WAVE INTERACTIONS AND VARIATION ESTIMATES FOR SELF-SIMILAR VISCOUS LIMITS IN SYSTEMS OF CONSERVATION LAWS

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Abstract. We consider the problem of self-similar viscous limits for systems of N conservation laws. First, we give general conditions so that the resulting boundary value problem admits solutions. The obtained existence theory covers a large class of systems; in particular the class of symmetric hyperbolic systems. Second, we show that if the system is strictly hyperbolic and the Riemann data are sufficiently close then the resulting family of solutions is of uniformly bounded variation and oscillation. Third, we construct solutions of the Riemann problem via self-similar viscous limits and study the structure of the emerging solution and the relation of self-similar viscous limits and shock profiles. The emerging solution consists of N wave fans separated by constant states. Each wave fan is associated with one of the characteristic fields and consists of a rarefaction, a shock, or an alternating sequence of shocks and rarefactions so that each shock adjacent to a rarefaction on one side is a contact discontinuity on that side. At shocks, the solutions of the self-similar viscous problem have the internal structure of a traveling wave.

1. Introduction

Consider the system of conservation laws in one space dimension,

$$\partial_t U + \partial_x F(U) = 0 \tag{1.1}$$

where $x \in \mathbb{R}$, $t > 0$, $U(x,t)$ takes values in $\mathbb{R}^N$ and the flux function $F : \mathbb{R}^N \to \mathbb{R}^N$ is assumed smooth. If the matrix $\nabla F(U)$ has real and distinct eigenvalues then (1.1) is called strictly hyperbolic and its eigenvalues (called characteristic speeds) may be ordered :

$$\lambda_1(U) < \lambda_2(U) < \dots < \lambda_N(U)\,. \tag{1.2}$$

Let $r_1(U)\,, \dots, r_N(U)$ and $l_1(U)\,, \dots, l_N(U)$ be the corresponding right and left eigenvectors. They are linearly independent and form a pair of local bases in the state space.

The Riemann problem consists of solving (1.1) with initial data a single jump discontinuity

$$U(x,0) = \begin{cases} U_- & x < 0 \\ U_+ & x > 0 \end{cases} \tag{1.3}$$

It describes the local structure of BV solutions at points of shock interactions (DiPerna [Dp], Liu [Li_4]) and serves as a building block for solving the Cauchy problem via the Glimm scheme (Glimm [G]). In solving the Riemann problem one encounters loss of uniqueness that has to be accounted for by imposing admissibility restrictions on solutions. For weak waves in strictly hyperbolic systems it suffices to impose such restrictions only at shocks. Lax [La_1] in the genuinely nonlinear case and Liu [Li_1, Li_2] in the general case provided comprehensive shock-admissibility criteria and obtained a unique solution of (1.1), (1.3) for weak waves. The reader is referred to Dafermos [D_3] for a thorough discussion of the issue of admissibility. The solution of the Riemann problem is based on the invariance of (1.1), (1.3) under dilations of the independent variables $(x,t) \mapsto (\alpha x\,, \alpha t)$, for $\alpha > 0$. Because of the expected uniqueness, one seeks for solutions $U = U(\xi)$ functions of the single variable $\xi = \frac{x}{t}$. The function U is a weak solution of the boundary value problem $(\mathcal{P})$

$$(\mathcal{P}) \qquad \begin{aligned} -\xi\, U' + F(U)' &= 0 \\ U(\pm\infty) &= U_\pm \end{aligned}$$

subject to admissibility conditions on shocks. The classical solution of $(\mathcal{P})$ proceeds in two steps: First special solutions of rarefaction waves, shock waves or contact discontinuities are studied, and are used to construct the elementary wave curves. There is one elementary curve associated with each characteristic field with the parametrization of the curve serving as a measure of the strength of the associated wave. Second, it is shown that the compound curves emanating from a fixed left state U_- give rise to an invertible map that covers a full neighborhood of right end states U_+ (*c.f.* [La_1, Li_4]).

The objective of this article is to obtain the complete solution of the Riemann problem for weak waves by an alternative approach, in the spirit of viscosity methods. Namely, admissible solutions of $(\mathcal{P})$ are constructed as $\varepsilon \searrow 0$ limits of solutions to the problem $(\mathcal{P}_\varepsilon)$

$$(\mathcal{P}_\varepsilon) \qquad \begin{aligned} -\xi\, U' + F(U)' &= \varepsilon U'' \\ U(\pm\infty) &= U_\pm\,, \end{aligned}$$

with $\varepsilon > 0$. The latter consists of an elliptic regularization of the Riemann operator in $(\mathcal{P})$. This approach was proposed by Dafermos [D_1], who motivated it by introducing an artificial "viscosity" regularization that preserves the invariance under dilations of coordinates. Solutions of $(\mathcal{P})$ are thus constructed as self-similar viscous limits, and the study of the Riemann problem amounts to performing the following steps:

(i) To construct solutions of the problem $(\mathcal{P}_\varepsilon)$, with $\varepsilon > 0$ fixed.

(ii) To construct solutions of $(\mathcal{P})$ as $\varepsilon \searrow 0$ limits of solutions of $(\mathcal{P}_\varepsilon)$.

(iii) To study the structure of the emerging solution.

Our interest in $(\mathcal{P}_\varepsilon)$ stems from the connection with the problem of viscous limits. For the system of viscous conservation laws

$$\partial_t U + \partial_x F(U) = \varepsilon\, \partial_x^2 U \tag{1.4}$$

subject to Riemann data, the invariance under dilations $(x,t) \mapsto (\alpha x, \alpha t)$, $\alpha > 0$, no longer holds. It is a simple calculation to see that the solution U^ε of $(1.3 - 1.4)$ can be expressed as

$$U^\varepsilon(x,t) = V(\frac{x}{t}, -\frac{\varepsilon}{t}) \tag{1.5}$$

where $V(\xi, s)$ satisfies

$$V_s - V_{\xi\xi} = \frac{1}{s}\big(- \xi V_\xi + F(V)_\xi \big) \tag{1.6}$$

for $-\infty < \xi < \infty$, $-\infty < s < 0$. Therefore, the viscous limit problem for Riemann data is a two parameter problem and studying the limit of U^ε as $\varepsilon \downarrow 0$ amounts to studying the limit of $V(\xi, s)$ as $s \uparrow 0-$. The problem $(\mathcal{P}_\varepsilon)$ arises when replacing the parabolic operator in (1.6) by an elliptic operator; its study is expected to provide insight on the difficult problem of viscous limits. The two regularizations have been compared for Burgers' equation (Slemrod [S_2]).

The notion of self-similar viscous limits appears in the articles [Ka], [Tu_1], [Tu_2], [D_1]. Tupciev [Tu_1, Tu_2] used them to formally motivate a shock admissibility condition for the Riemann problem, that amounts to the requirement that admissible shocks have associated shock profiles. The direct use of self-similar viscous limits is initiated by Dafermos [D_1, D_2] who proposed it as an admissibility criterion and devised a versatile framework for treating the analytical aspects of the problem. The approach has been tried on several examples of strictly hyperbolic 2×2 systems [D_1, DDp, KKr, STz_1, Tz_2], on a system of two equations that exhibits change of type [S_1, Fa_2], and on the fluid dynamic limit for the Broadwell model [STz_2, Tz_1]. It has been established at the level of such examples [D_2, Fa_1, Tz_2] that self-similar limits yield the same structure for the solution of the Riemann problem as the structure obtained by the shock-admissibility criteria of Lax [La_1] and Liu [Li_2], or by requesting that each admissible shock has an associated viscous shock profile. In contrast to most admissibility criteria, self-similar viscous limits penalize the whole wave-fan simultaneously. Based on that fact, a fitting terminology would be to call admissibility via self-similar viscous limits as the *viscous wave-fan admissibility criterion.*

Here, we pursue the method for strictly hyperbolic systems of more than two equations. We address the questions of existence, performing the $\varepsilon \to 0$ limit, and structure of the emerging solution. The key step lies in controlling the diffusion induced wave interactions and obtaining uniform variation estimates for solutions of $(\mathcal{P}_\varepsilon)$. The article is organized as follows:

In Section 2 we study the question of existence of solutions for $(\mathcal{P}_\varepsilon)$. We show that for any system equipped with an L^p estimate the problem $(\mathcal{P}_\varepsilon)$ admits solutions for each $\varepsilon > 0$. The analysis applies to the class of symmetric hyperbolic systems.

Sections 3 to 7 are the core of the article dealing with the question of obtaining uniform variation estimates for families of solutions to $(\mathcal{P}_\varepsilon)$. Even for Riemann data, waves of different families can interact through diffusion and contribute to the total variation. Therefore, one has to devise a scheme for measuring the variation of the solution (through the individual waves) and to calculate the effects of wave interactions. We refer to Section 3, which serves as an introduction to this part, for an outline of our strategy. The outcome is summarized in Theorem 3.1 and states that if (1.1) is strictly hyperbolic and the data $U_\pm$ are such that $|U_+ - U_-|$ is small, then $(\mathcal{P}_\varepsilon)$ has solutions that are of uniformly bounded (and small) oscillation and variation.

In Sections 8, 9 and 10 we develop an existence theory for the Riemann problem (1.1), (1.3) for strictly hyperbolic systems via self-similar viscous limits. Our approach differs from the existence theories of Lax [La$_1$] and Liu [Li$_1$, Li$_2$] in that it is analytical in nature and bypasses the construction (and hypotheses required thereto) of the wave curves. The variation estimates of Section 7 are used in Section 8 to establish the $\varepsilon \to 0$ limit and, more important, to study the structure of the emerging solution U of $(\mathcal{P})$. The existence result, Theorem 8.1, states that the Riemann problem is solvable under the sole hypotheses that (1.1) is strictly hyperbolic and $|U_+ - U_-|$ is small. The emerging solution U consists of N wave fans separated by constant states. Each wave fan is associated with one of the characteristic fields and is either a rarefaction, or a shock satisfying a weak form of the Lax conditions, or a composite wave consisting of an alternating sequence of shocks and rarefactions so that each shock adjacent to a rarefaction on one side is a contact discontinuity on that side. In Section 9 it is shown that for shocks that do not correspond to linearly degenerate characteristic fields solutions of $(\mathcal{P}_\varepsilon)$ have the internal structure of traveling waves. In Section 10 we compare the solution obtained via self-similar limits to the classical solution of the Riemann problem for genuinely nonlinear systems [La$_1$] or for general strictly hyperbolic systems [Li$_1$, Li$_2$]. In both cases the same structure results for the Riemann solution. The relation with the Liu shock admissibility criterion is indirect, and follows from the fact that (a strict version of) the Liu shock-admissibility criterion is equivalent to the requirement that admissible shocks have associated shock profiles (Liu [Li3], Majda and Pego [MP]).

