\infty \frac{p'(s)}{s}\,ds = \infty$. Given any ρ_-, $\rho_+ > 0$, there exists a pair of functions of bounded variation $\big(\rho(\xi), u(\xi)\big)$, defined on $(-\infty,\infty)$, with $\rho(\xi) \geq 0$, such that $\big(\rho(\frac{x}{t}), u(\frac{x}{t})\big)$ is a weak solution of the Riemann problem $(\mathcal{P})$.*

Note that in Theorem 4.5 no genuine nonlinearily assumption is required. In case the characteristic fields of (1.1) , (1.2) are genuinely nonlinear, we employ a different approach to obtain uniform bounds. This approach is motivated by the geometry of the shock and wave curves and parallels Lemma 4.1 in Dafermos and DiPerna [3].

Lemma 4.6. *Let $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ denote a solution of $(\mathcal{P}_\varepsilon)$ with $\rho_\varepsilon(\xi) > 0$. If, in addition,*

$$\frac{d}{d\rho}\big(\rho^2 p'(\rho)\big) > 0 \qquad \text{for } \rho > 0, \tag{4.26}$$

then the following hold:

(i) *If $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ is of class F_2,*

$$u_\varepsilon(\tau_\varepsilon) < u_- + \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(s)}}{s}\,ds. \tag{4.27}$$

(ii) *If $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ is of class F_3,*

$$\int_{\rho_\pm}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\,ds < \pm\big(u_\varepsilon(\tau_\varepsilon) - u_\pm\big). \tag{4.28}$$

(iii) *If $\big(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\big)$ is of class F_4,*

$$u_\varepsilon(\tau_\varepsilon) > u_+ - \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(s)}}{s}\,ds. \tag{4.29}$$

Proof. We prove (i) and (ii). The proof of (iii) is similar to the proof of (i) and is omitted.

Let $\xi_\varepsilon(\rho)$ denote the inverse of the function $\rho_\varepsilon(\xi)$. For solutions of class F_2 or F_4, $\rho_\varepsilon(\xi)$ is monotone and the inverse is well defined. By contrast, for solutions of class F_3, $\rho_\varepsilon(\xi)$ has a critical point at τ_ε and the inverse is well defined only in the semi-infinite intervals $(-\infty,\tau_\varepsilon]$ or $[\tau_\varepsilon,\infty)$.In what follows we work in either $(-\infty,\tau_\varepsilon]$ or $[\tau_\varepsilon,\infty)$ and $\xi_\varepsilon(\rho)$ will stand for the appropriate component of the inverse. By means of $\rho_\varepsilon(\xi)$ and $\xi_\varepsilon(\rho)$ one can change independent variables from ξ to ρ and vice versa. Thus, u_ε may be visualized as $u_\varepsilon(\xi)$ or $u_\varepsilon(\rho)$. Accordingly, m_ε is visualized as $m_\varepsilon(\xi) = \rho_\varepsilon(\xi)u_\varepsilon(\xi)$ or $m_\varepsilon(\rho) = \rho u_\varepsilon(\rho)$, respectively. With no danger of confusion we keep the same notation for both parametrizations. Using (4.1) and (4.2), a straightforward computation yields

$$\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) = \frac{m_\varepsilon'(\xi)}{\rho_\varepsilon'(\xi)}. \tag{4.30}$$

$$\begin{aligned} \varepsilon\frac{d}{d\xi}\left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right) &= \varepsilon\frac{m_\varepsilon''(\xi)\rho_\varepsilon'(\xi) - \rho_\varepsilon''(\xi)m_\varepsilon'(\xi)}{\rho_\varepsilon'^2(\xi)} \\ &= p'(\rho_\varepsilon(\xi)) - \left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) - \frac{m_\varepsilon(\xi)}{\rho_\varepsilon(\xi)}\right)^2, \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} &\varepsilon\frac{d^2}{d\xi^2}\left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right) \\ &= \left[p''(\rho_\varepsilon(\xi)) + \frac{2}{\rho_\varepsilon(\xi)}\left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) - \frac{m_\varepsilon(\xi)}{\rho_\varepsilon(\xi)}\right)^2\right]\rho_\varepsilon'(\xi) \\ &\quad - 2\left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) - \frac{m_\varepsilon(\xi)}{\rho_\varepsilon(\xi)}\right)\frac{d}{d\xi}\left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right). \end{aligned} \tag{4.32}$$