2. Existence of Connecting Trajectories for $(\mathcal{P}_\varepsilon)$

The objective of this section is to construct solutions of the problem $(\mathcal{P}_\varepsilon)$, for ε positive fixed. $(\mathcal{P}_\varepsilon)$ is a boundary-value problem for a system of non-autonomous ordinary differential equations. First, it is shown that L^∞ estimates are sufficient to establish existence of solutions for $(\mathcal{P}_\varepsilon)$. Then a construction scheme, originally proposed by Dafermos [D1], is presented in Section 2.2. Existence of connecting trajectories then relies on a priori estimates, which are established in Section 2.3 under various structural hypotheses on (1.1). Most notably, the analysis applies to the class of symmetric hyperbolic systems.

2.1. Preliminaries. Assume that U is a classical solution of $(\mathcal{P}_\varepsilon)$ satisfying the bound

$$\sup_{-\infty<\xi<\infty} |U(\xi)| \le M \,, \tag{2.1}$$

where M is a constant that may depend on ε. Integrating the differential equation

$$\varepsilon\, U'' = -\xi\, U' + F(U)' \tag{2.2}$$

it is easily seen that U satisfies the identities

$$U'(\xi) = U'(0)\, e^{-\frac{\xi^2}{2\varepsilon}} + \frac{1}{\varepsilon}\, e^{-\frac{\xi^2}{2\varepsilon}} \int_0^\xi e^{\frac{\zeta^2}{2\varepsilon}} \nabla F(U(\zeta))\, U'(\zeta)\, d\zeta \,, \tag{2.3}$$

and

$$\varepsilon\, U'(\xi) = \varepsilon\, U'(0) + F(U(\xi)) - \xi\, U(\xi) - F(U(0)) + \int_0^\xi U(\zeta)\, d\zeta \,. \tag{2.4}$$

Using (2.1), (2.3) and Gronwall's inequality, we obtain

$$|U'(\xi)| \le |U'(0)|\, e^{(2\alpha|\xi| - \xi^2)/2\varepsilon} \,, \tag{2.5}$$

where $\alpha := \sup_{|V|\le M} |\nabla F(V)|$.

Integrating (2.3) over $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ and performing a change of variables in the resulting integrals, we arrive at the identity

$$\begin{aligned} U'(0) \int_{-1}^1 e^{-\xi^2/2} d\xi &= \frac{1}{\sqrt{\varepsilon}} \big(U(\sqrt{\varepsilon}) - U(-\sqrt{\varepsilon}) \big) \\ &\quad + \frac{1}{\varepsilon} F(U(0)) \int_{-1}^1 e^{-\xi^2/2} d\xi - \frac{1}{\varepsilon} \int_{-1}^1 F\big(U(\sqrt{\varepsilon}\xi)\big)\, d\xi \\ &\quad + \frac{1}{\varepsilon} \int_{-1}^1 \int_0^\xi \zeta e^{(\zeta^2-\xi^2)/2}\, F\big(U(\sqrt{\varepsilon}\zeta)\big)\, d\zeta d\xi \,. \end{aligned} \tag{2.6}$$

In turn, this leads to

$$
\begin{aligned}
|U'(0)| \int_{-1}^{1} e^{-\xi^2/2} d\xi &\le \frac{1}{\varepsilon}\Big[\, \sqrt{\varepsilon}\, |U(\sqrt{\varepsilon}) - U(-\sqrt{\varepsilon})| \\
&\qquad + \sup_{-1\le\xi\le 1} |F(U(\sqrt{\varepsilon}\xi))| \left(4 + 2\int_0^1 \int_0^{\xi} \zeta e^{(\zeta^2-\xi^2)/2}\, d\zeta d\xi\,\right)\Big] \\
&\le \frac{6}{\varepsilon}\left(M + \sup_{|V|\le M} |F(V)|\right).
\end{aligned} \tag{2.7}
$$

On the other hand $(2,1)$, (2.4) and (2.7) give

$$
|U'(\xi)| \le \frac{C}{\varepsilon}\,(1+|\xi|)\,. \tag{2.8}
$$

Relations (2.5), (2.7) and (2.8) imply that any solution obeying the bound (2.1) will also satisfy the first derivative estimates

$$
|U'(\xi)| \le \begin{cases} \frac{C}{\varepsilon} & |\xi| \le 2\alpha \\ \frac{C}{\varepsilon}\, e^{(2\alpha|\xi|-\xi^2)/2\varepsilon} & |\xi| > 2\alpha \end{cases}\quad , \quad 0 < \varepsilon \le 1\,. \tag{2.9}
$$

In (2.9) the constants C and α depend only on $\sup_{-\infty<\xi<\infty} |U(\xi)|$, while the exponent becomes negative for $|\xi| > 2\alpha$. In addition, (2.2) yields

$$
|U''(\xi)| \le \frac{1}{\varepsilon}\left(\alpha + |\xi|\right) |U'(\xi)|\,, \tag{2.10}
$$

which in conjunction with (2.9) provides an estimate for the second derivatives.

2.2. The Construction Scheme. Let $\varepsilon \in (0,1]$ be fixed and consider the two-parameter family of boundary-value problems

$$
\begin{aligned}
-\xi\, U' + \mu\, F(U)' &= \varepsilon U'' \qquad -l < \xi < l \\
U(\pm l) &= \mu U_{\pm}
\end{aligned} \tag{2.11}
$$

with parameters $\mu \in [0,1]\,,\, l \ge 1$. The following theorem [D_1, p.3] provides sufficient conditions that guarantee the existence of solutions for $(\mathcal{P}_\varepsilon)$. We outline its proof for the sake of completeness.

Theorem 2.1 *Assume that there is a constant M depending at most on U_- , U_+, the function $F(U)$ and ε (but independent of μ and l), such that any solution $U(\xi)$ of* (2.11) *satisfies the bound*

$$
\sup_{-l\le\xi\le l} |U(\xi)| \le M\,. \tag{2.12}
$$

Then, there exists a classical solution of $(\mathcal{P}_\varepsilon)$ denoted again by $U(\xi)$ and defined on $(-\infty\,,\infty)$.

Proof. First, solutions of (2.11) are constructed by means of a continuation argument. Given a smooth function V the solution W of the boundary-value problem

$$
\begin{aligned}
&\varepsilon W''(\xi) + \xi W'(\xi) = F(V(\xi))' \qquad -l < \xi < l\,,\\
&W(-l) = U_-\,,\ W(+l) = U_+
\end{aligned}
\tag{2.13}
$$

is computed by the formula

$$
\begin{aligned}
W(\xi) = U_- + U_0 \int_{-l}^{\xi} e^{-\frac{\zeta^2}{2\varepsilon}}\,d\zeta + \frac{1}{\varepsilon}\int_{-l}^{\xi} F(V(\zeta))\,d\zeta\\
- \frac{1}{\varepsilon^2}\int_{-l}^{\xi}\int_0^{\zeta} \tau e^{\frac{\tau^2-\zeta^2}{2\varepsilon}} F(V(\tau))\,d\tau d\zeta\,,
\end{aligned}
\tag{2.14}
$$

where the constant $U_0 \in \mathbb{R}^N$ is calculated by

$$
\begin{aligned}
U_0 \int_{-l}^{l} e^{-\frac{\zeta^2}{2\varepsilon}}\,d\zeta = (U_+ - U_-) - \frac{1}{\varepsilon}\int_{-l}^{l} F(V(\zeta))\,d\zeta\\
+ \frac{1}{\varepsilon^2}\int_{-l}^{l}\int_0^{\zeta} \tau e^{\frac{\tau^2-\zeta^2}{2\varepsilon}} F(V(\tau))\,d\tau d\zeta\,.
\end{aligned}
\tag{2.15}
$$

Set $X = C^0([-l,l]\,;\mathbb{R}^N)$ and

$$
\Omega := \{U \in X : \sup_{-l\le\xi\le l} |U(\xi)| < M+1\}\,.
$$

X with the sup-norm is a Banach space and Ω is a bounded, open subset of X. Consider the map $T : \bar{\Omega} \to X$ carrying $V \in \bar{\Omega}$ to $W = T(V)$ defined by the relations (2.14) and (2.15). T is compact and continuous, and classical solutions of (2.11) are identified with fixed points of μT. The map $I - \mu T : \bar{\Omega} \times [0,1] \to X$ satisfies the hypotheses of the Schaeffer fixed-point theorem (*e.g.* Rabinowitz [R, Ch V]). Hence, for each $\mu \in (0,1]$ there is at least one solution of the equation $U - \mu T(U) = 0$ in the set Ω.

Let now $U(\cdot\,;l)$ denote a solution of (2.11) for $\mu = 1$. In the last step, solutions of $(\mathcal{P}_\varepsilon)$ are constructed as $l \to \infty$ limits of $U(\cdot\,;l)$. Proceeding as in the derivation of (2.9) and (2.10), it follows that such solutions satisfy the bounds (2.12) and

$$
\begin{aligned}
|U'(\xi\,;l)| &\le \frac{C}{\varepsilon}\, e^{(2\alpha|\xi|-\xi^2)/2\varepsilon}\\
|U''(\xi\,;l)| &\le \frac{C}{\varepsilon^2}\,(1+|\xi|)\, e^{(2\alpha|\xi|-\xi^2)/2\varepsilon}
\end{aligned}
\tag{2.16}
$$

with C and α depending on M but not on l. Extend $U(\cdot\,;l)$ outside $[-l,l]$ by setting $U(\xi\,;l) = U_-$ for $\xi < -l$ and $U(\xi\,;l) = U_+$ for $\xi > l$. The Ascoli-Arzela theorem, together with a diagonal argument, implies the existence of a sequence $\{l_n\}$, $l_n \to \infty$, and a function $U \in C^1((-\infty,\infty)\,;\mathbb{R}^N)$ such that $U(\cdot\,;l_n) \to U$ and $U'(\cdot\,;l_n) \to U'$ uniformly on compact subsets of $\mathbb{R}$. Because of (2.16) the convergence is uniform on $\mathbb{R}$ and $U(\pm\infty) = U_\pm$. Passing to the limit $l_n \to \infty$ shows that U is a classical solution of $(\mathcal{P}_\varepsilon)$. ∎

2.3 The a priori estimates. The scope of this section is to provide the sup-norm estimates that authorize application of Theorem 2.1. In the sequel, $U(\xi)$ stands for a solution of the family of boundary-value problems (2.11) defined on $[-l,l]$ and depending implicitly on μ , l and ε. In the process of estimating $U(\xi)$ we pursue ideas that were developed by Dafermos and DiPerna [DDp] in the context of 2×2 systems and use the concept of entropy-entropy flux pairs (Lax [La$_2$]).

A scalar-valued function $\eta(U)$ is called an entropy for (1.1), with corresponding entropy flux $q(U)$, if every smooth solution satisfies the additional conservation law

$$\partial_t \eta(U) + \partial_x q(U) = 0\,. \tag{2.17}$$

Such pairs $(\eta(U), q(U))$ are generated by solving the system of (linear) differential equations

$$\nabla q(U) = \nabla \eta(U)\, \nabla F(U)\,. \tag{2.18}$$

Trivial examples of solutions are provided by $(c\cdot U\,,\, c\cdot F(U))$, with c any constant vector in $\mathbb{R}^N$. Since (2.18) is overdetermined for $N \geq 3$, for systems of three or more equations the existence of (nontrivial) entropies is the exception rather than the rule. Nevertheless, specific systems that arise in applications are often naturally endowed with some entropy-entropy flux pairs. Also, the class of symmetric hyperbolic systems, that is systems for which $\nabla F(U)$ is a symmetric matrix, admits the pair

$$\eta(U) = \frac{1}{2}\,|U|^2 \qquad q(U) = U\cdot F(U) - g(U)\,, \tag{2.19}$$

where g is a potential for F satisfying $F(U) = \nabla g(U)$.

Let $\big(\eta(U), q(U)\big)$ be an entropy-entropy flux pair for (1.1). Using (2.18) we deduce that solutions of (2.11) satisfy the identity

$$-\xi\,\eta' + \mu\, q' = \varepsilon\,\eta'' - \varepsilon\, U'\cdot(\nabla^2\eta)\,U' \tag{2.20}$$

where $\eta = \eta\big(U(\xi)\big)$, $q = q\big(U(\xi)\big)$. In exploiting (2.20), it is helpful to use entropy functions $\eta(U)$ that are convex (or linear). The following lemma indicates how to bound the total entropy production. Given a constant entropy level $\bar\eta$, consider the level set

$$\mathcal{C}_{\bar\eta} = \big\{U \in \mathbb{R}^N : \eta(U) = \bar\eta\big\}\,. \tag{2.21}$$

If $\mathcal{C}_{\bar\eta}$ is nonempty, let

$$Q_{\bar\eta} = \sup_{U_1, U_2 \in \mathcal{C}_{\bar\eta}} |q(U_1) - q(U_2)| \tag{2.22}$$

be the oscillation of $q(U)$ on the level set $\mathcal{C}_{\bar\eta}$.

Lemma 2.2. *Assume that $\eta(U)$ is a convex entropy with corresponding entropy flux $q(U)$. If $\bar\eta$ is any constant such that*

$$\bar\eta > \max_{0 \le \mu \le 1} \{\eta(\mu U_-), \eta(\mu U_+)\} \tag{2.23}$$

then for any $(\alpha, \beta) \subset (-l, l)$

$$\int_\alpha^\beta \big(\eta(U(\xi)) - \bar\eta\big)\, d\xi \le K\,, \tag{2.24}$$

where $K = Q_{\bar\eta}$ if $\eta(U(\xi)) > \bar\eta$ for some $\xi \in (\alpha, \beta)$, and $K = 0$ otherwise.

Proof. The proof is based on the following observation. Let $\bar\eta$ be a fixed entropy level and suppose that a , b are two points in $(-l, l)$ with the properties $a < b$ and

$$\eta(U(a)) = \eta(U(b)) = \bar\eta \qquad \text{with} \quad (\eta \circ U)'(a) \ge 0\,, \quad (\eta \circ U)'(b) \le 0\,. \tag{2.25}$$

Integrating (2.20) over $[a, b]$, we obtain

$$\begin{aligned} \int_a^b \big(\eta(U(\xi)) - \bar\eta\big)\, d\xi + \varepsilon \int_a^b U'(\xi) \cdot \nabla^2 \eta(U(\xi))\, U'(\xi)\, d\xi \\ \le\, -\mu\, \big[q(U(b)) - q(U(a))\big] \,\le\, Q_{\bar\eta}\,, \end{aligned} \tag{2.26}$$

which, upon using the convexity of $\eta(U)$, yields

$$\int_a^b \big(\eta(U(\xi)) - \bar\eta\big)\, d\xi \le\, Q_{\bar\eta}\,. \tag{2.27}$$

If $\eta(U(\xi)) \le \bar\eta$ for $-l \le \xi \le l$, then (2.24) is trivially true with $K = 0$. So suppose that the set $\{\xi \in (-l, l) : \eta(U(\xi)) > \bar\eta\}$ is nonempty. It is also open and thus admits a decomposition into a countable union of disjoint subintervals

$$\big\{\xi \in (-l, l) : \quad \eta(U(\xi)) > \bar\eta\big\} = \bigcup_{k \in I} (a_k, b_k)\,, \tag{2.28}$$

where k ranges over an index set I (either a finite set or the integers). For $\bar\eta$ restricted by (2.23) the points a_k and b_k lie in $(-l, l)$. Also, since $\eta(U(\xi)) > \bar\eta$ for $a_k < \xi < b_k$ with $k \in I$, relations (2.25) are satisfied at the endpoints a_k , b_k.

Given any $(\alpha, \beta) \subset (-l, l)$, choose a , b as follows: If $\eta(U(\alpha)) > \bar{\eta}$ set $a = \sup\{a_k < \alpha\}$, while if $\eta(U(\alpha)) \leq \bar{\eta}$ set $a = \inf\{a_k > \alpha\}$; if $\eta(U(\beta)) > \bar{\eta}$ set $b = \inf\{b_k > \beta\}$, while if $\eta(U(\beta)) \leq \bar{\eta}$ set $b = \sup\{b_k < \beta\}$. If $\eta(U(\xi)) > \bar{\eta}$ at some $\xi \in (\alpha, \beta)$, a and b are well defined, $a < b$, relations (2.25) are satisfied at a, b and

$$\int_{\alpha}^{\beta} \big(\eta(U(\xi)) - \bar{\eta}\big)\, d\xi \leq \int_{a}^{b} \big(\eta(U(\xi)) - \bar{\eta}\big)\, d\xi \leq Q_{\bar{\eta}} \,. \tag{2.29}$$

Otherwise (2.24) holds with $K = 0$. ∎

In general the quantity $Q_{\bar{\eta}}$ depends on the form of the level set $\mathcal{C}_{\bar{\eta}}$ as well as the function $q(U)$ and may be infinite. If it happens that $\mathcal{C}_{\bar{\eta}}$ is a compact set, then $Q_{\bar{\eta}}$ is finite and (2.24) provides an integral estimate independent of μ , l and ε. An entropy is called normal if $\eta(U) \to \infty$ as $|U| \to \infty$. If the system (1.1) is endowed with a convex normal entropy, then nonempty level sets $\mathcal{C}_{\bar{\eta}}$ are compact, and that leads to integral estimates of the type (2.24). For a symmetric hyperbolic system $\eta(U) = \frac{1}{2}|U|^2$ is an example of a convex normal entropy.

Next, we present two approaches for obtaining the sup-norm estimates (2.12). The first exploits the entropy identity (2.20), and requires the existence of a strictly convex, normal entropy function $\eta(U)$, defined (only) on the exterior of some open ball in the state space.

Proposition 2.3. *Assume that* (1.1) *admits a strictly convex, normal entropy* $\eta(U)$ *defined on the exterior of a ball and satisfying the growth restriction*: *There are* $q > 0$ *and positive constants* C *and* r_0 *such that*

$$|\nabla \eta(U)|^2 \leq C\, \nu(U)\, \eta(U)^{3-q} \qquad \text{for} \quad |U| \geq r_0 \,, \tag{H}$$

where $\nu(U)$ *is the smallest eigenvalue of the Hessian* $\nabla^2 \eta(U)$. *Then solutions of* $(\mathcal{P}_\varepsilon)$ *exist for every* $\varepsilon > 0$.

Proof. Let $\eta(U)$ be a strictly convex, normal entropy defined for $\{U \in \mathbb{R}^N : |U| \geq r_0\}$ and satisfying (H) for some $q > 0$. Without loss of generality we may assume that $\eta(U)$ is positive. Let $U(\xi)$ be a solution of (2.11) on $(-l, l)$. For those ξ that $|U(\xi)| > r_0$ equation (2.20) is satisfied.

Let $r > \max\{|U_+| \,, |U_-| \,, r_0\}$ and $\bar{\eta}_r = \max_{|U|=r} \eta(U)$ be fixed and choose two entropy levels $\bar{\eta}_2 > \bar{\eta}_1 > \bar{\eta}_r > 0$. Consider the set

$$\mathcal{A} = \big\{\xi \in (-l, l) : \eta(U(\xi)) > \bar{\eta}_2 \,, \, |U(\xi)| > r \,\big\} \,. \tag{2.30}$$

Since $\eta(U) \to \infty$ as $|U| \to \infty$, if the set $\mathcal{A}$ is empty then $\sup_{-l \leq \xi \leq l} |U(\xi)| \leq M$, for some M depending on $\bar{\eta}_2$ and r, and thus (2.12) holds in this case. So, assume that $\mathcal{A}$ is nonempty. It is also open and thus admits the decomposition $\mathcal{A} = \bigcup_{k\in I}(a_k, b_k)$ into a countable (or finite) union of disjoint intervals. In addition the choice $\bar\eta_2 > \bar\eta_r$ implies that, for any $k \in I$,

$$
\begin{aligned}
\eta(U(\xi)) > \bar\eta_2 \quad &\text{for} \quad a_k < \xi < b_k\,,\\
\eta(U(a_k)) = \eta(U(b_k)) = \bar\eta_2 \quad &\text{and} \quad (\eta\circ U)'(a_k) \geq 0\,, \quad (\eta\circ U)'(b_k) \leq 0\,.
\end{aligned}
\tag{2.31}
$$