Also, observe that

$$\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) - \frac{m_\varepsilon(\xi)}{\rho_\varepsilon(\xi)} = \rho_\varepsilon(\xi)\frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)). \tag{4.33}$$

With the remarks above in order, we proceed to establish (4.27). Here, $\rho_\varepsilon(\xi)$ is decreasing, while $u_\varepsilon(\xi)$ attains a maximum at τ_ε. Let

$$\xi_1 = \inf\left\{\zeta \in (-\infty,\tau_\varepsilon] : p'(\rho_\varepsilon(\xi)) - \left(\rho_\varepsilon(\xi)\frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right)^2 > 0 \quad \text{for } \zeta < \xi \le \tau_\varepsilon\right\}.$$

Clearly, ξ_1 is well defined and $-\infty \leq \xi_1 < \tau_\varepsilon$. If $\xi_1 > -\infty$, then, by (4.31) and (4.33), it follows that $\frac{d}{d\xi}\big((dm_\varepsilon/d\rho)(\rho_\varepsilon(\xi))\big) = 0$ at ξ_1, while $(d^2/d\xi^2) \times \big((dm_\varepsilon/d\rho)(\rho_\varepsilon(\xi))\big) \geq 0$ at ξ_1. But (4.32) implies that

$$\varepsilon \frac{d^2}{d\xi^2}\left(\frac{dm_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right)\bigg|_{\xi=\xi_1} = \left(p''(\rho_\varepsilon(\xi_1)) + \frac{2}{\rho_\varepsilon(\xi_1)}p'(\rho_\varepsilon(\xi_1))\right)\rho'_\varepsilon(\xi_1) < 0, \tag{4.34}$$

where in (4.34) we used the monotonicity properties of ρ_ε together with (4.26). Hence, we conclude that $\xi_1 = -\infty$ and that

$$-\frac{\sqrt{p'(\rho_\varepsilon(\xi)}}{\rho_\varepsilon(\xi)} < \frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) < 0 \quad \text{for } -\infty < \xi < \tau_\varepsilon. \tag{4.35}$$

By virtue of (4.35),

$$u_\varepsilon(\tau_\varepsilon) = u_- + \int_{-\infty}^{\tau_\varepsilon} \frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\rho'_\varepsilon(\xi)\,d\xi \leq u_- + \int_{\rho_\varepsilon(\tau_\varepsilon)}^{\rho_-} \frac{\sqrt{p'(s)}}{s}\,ds, \tag{4.36}$$

which leads to (4.27).

We now take up the case that $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ is of class F_3, i.e., $u_\varepsilon(\xi)$ is decreasing, while $\rho_\varepsilon(\xi)$ has a maximum at τ_ε. Set

$$\xi_1 = \inf\left\{\zeta \in (-\infty, \tau_\varepsilon] : p'(\rho_\varepsilon(\xi)) - \left(\rho_\varepsilon(\xi)\frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right)^2 < 0 \quad \text{for } \zeta < \xi \leq \tau_\varepsilon\right\}$$

and

$$\xi_2 = \sup\left\{\zeta \in [\tau_\varepsilon, \infty) : p'(\rho_\varepsilon(\xi)) - \left(\rho_\varepsilon(\xi)\frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi))\right)^2 < 0 \quad \text{for } \tau_\varepsilon \leq \xi < \zeta\right\}.$$

Since $\rho'_\varepsilon(\tau_\varepsilon) = 0$, the sets above are nonempty and $-\infty \leq \xi_1 < \tau_\varepsilon$, $\tau_\varepsilon < \xi_2 \leq \infty$. An argument similar to the one leading to the proof of (4.35) shows that $\xi_1 = -\infty$ and $\xi_2 = \infty$. Thus, taking account of the monotonicity properties of $u_\varepsilon(\xi)$ and $\rho_\varepsilon(\xi)$, we conclude

$$\frac{u'_\varepsilon(\xi)}{\rho'_\varepsilon(\xi)} = \frac{du_\varepsilon}{d\rho}(\rho_\varepsilon(\xi)) < -\frac{\sqrt{p'(\rho_\varepsilon(\xi))}}{\rho_\varepsilon(\xi)} < 0, \quad \text{for } -\infty < \xi < \tau_\varepsilon, \tag{4.37}$$

while

$$\frac{u_\varepsilon'(\xi)}{\rho_\varepsilon'(\xi)} = \frac{du_\varepsilon}{d\rho}\big(\rho_\varepsilon(\xi)\big) > +\frac{\sqrt{p'\big(\rho_\varepsilon(\xi)\big)}}{\rho_\varepsilon(\xi)} > 0, \tag{4.38}$$

for $\tau_\varepsilon < \xi < \infty$.