Henceforth we focus on a fixed interval (a_k, b_k). Let τ_k be a point where $\eta(U(\xi))$ assumes its maximum in the closed interval $[a_k, b_k]$. Using Schwarz's inequality, the strict convexity of η, hypothesis (H) and relations (2.31), (2.25) and (2.26) we obtain

$$
\begin{aligned}
(2/q)\,\big[\,\eta(U(\tau_k))^{\frac{q}{2}} - \eta(U(a_k))^{\frac{q}{2}}\big] &= \int_{a_k}^{\tau_k} \eta(U(\zeta))^{\frac{q}{2}-1}\,\nabla\eta(U(\zeta))\,U'(\zeta)\,d\zeta\\
&\leq \big[\int_{a_k}^{\tau_k} \frac{\eta(U(\zeta))^{q-2}}{\nu(U(\zeta))}\,|\nabla\eta(U(\zeta))|^2 d\zeta\,\big]^{\frac12}\,\big[\int_{a_k}^{\tau_k} U'(\zeta)\cdot\nabla^2\eta(U(\zeta))U'(\zeta)\,d\zeta\big]^{\frac12}\\
&\leq \big[C\int_{a_k}^{\tau_k}\eta(U(\zeta))\,d\zeta\,\big]^{\frac12}\big(\frac{1}{\varepsilon}\,Q_{\bar\eta_2}\big)^{\frac12}\,.
\end{aligned}
\tag{2.32}
$$

For those $U \in \mathbb{R}^N$ that $\eta(U) > \bar\eta_2 > \bar\eta_1 > 0$, it is

$$
\eta(U) - \bar\eta_1 > \frac{\bar\eta_2 - \bar\eta_1}{\bar\eta_2}\,\eta(U)\,. \tag{2.33}
$$

Then (2.32) yields the estimate

$$
\eta(U(\tau_k))^{\frac{q}{2}} \leq\, (\bar\eta_2)^{\frac{q}{2}} + (q/2)\,\big(\frac{C\,\bar\eta_2}{\varepsilon\,(\bar\eta_2-\bar\eta_1)}\,Q_{\bar\eta_2}\,\big)^{1/2}\,\big(\int_{a_k}^{\tau_k}(\,\eta(U(\zeta)) - \bar\eta_1\,)\,d\zeta\,\big)^{\frac12}\,. \tag{2.34}
$$

Set $a = \inf\{\xi \in (-l, a_k) :\ \eta(U(\zeta)) > \bar\eta_1 \,\text{on}\, (\xi, a_k)\}$, $b = \sup\{\xi \in (b_k, l) :\ \eta(U(\zeta)) > \bar\eta_1 \,\text{on}\, (b_k, \xi)\}$. Since $|U(\pm l)| < r$ and $\bar\eta_1 > \bar\eta_r$, a, b are well defined and satisfy $-l < a < a_k < b_k < b < l$. In addition (2.25) holds and, as in the proof of Lemma 2.2,

$$
\int_{\alpha_k}^{\tau_k} \big(\eta(U(\xi)) - \bar\eta_1\big)\,d\xi \leq \int_a^b \big(\eta(U(\xi)) - \bar\eta_1\big)\,d\xi \leq\, Q_{\bar\eta_1}\,. \tag{2.35}
$$

As a consequence, the right-hand side of (2.34) is bounded independently of k, and (2.12) holds in the case that $\mathcal{A}$ is nonempty too. The conclusion now follows from Theorem 2.1. ∎

Regarding the growth assumption (H) the following remarks are in order. If the strictly convex, normal entropy function is of the form $\eta(U) = (1/p)\,|U|^p$, with $p > 1$, one easily calculates

$$
\nabla\eta(U) = |U|^{p-2}\,U\,, \qquad \nabla^2\eta(U) = |U|^{p-2}\,I + (p-2)\,|U|^{p-4}\,U\otimes U\,. \tag{2.36}
$$

The Hessian of η is a positive definite matrix having eigenvalues: $(p-1)|U|^{p-2}$ with corresponding eigenvector U, and $|U|^{p-2}$ of multiplicity $N-1$ with corresponding eigenvectors $U^\perp$ any vector orthogonal to U. Hypothesis (H) is then satisfied with $q = 2$. On the other hand, if $\eta(U)$ grows like a power up to first order derivatives, i.e.,

$$\frac{1}{c}|U|^p \leq \eta(U) \leq c|U|^p\,, \quad |\nabla\eta(U)| \leq c|U|^{p-1}\,, \tag{2.37}$$

for some positive constant c, then (H) becomes a restriction on the decay of the minimum eigenvalue for U large and is satisfied provided that $\nu(U) \geq |U|^{-s}$ for some $s < p+2$.

As a consequence of the above remarks in conjunction with Proposition 2.3, we have.

Theorem 2.4. *If* (1.1) *is a symmetric hyperbolic system, then solutions of* $(\mathcal{P}_\varepsilon)$ *exist for every* $\varepsilon > 0$.

In the interest of developing technique, we present an alternative way for establishing (2.12) for symmetric hyperbolic systems. The actual result is weaker than Theorem 2.4, as it requires a growth assumption on the flux $F(U)$, but the approach may be useful for other problems.

Proposition 2.5. *Suppose that* (1.1) *is a symmetric hyperbolic system, such that the flux function satisfies the growth assumption*

$$|F(U)| \leq C\,(1+|U|)^p \tag{2.38}$$

for some positive constants C *and* $p \leq 3$. *Then solutions of* $(\mathcal{P}_\varepsilon)$ *exist for every* $\varepsilon > 0$.

Proof. Symmetric hyperbolic systems are endowed with the entropy - entropy flux pair (2.19), for which (2.20) takes the form

$$-\xi\,(|U|^2)' + 2\mu\,(U\cdot F(U) - g(U))' = \varepsilon\,(|U|^2)'' - 2\varepsilon\,|U'|^2\,. \tag{2.39}$$

The function g is a potential for F satisfying $F(U) = \nabla g(U)$. It can be defined by the formula

$$g(U) = \int_0^1 \frac{d}{dt} g(t\,U)\,dt = \int_0^1 F(t\,U)\cdot U\,dt\,, \tag{2.40}$$

where g has been normalized by setting $g(0) = 0$. Assumption (2.38) induces a growth restriction on g as follows:

$$|g(U)| = |\int_0^1 F(tU)\cdot U\,dt\,| \leq C\,(1+|U|)^{p+1}\,. \tag{2.41}$$

Set $r = \max\{|U_-|, |U_+|\}$ and consider any point $\xi \in (-l,l)$ such that $|U(\xi)| > r$ and $(d|U|^2/d\xi)(\xi) > 0$. Define $\xi' = \inf\{\zeta \in (\xi, l] : |U(\zeta)| < |U(\xi)|\}$ and observe that ξ' is well defined with $\xi < \xi' < l$. Moreover, $|U(\xi')| = |U(\xi)|$, $(d|U|^2/d\xi)(\xi') \le 0$ and $|U(\zeta)| \ge |U(\xi)|$ for $\xi \le \zeta \le \xi'$. Integrating (2.39) over $[\xi, \xi']$, we obtain

$$\begin{aligned} &-\int_\xi^{\xi'} \zeta \left(|U|^2\right)'(\zeta)\, d\zeta + 2\varepsilon \int_\xi^{\xi'} |U'(\zeta)|^2\, d\zeta \\ &+ 2\mu \left[U(\xi') \cdot F(U(\xi')) - g(U(\xi')) - U(\xi) \cdot F(U(\xi)) + g(U(\xi)) \right] \\ &= \varepsilon\big(d|U|^2/d\xi\big)(\xi') - \varepsilon\big(d|U|^2/d\xi\big)(\xi)\ . \end{aligned} \tag{2.42}$$

Since

$$-\int_\xi^{\xi'} \zeta \left(|U|^2\right)'(\zeta)\, d\zeta = \int_\xi^{\xi'} \left(|U(\zeta)|^2 - |U(\xi)|^2\right) d\zeta \ge 0\,, \tag{2.43}$$

(2.42) together with (2.38) and (2.41) yield

$$\varepsilon \frac{d|U|^2}{d\xi}(\xi) \le\ 8C\,(1 + |U(\xi)|)^{p+1}\,. \tag{2.44}$$

Note that the bound (2.44) holds for any $\xi \in (-l, l)$ such that $|U(\xi)| > r$.

To conclude the proof fix two levels r_1 and r_2, with $r_2 > r_1 > r$, and consider the set $\mathcal{B} = \{\xi \in (-l, l) : |U(\xi)| > r_2\}$. If $\mathcal{B}$ is empty then (2.12) holds and Theorem 2.1 implies the desired result. If $\mathcal{B}$ is nonempty, then it can be decomposed into an at most countable union of disjoint subintervals (a_k, b_k) such that $|U(a_k)| = |U(b_k)| = r_2$ and $|U(\xi)| > r_2$ for $a_k < \xi < b_k$. In each of the intervals $[a_k, b_k]$ the differential inequality (2.44) is satisfied. Next, fix k and let $\tau_k \in [a_k, b_k]$ be a point where $|U(\tau_k)| = \max_{a_k \le \xi \le b_k} |U(\xi)|$. Lemma 2.2, applied for the entropy $\eta(U) = |U|^2$ and the level $\bar\eta = r_1^2$, implies

$$\int_{a_k}^{\tau_k} \left(|U(\xi)|^2 - r_1^2\right) d\xi \le Q_{r_1^2} < \infty\,. \tag{2.45}$$

Since the ratio $|U|^{1-p}(1 + |U|)^{1+p}/(|U|^2 - r_1^2)$ remains bounded for $|U| \ge r_2$, using (2.44) and (2.45) we deduce

$$\varepsilon \int_{a_k}^{\tau_k} |U(\xi)|^{1-p}\, \frac{d|U|^2}{d\xi}\, d\xi \le C' \int_{a_k}^{\tau_k} \left(|U(\xi)|^2 - r_1^2\right) d\xi \le C'\, Q_{r_1^2}\,. \tag{2.46}$$

In turn, performing the integration in (2.46) yields

$$|U(\tau_k)|^{3-p} \le (r_2)^{3-p} + \frac{3-p}{2\varepsilon} C'\, Q_{r_1^2}\,, \tag{2.47}$$

for $p < 3$, and

$$|U(\tau_k)| \le r_2\, e^{C' Q_{r_1^2}/2\varepsilon}\,, \tag{2.48}$$

for $p = 3$. In either case (2.12) holds and proof is complete. ∎

3. Solution decomposition - The main result

The aim of this article is to construct solutions of the Riemann problem $(\mathcal{P})$ as $\varepsilon \to 0$ limits of solutions of $(\mathcal{P}_\varepsilon)$. The central difficulty lies in obtaining ε-independent variation estimates for families of solutions of the problems $(\mathcal{P}_\varepsilon)$. The reason is that, even for Riemann data, there are wave interactions induced by the coupling through the self-similar viscosity that need to be accounted for. The derivation of the variation estimates follows from a lengthy analysis, carried out in Sections 3-7. The present section serves as an introduction, where we outline the general strategy, introduce the main hypotheses, and present certain interesting geometric properties.

Our approach is motivated by a detailed study of the following question: Suppose we are given a family of solutions to $(\mathcal{P}_\varepsilon)$ of uniformly small oscillation :

$$(C_o) \qquad \sup_{-\infty<\xi<\infty} |U_\varepsilon(\xi) - U_-| \le \mu \ .$$

Such a family would also satisfy uniform L^∞ bounds

$$(C_b) \qquad \sup_{-\infty<\xi<\infty} |U_\varepsilon(\xi)| \le M \,,$$

where the constants M and μ are independent of ε and μ is also small. Examine under what structural hypotheses on (1.1) the family $\{U_\varepsilon\}_{\varepsilon>0}$ is of uniformly bounded variation :

$$(S) \qquad TV_{(-\infty,\infty)}\, U_\varepsilon \le C \,.$$

It is instructive to give a proof of (S) for the single conservation law, which contains some ingredients of the approach followed for systems. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of scalar-valued functions satisfying

$$(3.1) \qquad \begin{gathered} \varepsilon\, u_\varepsilon'' = -\xi\, u_\varepsilon' + f(u_\varepsilon)' \\ u_\varepsilon(\pm\infty) = u_\pm \,. \end{gathered}$$

and the uniform bounds (C_b) (which are easily justifiable in this case). Let $\lambda(u) = f'(u)$ be the characteristic speed of the associated hyperbolic equation. It is easy to see that solutions of (3.1) satisfy the representation formula

$$(3.2) \qquad u_\varepsilon'(\xi) = (u_+ - u_-)\ \frac{e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta}$$

where

$$(3.3) \qquad g_\varepsilon(\xi) = \int_\alpha^\xi s - \lambda\big(u_\varepsilon(s)\big)\, ds \,.$$

From the form of (3.2), it follows that $\{u'_\varepsilon\}$ are uniformly bounded in L^1 and thus $\{u_\varepsilon\}$ is of uniformly bounded variation.

Returning to the general case, we note that the system (1.1) is assumed strictly hyperbolic, but no other structural assumptions are imposed. The eigenvalues of $\nabla F(U)$ are denoted by

$$\lambda_1(U) < \lambda_2(U) < \dots < \lambda_N(U) \tag{3.4}$$

and are ordered. The corresponding right eigenvectors $r_1(U)\,, \dots, r_N(U)$ and left eigenvectors $l_1(U)\,, \dots, l_N(U)$ are linearly independent and satisfy the relations

$$\nabla F(U)\, r_i(U) = \lambda_i(U)\, r_i(U)\,, \tag{3.5}$$

$$l_i(U) \cdot \nabla F(U) = \lambda_i(U)\, l_i(U)\,, \tag{3.6}$$

$$l_i(U) \cdot r_j(U) = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases} . \tag{3.7}$$

$\{r_i\}$ and $\{l_i\}$ form a pair of local bases in the state space $\mathbb{R}^N$. By normalizing one of these bases we can attain

$$l_i(U) \cdot r_j(U) = \delta_{ij}\,. \tag{3.8}$$

The family $\{U_\varepsilon\}_{\varepsilon>0}$ consists of solutions to the boundary value problem $(\mathcal{P}_\varepsilon)$ that connect two fixed end-states U_- and U_+. Conditions that guarantee existence of solutions for $(\mathcal{P}_\varepsilon)$ are given in Section 2; nevertheless, the forthcoming analysis is independent of such considerations, and eventually it will also suggest a construction scheme. We assume the members of $\{U_\varepsilon\}_{\varepsilon>0}$ satisfy the hypothesis (C_o) of uniformly in ε small oscillation and (a-fortiori) the uniform bound (C_b). That restricts the data $U_\pm$ to satisfy

$$|U_+ - U_-| < r \tag{H_D}$$

with r sufficiently small. Also, each wave speed is bounded

$$\lambda_{k-} \leq \lambda_k\big(U_\varepsilon(\xi)\big) \leq \lambda_{k+} \tag{3.9}$$

by constants λ_{k-}, λ_{k+} independent of ε. By choosing μ sufficiently small, we guarantee that the wave speeds are *totally separated* along the family $\{U_\varepsilon\}_{\varepsilon>0}$, that is

$$\begin{aligned} &\lambda_{1-} \leq \lambda_1\big(U_\varepsilon(\xi)\big) \leq \lambda_{1+} < \lambda_{2-} \leq \lambda_2\big(U_\varepsilon(\xi)\big) \leq \lambda_{2+} < \dots \\ &\quad < \lambda_{(N-1)-} \leq \lambda_{N-1}\big(U_\varepsilon(\xi)\big) \leq \lambda_{(N-1)+} < \lambda_{N-} \leq \lambda_N\big(U_\varepsilon(\xi)\big) \leq \lambda_{N+}\,. \end{aligned} \tag{3.10}$$

The bound (C_b) implies that the derivatives of U_ε satisfy the estimates (2.9) and (2.10) with the constants C and α independent of ε. In the sequel we use the following conventions on notation : The ε-dependence will be suppressed from functions, except at places where emphasis is needed. By contrast, any ε-dependence of constants will be explicitly indicated by either recording the precise dependence or by using ε as a subscript.

Consider the decomposition of U'_ε in the basis of right eigenvectors evaluated at the local value of the solution U_ε :

$$U'_\varepsilon(\xi) = \sum_{k=1}^{N} a_k(\xi)\, r_k\big(U_\varepsilon(\xi)\big)\,. \tag{3.11}$$

The amplitudes a_k can be recovered by using (3.8)

$$a_k(\xi) = l_k(U_\varepsilon(\xi)) \cdot U'_\varepsilon(\xi)\,. \tag{3.12}$$

Also, integrating (3.11) over $(-\infty, \infty)$, we have

$$U_+ - U_- = \sum_{k=1}^{N} \int_{-\infty}^{\infty} a_k(\zeta)\, r_k\big(U_\varepsilon(\zeta)\big)\, d\zeta\,. \tag{3.13}$$

To compute the equations that a_k satisfy, take the inner product of (2.2) with $l_k(U_\varepsilon)$ to obtain

$$\begin{aligned} -\xi\, a_k + \lambda_k(U_\varepsilon(\xi))\, a_k &= \varepsilon\, l_k(U_\varepsilon(\xi)) \cdot U''_\varepsilon \\ &= \varepsilon\, a'_k - \varepsilon\, \nabla l_k(U_\varepsilon(\xi))\, U'_\varepsilon \cdot U'_\varepsilon\,, \end{aligned} \tag{3.14}$$

and hence

$$\varepsilon\, a'_k + \big[\xi - \lambda_k(U_\varepsilon(\xi))\big]\, a_k = \varepsilon \sum_{m=1}^{N}\sum_{n=1}^{N} \big[\nabla l_k(U_\varepsilon(\xi))\, r_m(U_\varepsilon(\xi)) \cdot r_n(U_\varepsilon(\xi))\big]\, a_m\, a_n\,. \tag{3.15}$$

If we introduce the notation

$$\lambda_k = \lambda_k(U_\varepsilon(\xi)) \tag{3.16}$$

$$\beta_{k,mn} = \beta_{k,mn}(U_\varepsilon(\xi)) = \nabla l_k(U_\varepsilon(\xi))\, r_m(U_\varepsilon(\xi)) \cdot r_n(U_\varepsilon(\xi)) \tag{3.17}$$

then a_k satisfy the coupled system of ordinary differential equations with variable coefficients

$$\varepsilon\, a'_k + (\xi - \lambda_k)\, a_k = \varepsilon \sum_{m=1}^{N}\sum_{n=1}^{N} \beta_{k,mn}\, a_m\, a_n\,. \tag{3.18}$$

At this point several remarks are in order. First, the decomposition (3.11) is partly motivated by the classical solution of the Riemann problem (Lax [La$_1$], Liu [Li$_2$]). It is expected to capture the behavior near rarefactions, but it is not a priori clear that it should work well near shocks. Good overall performance would indicate that (3.11) captures the nature of diffusion induced averaging at a shock. The quadratic terms in (3.18) represent the effect induced on the k-family by interactions of waves of all the families, and $\beta_{k,mn}$ measure the weights of such contributions. By virtue of (C_b), $\beta_{k,mn}$ are uniformly bounded

$$|\beta_{k,mn}| \leq B\,. \tag{3.19}$$

Let g_k be the antiderivative of

$$g_k' = \xi - \lambda_k = \xi - \lambda_k(U_\varepsilon(\xi)) \tag{3.20}$$

defined within an arbitrary constant of integration by

$$g_k = \int_\alpha^\xi s - \lambda_k(U_\varepsilon(s))\,ds\ . \tag{3.21}$$

In view of (3.9), it is

$$s - \lambda_{k+} \leq s - \lambda_k(U_\varepsilon(s)) \leq s - \lambda_{k-}\ , \tag{3.22}$$

which in turn implies $g_k' > 0$ for $\xi > \lambda_{k+}$, $g_k' < 0$ for $\xi < \lambda_{k-}$, and g_k looks like a potential-well function (see Figure 1). Let $\rho_{k\varepsilon}$ be a point where g_k attains its global minimum, $g_k(\rho_{k\varepsilon}) = \min g_k(\xi)$. Then $\lambda_{k-} \leq \rho_{k\varepsilon} \leq \lambda_{k+}$ while the value of $g_k(\rho_{k\varepsilon})$ depends on the choice of the arbitrary constant in (3.21). By setting

$$g_k(\xi) := \int_{\rho_{k\varepsilon}}^\xi s - \lambda_k(U_\varepsilon(s))\,ds \tag{3.23}$$

we attain $g_k(\xi) \geq g_k(\rho_{k\varepsilon}) = 0$ for $\xi \in \mathbb{R}$. Furthermore, $\lambda_k(U_\varepsilon(\rho_{k\varepsilon})) = \rho_{k\varepsilon}$ and $g_k(\xi) = O(|\xi|^2)$ as $|\xi| \to \infty$.