Finally, using (4.37),

$$\begin{aligned} u_\varepsilon(\tau_\varepsilon) - u_- &= \int_{-\infty}^{\tau_\varepsilon} u_\varepsilon'(\xi)\, d\xi \\ &< -\int_{-\infty}^{\tau_\varepsilon} \frac{\sqrt{p'\big(\rho_\varepsilon(\xi)\big)}}{\rho_\varepsilon(\xi)} \rho_\varepsilon'(\xi)\, d\xi \\ &= -\int_{\rho_-}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\, ds; \end{aligned} \tag{4.39}$$

similarly, using (4.38),

$$\begin{aligned} u_+ - u_\varepsilon(\tau_\varepsilon) &= \int_{\tau_\varepsilon}^{\infty} u_\varepsilon'(\xi)\, d\xi \\ &< \int_{\tau_\varepsilon}^{\infty} \frac{\sqrt{p'\big(\rho_\varepsilon(\xi)\big)}}{\rho_\varepsilon(\xi)} \rho_\varepsilon'(\xi)\, d\xi \\ &= -\int_{\rho_+}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\, ds; \end{aligned} \tag{4.40}$$

and the proof of the lemma is complete. □

Relations (4.27)–(4.29) give rise to ε-independent bounds for solutions of $(\mathcal{P}_\varepsilon)$. Since ρ_-, $\rho_+ > 0$, (4.27) and (4.29) imply that $u_\varepsilon(\tau_\varepsilon)$ is uniformly bounded for solutions of class F_2 and F_4. In addition, (4.28) yields

$$\int_{\rho_-}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\, ds + \int_{\rho_+}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\, ds < u_- - u_+. \tag{4.41}$$

Provided

$$\int_{\min\{\rho_-,\rho_+\}}^{\infty} \frac{\sqrt{p'(s)}}{s}\, ds > \frac{u_- - u_+}{2}, \tag{4.42}$$

(4.41) implies that $\rho_\varepsilon(\tau_\varepsilon)$ is uniformly bounded for solutions of class F_3. This last hypothesis can be viewed as a restriction on the Riemann data; it is certainly satisfied for all data if

$$\int_1^\infty \frac{\sqrt{p'(s)}}{s}\,ds = \infty. \tag{4.43}$$

The remarks above lead to an alternative existence theorem for the Riemann problem $(\mathcal{P})$; the simplest set of sufficient hypotheses being (1.3), (4.26) and (4.43).

The hypotheses on $p(\rho)$ may be reformulated in terms of dependence on the specific volume $v = \frac{1}{\rho}$. For if we set $\bar{p}(v) = p(\frac{1}{v})$, the hypotheses (1.3), (4.26), (4.17) and (4.43) readily translate into $\bar{p}'(v) < 0$ for $v < 0$, $\bar{p}''(v) > 0$ for $v > 0$, $-\int_0^1 v\bar{p}'(v)\,dv = +\infty$ and $\int_0^1 \sqrt{-\bar{p}'(v)}dv = +\infty$, respectively. Finally, note that (1.3), (4.26), (4.17) and (4.43) are satisfied by the equation of state $p(\rho) = k\rho^\gamma, \gamma \geq 1$.

5. Riemann Data Containing a Vacuum State. In their study of the vacuum problem for the equations of isentropic gas dynamics, Liu and Smoller [15] use the shock and wave curves for (1.1)–(1.2) to solve the Riemann problem when one of the states in the Riemann data is a vacuum state. We consider this problem from the viewpoint of the method we are studying here.