Consider the linearization of the system (3.18). It consists of the decoupled system of equations

$$\varepsilon\,\varphi_k' + (\xi - \lambda_k)\,\varphi_k = 0\,, \tag{3.24}$$

whose solutions are constant multiples of

$$\varphi_k = \frac{e^{-\frac{1}{\varepsilon}g_k}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}g_k}\,d\zeta} = \frac{e^{-\frac{1}{\varepsilon}\int_{\rho_{k\varepsilon}}^{\xi} s-\lambda_k(U_\varepsilon(s))\,ds}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_{\rho_{k\varepsilon}}^{\zeta} s-\lambda_k(U_\varepsilon(s))\,ds}\,d\zeta} \tag{3.25}$$

Due to their form $\{\varphi_{k\varepsilon}\}$ are strictly positive functions that are uniformly (in ε) bounded in L^1.

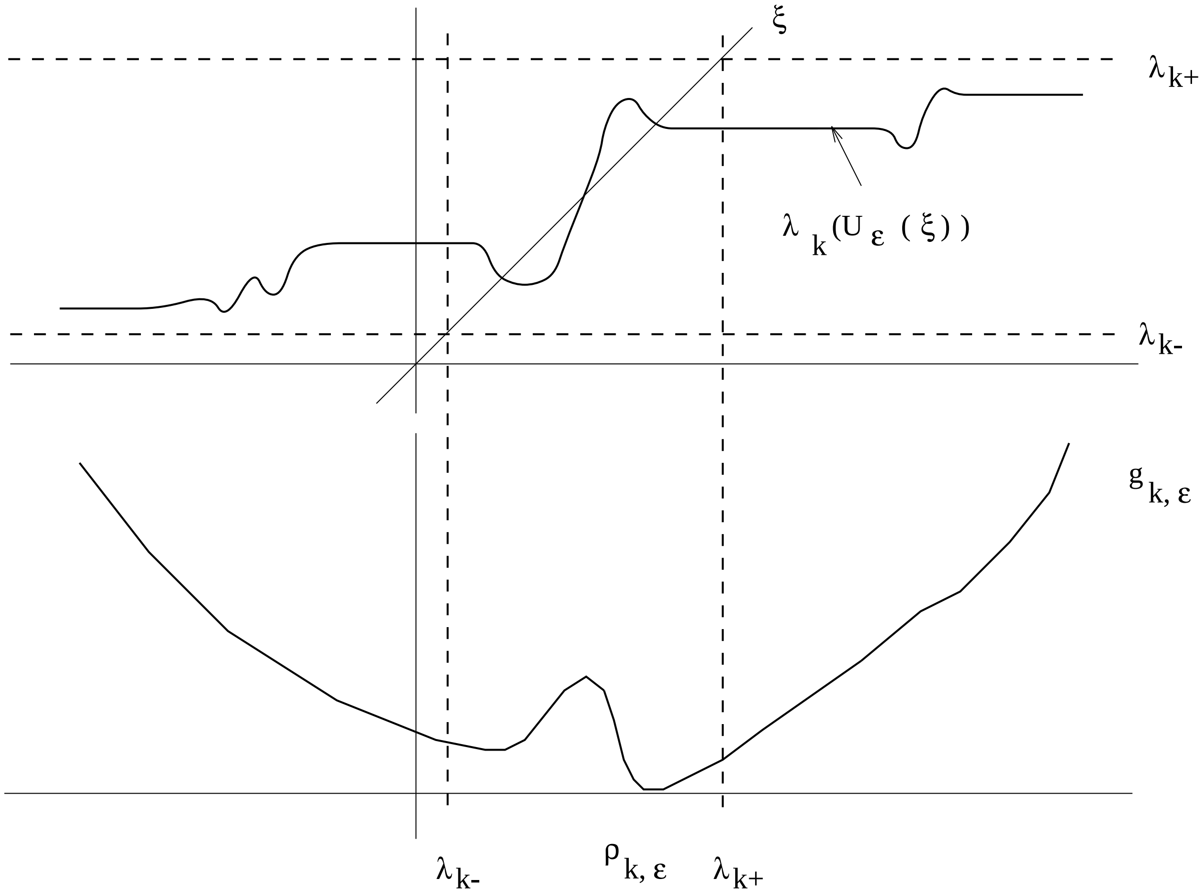

FIGURE 1.

A comparison of (3.24) with (3.1) and (3.25) with the representation formula (3.2) shows that, for the case of the single equation, it is precisely the above step that provides the variation bounds. Due to the quadratic terms in (3.18) though, this is insufficient for systems of conservation laws. There are two problems that we need to account for, in the case of systems. First to understand the effect of the quadratic terms. Second, differential systems like (3.18) are best handled with pointwise conditions. On the other hand the only existing information (3.13), relating the data $U_\pm$ with the amplitudes a_k, is of integral type. It is thus necessary to devise a scheme that connects pointwise with integral information.

We proceed by introducing a decomposition of a_k in the form

$$a_k = \tau_k \, \varphi_k + \theta_k \, , \tag{3.26}$$

where φ_k is given by (3.25) and θ_k satisfies the system of differential equations

$$\varepsilon\,\theta_k' + (\xi - \lambda_k)\,\theta_k = \varepsilon \sum_{m=1}^{N} \sum_{n=1}^{N} \beta_{k,mn}\,(\tau_m\,\varphi_m + \theta_m)\,(\tau_n\,\varphi_n + \theta_n)\,. \tag{3.27}$$

Then the sum $\tau_k\,\varphi_k + \theta_k$ is a solution of (3.18). The idea is to seek an asymptotic expansion of the wave amplitude a_k in a parameter $\tau = (\tau_1, \dots, \tau_N)$, where τ_k is thought as a measure of the strength of the k-th wave, and to construct an expansion uniform in ε in the L^1-norm. In this expansion $\tau_k\varphi_k$ is the leading term and θ_k is the error, which should be of order $O(|\tau|^2)$ as $|\tau| = |\tau_1| + \dots + |\tau_N| \to 0$. Clearly, such an expansion depends on the provided data, and the key question becomes under what conditions to solve (3.27).

Next, we outline the strategy we follow and the attained results concerning those problems : Fix $c_1, c_2, \dots, c_N$ to be the respective middle points of the intervals $[\lambda_{1-}, \lambda_{1+}]$, $[\lambda_{2-}, \lambda_{2+}]$, ... , $[\lambda_{N-}, \lambda_{N+}]$. Given a constant vector $\tau = (\tau_1, \tau_2, \dots, \tau_N) \in \mathbb{R}^N$, we consider (3.27) subject to the conditions

$$\theta_k(c_k) = 0\,, \tag{3.28}$$

and for $|\tau|$ sufficiently small construct a solution $\theta_k(\xi;\,\tau)$ that satisfies the estimate

$$|\theta_k(\cdot\,;\,\tau)| \le C\,|\tau|^2 \sum_{m=1}^{N} \varphi_m\,. \tag{3.29}$$

This construction is performed in Section 5. It is based on detailed estimates, that are presented in Section 4, on the functions φ_k and on integrals involving $\varphi_m\varphi_n$ and capturing wave interactions. The method is to apply the uniform contraction principle to a weighted space of continuous functions. The selection of the weight is motivated by the analysis of Section 4. The analysis of Section 5 validates the asymptotic expansion

$$a_k(\cdot\,;\,\tau) = \tau_k\,\varphi_k(\cdot) + \theta_k(\cdot\,;\,\tau) \tag{3.30}$$

for the amplitude a_k in the parameter τ. Note that a_k satisfy the pointwise information

$$a_k(c_k\,;\,\tau) = \tau_k\,\varphi_k(c_k) \tag{3.31}$$

and solve (3.18) but not necessarily (3.13).

The objective of Section 6 is then to show that there exists a choice of $\tau = (\tau_1, \dots, \tau_N)$ such that (3.13) is fulfilled. To this end, we consider the map $S : \mathbb{R}^N \to \mathbb{R}^N$ that connects the wave strengths to the boundary data by taking τ to

$$S(\tau) = U_- + \sum_{k=1}^{N} \int_{-\infty}^{\infty} \left[\tau_k\,\varphi_k(\zeta) + \theta_k(\zeta;\,\tau)\right] r_k(U_\varepsilon(\zeta))\,d\zeta\,. \tag{3.32}$$

We show in Section 6 that S is locally invertible in a neighborhood of $\tau = 0$, and that the inverse map S^{-1} is uniformly bounded independent of ε.

In Sections 4 to 6, we identify the precise hypotheses (supplementary to (3.10)), on the behavior of the wave speeds λ_k and the right and left eigenvectors r_k and l_k along solutions U_ε, that are necessary to carry out the intermediate steps. All these hypotheses are fulfilled if the oscillation of the family $\{U_\varepsilon\}_{\varepsilon>0}$ is restricted, uniformly in ε. It is convenient to phrase the analysis by using a general function V of restricted oscillation, $\sup_{\xi\in\mathbb{R}} |V(\xi) - U_-| \le \mu$, in the place of a member of $\{U_\varepsilon\}_{\varepsilon>0}$ Apart from splitting naturally the various parts of the analysis, this has another advantage. The considerations of Sections 4 to 6 motivate a construction scheme that enables us, given Riemann data $U_\pm$ with $|U_+ - U_-|$ small, to use the Schauder fixed point theorem and construct solutions U_ε of $(\mathcal{P}_\varepsilon)$ that are of uniformly small oscillation as well as of uniformly small variation. One interesting feature of the scheme is that it is based on the quadratic equation (3.18) rather than on a linearized equation. This final part of the analysis is carried out in Section 7. It justifies in particular Hypothesis (C_o) and leads to the following theorem.