For concreteness, let $\rho_- > 0$, $\rho_+ = 0$ and consider the problem $(\mathcal{P})$ with Riemann data (ρ_-, u_-) and $(0, u_+)$. Our goal is to construct solutions of $(\mathcal{P})$ as $\varepsilon \searrow 0$ limits of $\left(\rho_\varepsilon(\frac{x}{t}), u_\varepsilon(\frac{x}{t})\right)$, where $\left(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\right)$ solve (4.1)–(4.2) with boundary conditions

$$\begin{aligned} \rho(-\infty) &= \rho_- \qquad & \rho(+\infty) &= \rho_+(\varepsilon) \\ u(-\infty) &= u_- \qquad & u(+\infty) &= u_+ \end{aligned} \tag{5.1}$$

with $\rho_+(\varepsilon) > 0$ for $\varepsilon > 0$ and $\lim_{\varepsilon\searrow 0}\rho_+(\varepsilon) = 0$. Under (1.3), the boundary value problem (4.1), (4.2) and (5.1) admits, for each fixed $\varepsilon > 0$, a solution $\left(\rho_\varepsilon(\xi), u_\varepsilon(\xi)\right)$ defined for $-\infty < \xi < \infty$, with the property $\rho_\varepsilon(\xi) > 0$ for $-\infty < \xi < \infty$.

In the sequel, we seek conditions on $p(\rho)$ that guarantee that the family of functions $\{\rho_\varepsilon(\cdot), u_\varepsilon(\cdot);\ 0 < \varepsilon \leq 1\}$ is uniformly bounded and of uniformly bounded variation. Then, Theorem 4.1 asserts that there is a subsequence of functions $\left(\rho_{\varepsilon_n}(\xi), u_{\varepsilon_n}(\xi)\right)$, $n = 1, 2, \ldots$, and a pair of functions of bounded variation $\left(\rho(\xi), u(\xi)\right)$, with $\rho(\xi) \geq 0$, such that $\left(\rho_{\varepsilon_n}(\frac{x}{t}), u_{\varepsilon_n}(\frac{x}{t})\right) \to \left(\rho(\frac{x}{t}), u(\frac{x}{t})\right)$ for a.e. $(x,t) \in (-\infty,\infty) \times (0,\infty)$ and $\left(\rho(\frac{x}{t}), u(\frac{x}{t})\right)$ is a weak solution of (1.1)–(1.2). Moreover, there is a constant α, such that $\rho(\xi) = \rho_-$, $m(\xi) = m_-$,

$u(\xi) = m_-/\rho_- = u_-$ for $\xi < -2\alpha$ and $\rho(\xi) = 0$, $m(\xi) = 0$ for $\xi > 2\alpha$. However, the value of $u(\xi)$ for ξ large is not determined by this argument. In particular,

$$\lim_{t\searrow 0}\rho\left(\frac{x}{t}\right) = \begin{cases}\rho_-, & x<0,\\ 0, & x>0,\end{cases} \quad \lim_{t\searrow 0} m\left(\frac{x}{t}\right) = \begin{cases}\rho_- u_-, & x<0,\\ 0, & x>0,\end{cases}$$

and $\lim_{t\searrow 0} u\left(\frac{x}{t}\right) = u_-$ for $x < 0$, but is not determined for $x > 0$.

We claim that if $p(\rho)$ satisfies (1.3), (4.26), (4.43) and

$$\int_0^{\rho_-} \frac{\sqrt{p'(s)}}{s}\, ds < +\infty, \tag{5.2}$$

then $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ are uniformly bounded and of uniformly bounded variation for ε small. Indeed, as long as $\rho_+(\varepsilon) < \rho_-$, either (i) ρ_ε is strictly decreasing and u_ε is strictly monotone, or (ii) ρ_ε and u_ε belong to one of the classes F_1, F_2 or F_3. By Lemma 4.6, if $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ is of class F_2, then

$$u_\varepsilon(\tau_\varepsilon) < u_- + \int_0^{\rho_-} \frac{\sqrt{p'(s)}}{s}\, ds, \tag{5.3}$$

while, if $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ is of class F_3,

$$\int_{\rho_-}^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\, ds + \int_0^{\rho_\varepsilon(\tau_\varepsilon)} \frac{\sqrt{p'(s)}}{s}\, ds < u_- - u_+. \tag{5.4}$$

In view of (4.43) and (5.2) the desired bounds are true in all cases.

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