Theorem 3.1 *Assume that* (1.1) *is strictly hyperbolic and let U_- be fixed. There exists r sufficiently small such that for $\varepsilon > 0$ and $|U_+ - U_-| < r$ the problem $(\mathcal{P}_\varepsilon)$ has a solution U_ε with the properties*:

(i) *The family $\{U_\varepsilon\}_{\varepsilon>0}$ satisfies (C_o) with some μ independent of ε.*

(ii) *The solutions U_ε satisfy the representation formula*

$$U'_\varepsilon = \sum_{k=1}^{N} \left[\, \tau_{k,\varepsilon}\varphi_k + \theta_k(\cdot;\tau_\varepsilon)\right] r_k(U_\varepsilon)\,, \tag{3.33}$$

where φ_k is given by (3.25), *$\theta_k(\cdot;\tau)$ satisfies* (3.29), *and τ_ε solves $S(\tau_\varepsilon) = U_+$.*

(iii) *The family $\{U'_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^1(\mathbb{R})$ and $\{U_\varepsilon\}_{\varepsilon>0}$ is of uniformly bounded (and small) variation.*

We list below certain properties of $S = F(U)$ relating to the coefficients $\beta_{k,mn} = \nabla l_k\, r_m \cdot r_n$. First, $\{r_k\}$ and $\{l_k\}$ form bases of the (trivial) tangent and cotangent spaces of the state space at each U. Let f^j be the components of F and consider the action of the Hessian $\nabla^2 F(a,b)$ on the vectors $a\,,\, b \in \mathbb{R}^N$. $\nabla^2 F(a,b)$ is vector-valued with components $a \cdot \nabla^2 f^j\, b$. Since $\nabla^2 f^j$ is symmetric, it follows that $\nabla^2 F(a,b) = \nabla^2 F(b,a)$. For U fixed, $t \in \mathbb{R}$ and $a\,,\, b \in \mathbb{R}^N$ equation (3.6) implies

$$l_k(U + ta) \cdot \nabla F(U + ta)\, b = \lambda_k(U + ta)\, l_k(U + ta) \cdot b\,. \tag{3.34}$$

Differentiating (3.34) with respect to t and setting $t = 0$ in the resulting equation, we deduce the identity

$$l_k \cdot \nabla^2 F(a,b) = (\nabla\lambda_k \cdot a)\,(l_k \cdot b) - (\nabla l_k\, a)\cdot(\nabla F - \lambda_k I)b\,, \tag{3.35}$$

which, in turn, yields the well-known identities

$$\begin{aligned} l_k \cdot \nabla^2 F(r_m, r_n) &= (\nabla\lambda_k \cdot r_m)\,(l_k \cdot r_n) + (\lambda_k - \lambda_n)(\nabla l_k\, r_m \cdot r_n) \\ &= \begin{cases} (\lambda_k - \lambda_n)(\nabla l_k\, r_m \cdot r_n) & k \neq n, \\ (\nabla\lambda_k \cdot r_m)\,(l_k \cdot r_k) & k = n. \end{cases} \end{aligned} \tag{3.36}$$

The coefficients $\beta_{k,mn}$ are related to the second derivatives $l_k \cdot \nabla^2 F(r_m, r_n)$ whenever $k \neq m$ or $k \neq n$. There is also the formula

$$(\lambda_k - \lambda_n)\,(\nabla l_k\, r_k \cdot r_n) = (\nabla\lambda_k \cdot r_n)\,(l_k \cdot r_k)\,, \qquad n \neq k\,. \tag{3.37}$$

The coefficient $\beta_{k,kk} = \nabla l_k\, r_k \cdot r_k$ does not appear in the above relations. To explain that, consider the effect of renormalizing the eigenvectors on the coefficients $\beta_{k,mn}$ and especially to $\beta_{k,kk}$. Let $\{\hat r_k\}$ and $\{\hat l_k\}$ be a given set of right and left eigenvectors and set $r_k = \tau_k \hat r_k$, $l_k = s_k \hat l_k$ where $\tau_k = \tau_k(U)$ and $s_k = s_k(U)$ are renormalizing factors with $\tau_k > 0$, $s_k > 0$. A simple computation shows $\nabla l_k = \hat l_k \otimes \nabla s_k + s_k \nabla \hat l_k$ and thus

$$\begin{aligned} \beta_{k,mn} = \nabla l_k\, r_m \cdot r_n &= \tau_m \tau_n \left[(\hat r_m \cdot \nabla s_k)\,(\hat l_k \cdot \hat r_n) + s_k \nabla \hat l_k\, \hat r_m \cdot \hat r_n \right] \\ &= \tau_m \tau_n \left[(\hat r_m \cdot \nabla s_k)\,(\hat l_k \cdot \hat r_n) + s_k \hat\beta_{k,mn} \right]. \end{aligned} \tag{3.38}$$

If $k \neq n$ the renormalization has no effect on the sign of $\beta_{k,mn}$. However, if $k = n$ the renormalization of the left eigenvectors affects $\beta_{k,mk}$ and can make it to be zero.

In particular, on a small neighborhood of some state U, we can choose a renormalization so that the resulting eigenvectors satisfy simultaneously

$$l_k \cdot r_k = 1\,, \qquad \nabla l_k\, r_k \cdot r_k = 0\,. \tag{3.39}$$

To this end, choose first s_k so that

$$(\hat r_k \cdot \nabla s_k)\,(\hat l_k \cdot \hat r_k) + s_k \nabla \hat l_k\, \hat r_k \cdot \hat r_k = 0\,. \tag{3.40}$$

(3.40) is a hyperbolic equation for s_k. If we assign data for s_k on a hypersurface $\mathcal{S}$ transversal to the vector field $\hat r_k$, the Cauchy problem for (3.40) has locally a unique solution. If the data are positive then $s_k > 0$. Following that, τ_k is chosen so that $\tau_k s_k \hat l_k \cdot \hat r_k = 1$. The resulting $\{r_k\}$, $\{l_k\}$ have the desired properties.

4. Properties of the functions φ_k - Wave interaction estimates

Let $C^0(-\infty,\infty)$ stand for the space of the continuous, bounded (scalar or vector-valued) functions. Consider the set

$$\bar{\Omega} = \{ V \in C^0(-\infty,\infty) \ : \quad \sup_{\xi\in\mathbb{R}} |V(\xi) - U_-| \le \mu \} \tag{4.1}$$

and suppose that μ is sufficiently small so that the wave speeds $\lambda_k(V)$ are bounded and totally separated for $V \in \bar{\Omega}$:

$$\lambda_{k-} \le \lambda_k\big(V(\xi)\big) \le \lambda_{k+} \tag{A_1}$$

$$\begin{aligned} &\lambda_{1-} \le \lambda_1\big(V(\xi)\big) \le \lambda_{1+} < \lambda_{2-} \le \lambda_2\big(V(\xi)\big) \le \lambda_{2+} < \dots \\ &\quad < \lambda_{(N-1)-} \le \lambda_{N-1}\big(V(\xi)\big) \le \lambda_{(N-1)+} < \lambda_{N-} \le \lambda_N\big(V(\xi)\big) \le \lambda_{N+} \,. \end{aligned} \tag{A_2}$$

Consider the linearized equation

$$\varepsilon\, \varphi_k' + \big(\xi - \lambda_k(V(\xi))\big)\, \varphi_k = 0 \,. \tag{4.2}$$

The fundamental solution of (4.2) may be written in the form

$$\varphi_k = \frac{e^{-\frac{1}{\varepsilon} g_k}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_k(\zeta)}\, d\zeta} = \frac{1}{I_{k\varepsilon}}\ e^{-\frac{1}{\varepsilon}\int_{\rho_k}^{\xi} s - \lambda_k(V(s))\, ds} \,, \tag{4.3}$$

where

$$\begin{aligned} g_k &= \int_{\rho_k}^{\xi} \big[\,\zeta - \lambda_k(V(\zeta))\,\big]\, d\zeta \,, \\ I_{k\varepsilon} &= \int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_k}\, d\zeta = \int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_{\rho_k}^{\zeta} s - \lambda_k(V(s))\, ds}\, d\zeta \,. \end{aligned} \tag{4.4}$$

Recall that g_k has the form of a potential-well function (*cf.* Figure 1) and that ρ_k is selected as a point where g_k achieves its global minimum. As a result ρ_k satisfies $\lambda_{k-} \le \rho_k \le \lambda_{k+}$, $\lambda_k(V(\rho_k)) = \rho_k$ and

$$g_k(\xi) \ge g_k(\rho_k) = 0 \,, \qquad \xi \in \mathbb{R} \,. \tag{4.5}$$

The aim of this section is to establish various estimates on the functions φ_k and integrals involving them, that are needed in the forthcoming constructions.

We begin with a careful analysis of the behavior of φ_k in the limit $\varepsilon \to 0$. Given a positive function $h(\varepsilon)$, we use the customary notation $f(\varepsilon) = O(h(\varepsilon))$ as $\varepsilon \to 0$ to mean that there are constants ε_0 sufficiently small and C such that $|f(\varepsilon)| \le C\, h(\varepsilon)$ for $0 < \varepsilon \le \varepsilon_0$.

Lemma 4.1. *Suppose the wave speed* $\lambda_k(V)$ *satisfies* (A_1).

(i) *If* $d_k = \lambda_{k+} - \lambda_{k-} > 0$, *then as* $\varepsilon \to 0$:

$$\frac{1}{O(1)}\,\frac{\varepsilon}{d_k} \le I_{k\varepsilon} \le d_k + \sqrt{2\,\pi\,\varepsilon}\,, \tag{4.6}$$

$$0 < \varphi_k(\xi) \le O(1)\,\frac{d_k}{\varepsilon}\,, \qquad \text{for } \xi \in \mathbb{R}\,, \tag{4.7}$$

and

$$\begin{aligned} &\varphi_k(\xi) \le O(1)\,\frac{d_k}{\varepsilon}\,e^{-\frac{1}{2\varepsilon}(\xi-\lambda_{k-})^2}\,, &&\text{for } \xi < \lambda_{k-}\,,\\ &\varphi_k(\xi) \le O(1)\,\frac{d_k}{\varepsilon}\,e^{-\frac{1}{2\varepsilon}(\xi-\lambda_{k+})^2}\,, &&\text{for } \xi > \lambda_{k+}\,. \end{aligned} \tag{4.8}$$

(ii) *If* $d_k = \lambda_{k+} - \lambda_{k-} = 0$, *then*

$$I_{k\varepsilon} = \sqrt{2\pi\varepsilon}\,, \qquad \varphi_k(\xi) = \frac{1}{\sqrt{2\pi\varepsilon}}\,e^{-\frac{1}{2\varepsilon}(\xi-\lambda_{k-})^2}\,. \tag{4.9}$$

Proof. Assume first that $d_k > 0$. Performing the change of variable $\zeta = \rho_k + \sqrt{\varepsilon}\,\eta$ in the integral (4.4), we obtain

$$\begin{aligned} I_{k\varepsilon} &= \int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}g_k(\zeta)}\,d\zeta = \sqrt{\varepsilon}\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}g_k(\rho_k+\sqrt{\varepsilon}\,\eta)}\,d\eta \\ &= \sqrt{\varepsilon}\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_{\rho_k}^{\rho_k+\sqrt{\varepsilon}\,\eta} s-\lambda_k(V(s))\,ds}\,d\eta\,. \end{aligned} \tag{4.10}$$

Using again the change of variable $s = \rho_k + \sqrt{\varepsilon}\,\tau$ and (4.5), we have

$$\begin{aligned} \frac{1}{\varepsilon}\,g_k(\rho_k+\sqrt{\varepsilon}\,\eta) &= \frac{1}{\varepsilon}\int_{\rho_k}^{\rho_k+\sqrt{\varepsilon}\,\eta} s-\lambda_k(V(s))\,ds \\ &= \int_0^{\eta}\Big\{\tau - \frac{1}{\sqrt{\varepsilon}}\big[\lambda_k(V(\rho_k+\sqrt{\varepsilon}\,\tau)) - \lambda_k(V(\rho_k))\big]\Big\}\,d\tau \\ &\ge 0\,, \qquad\qquad \text{for } \eta \in \mathbb{R}\,. \end{aligned} \tag{4.11}$$

We notice that for $\eta > 0$

$$\int_0^{\eta} \tau - \frac{1}{\sqrt{\varepsilon}}\big[\lambda_k(V(\rho_k+\sqrt{\varepsilon}\,\tau)) - \lambda_k(V(\rho_k))\big]\,d\tau \le \frac{\eta^2}{2} + \frac{1}{\sqrt{\varepsilon}}\,(\lambda_{k+}-\lambda_{k-})\,\eta \tag{4.12}$$

while for $\eta < 0$

$$\int_0^{\eta} \tau - \frac{1}{\sqrt{\varepsilon}}\big[\lambda_k(V(\rho_k+\sqrt{\varepsilon}\,\tau)) - \lambda_k(V(\rho_k))\big]\,d\tau \le \frac{\eta^2}{2} - \frac{1}{\sqrt{\varepsilon}}\,(\lambda_{k+}-\lambda_{k-})\,\eta\,. \tag{4.13}$$

Therefore (4.10), (4.11), (4.12) and (4.13) provide the estimate

$$
\begin{aligned}
I_{k\varepsilon} &= \sqrt{\varepsilon} \int_{-\infty}^{0} \exp\left\{ - \int_0^{\eta} \tau - \frac{1}{\sqrt{\varepsilon}} \left[\lambda_k(V(\rho_k + \sqrt{\varepsilon}\,\tau)) - \lambda_k(V(\rho_k)) \right] d\tau \right\} d\eta \\
&+ \sqrt{\varepsilon} \int_{0}^{\infty} \exp\left\{ - \int_0^{\eta} \tau - \frac{1}{\sqrt{\varepsilon}} \left[\lambda_k(V(\rho_k + \sqrt{\varepsilon}\,\tau)) - \lambda_k(V(\rho_k)) \right] d\tau \right\} d\eta \\
&\geq \sqrt{\varepsilon}\, e^{\frac{d_k^2}{2\varepsilon}} \int_{-\infty}^{0} e^{-\frac{1}{2}\left(\eta - \frac{d_k}{\sqrt{\varepsilon}}\right)^2} d\eta + \sqrt{\varepsilon}\, e^{\frac{d_k^2}{2\varepsilon}} \int_{0}^{\infty} e^{-\frac{1}{2}\left(\eta + \frac{d_k}{\sqrt{\varepsilon}}\right)^2} d\eta \\
&= \sqrt{\varepsilon}\, e^{\frac{d_k^2}{2\varepsilon}} \left(\int_{-\infty}^{-\frac{d_k}{\sqrt{\varepsilon}}} e^{-\frac{\zeta^2}{2}}\, d\zeta + \int_{\frac{d_k}{\sqrt{\varepsilon}}}^{\infty} e^{-\frac{\zeta^2}{2}}\, d\zeta \right) .
\end{aligned}
\tag{4.14}
$$

The asymptotic behavior of the last integrals can be evaluated by using the limits

$$
\lim_{x\to\infty} \frac{\int_x^{\infty} e^{-\frac{\zeta^2}{2}}\, d\zeta}{\frac{1}{x} e^{-\frac{x^2}{2}}} = \lim_{x\to\infty} \frac{-e^{-\frac{x^2}{2}}}{-\frac{1}{x^2} e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}}} = 1
\tag{4.15}
$$

$$
\lim_{x\to-\infty} \frac{\int_{-\infty}^{x} e^{-\frac{\zeta^2}{2}}\, d\zeta}{-\frac{1}{x} e^{-\frac{x^2}{2}}} = \lim_{x\to-\infty} \frac{e^{-\frac{x^2}{2}}}{\frac{1}{x^2} e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}} = 1
\tag{4.16}
$$

and yields for small ε

$$
I_{k\varepsilon} \geq \sqrt{\varepsilon}\, e^{\frac{d_k^2}{2\varepsilon}} \left(\frac{1}{O(1)} \frac{\sqrt{\varepsilon}}{d_k} e^{-\frac{d_k^2}{2\varepsilon}} + \frac{1}{O(1)} \frac{\sqrt{\varepsilon}}{d_k} e^{-\frac{d_k^2}{2\varepsilon}} \right) = \frac{1}{O(1)} \frac{\varepsilon}{d_k} .
\tag{4.17}
$$

Next, observe that for $\xi > \lambda_{k+} \geq \rho_k$

$$
\begin{aligned}
g_k(\xi) &= \int_{\lambda_{k+}}^{\xi} s - \lambda_k(V(s))\, ds + g_k(\lambda_{k+}) \\
&\geq \int_{\lambda_{k+}}^{\xi} (s - \lambda_{k+})\, ds = \frac{1}{2}(\xi - \lambda_{k+})^2 ,
\end{aligned}
\tag{4.18}
$$

while for $\xi < \lambda_{k-} \leq \rho_k$

$$
\begin{aligned}
g_k(\xi) &= \int_{\lambda_{k-}}^{\xi} s - \lambda_k(V(s))\, ds + g_k(\lambda_{k-}) \\
&\geq - \int_{\xi}^{\lambda_{k-}} (s - \lambda_{k-})\, ds = \frac{1}{2}(\xi - \lambda_{k-})^2 .
\end{aligned}
\tag{4.19}
$$

Therefore, (4.4) and (4.5) imply

$$
\begin{aligned}
I_{k\varepsilon} &\leq \int_{-\infty}^{\lambda_{k-}} e^{-\frac{1}{2\varepsilon}(\zeta - \lambda_{k-})^2}\, d\zeta + d_k + \int_{\lambda_{k+}}^{\infty} e^{-\frac{1}{2\varepsilon}(\zeta - \lambda_{k+})^2}\, d\zeta \\
&= d_k + \sqrt{\varepsilon} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\eta^2}\, d\eta = d_k + \sqrt{2\,\pi\,\varepsilon}
\end{aligned}
\tag{4.20}
$$

which together with (4.17) complete the proof of (4.6).

Estimates (4.7) and (4.8) follow from

$$\varphi_k(\xi) = \frac{e^{-\frac{1}{\varepsilon} g_k(\xi)}}{I_{k\varepsilon}} \leq O(1)\, \frac{d_k}{\varepsilon}\, e^{-\frac{1}{\varepsilon} g_k(\xi)}\,, \tag{4.21}$$

a consequence of (4.3) and (4.6), in conjunction with (4.5), (4.18) and (4.19). Finally, if $d_k = 0$ then $\lambda_k(V)$ remains constant, say λ_{k-}, and (4.9) follows from (4.3) and (4.4) via a direct calculation. ∎

Remark. To expand on the implications of the lemma, suppose we are given a family of functions $\{U_\varepsilon\}_{\varepsilon>0} \subset \bar{\Omega}$ and that for each U_ε we define the corresponding solution $\varphi_{k\varepsilon}$ of (3.24). Then (4.8) implies that $\varphi_{k\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly on any interval of the form $(-\infty, a_k] \cup [b_k, \infty)$ with $a_k < \lambda_{k-} \leq \lambda_{k+} < b_k$. The family $\{\varphi_{k\varepsilon}\}_{\varepsilon>0}$ is uniformly bounded in L^1 and thus there exists a subsequence $\varphi_{k\varepsilon_n}$ with $\varepsilon_n \to 0$ and a finite Borel measure ϕ_k with $\mathrm{supp}\,\phi_\mathrm{k} \subset [\lambda_{\mathrm{k}-}, \lambda_{\mathrm{k}+}]$ such that $\varphi_{k\varepsilon_n} \rightharpoonup \phi_k$ weak-$\star$ in measures. For the single conservation law or the equations of isothermal elasticity, objects similar to ϕ_k yield the same structure for the Riemann problem solution as that obtained by the Liu shock admissibility criterion. (*cf* Tzavaras [Tz$_2$]).

Our next task is to study certain integrals involving φ_m and φ_n that calculate the effect of interactions between elementary waves. It is convenient to introduce the notation:
d_k = length of the interval $[\lambda_{k-}, \lambda_{k+}]$,
c_k = middle point of the interval $[\lambda_{k-}, \lambda_{k+}]$,
$d(\xi, \lambda_k)$ = distance between the point ξ and the interval $[\lambda_{k-}, \lambda_{k+}]$,
$D_{mn} = d(\lambda_m, \lambda_n)$ = distance between the intervals $[\lambda_{m-}, \lambda_{m+}]$ and $[\lambda_{n-}, \lambda_{n+}]$.
Because of (A_2), $D_{mn} > 0$. Also, we may assume without loss of generality that $d_k > 0$ by replacing (4.9) with the weaker estimates $(4.6 - 4.8)$. Lemma 4.1 indicates that φ_k has the form shown in Figure 2. The behavior of φ_k is uncontrolled in the interval $[\lambda_{k-}, \lambda_{k+}]$, where the wave speed $\lambda_k(V)$ takes values, but its amplitude is at most of order $O(\frac{1}{\varepsilon})$. For $\xi \notin [\lambda_{k-}, \lambda_{k+}]$, φ_k decays like $O(\frac{1}{\varepsilon}\, e^{-\frac{1}{2\varepsilon} d(\xi,\lambda_k)^2})$. It is expedient to fix points a_k, b_k, $k = 1, \ldots, N$, such that

$$\begin{aligned} &a_1 < \lambda_{1-} \leq \lambda_{1+} < b_1 < a_2 < \lambda_{2-} \leq \lambda_{2+} < b_2 < \ldots \\ &\qquad < a_{N-1} < \lambda_{(N-1)-} \leq \lambda_{(N-1)+} < b_{N-1} < a_N < \lambda_{N-} \leq \lambda_{N+} < b_N\,, \end{aligned} \tag{4.22}$$

and introduce the notation

$$s_k(\varepsilon) = \max_{\xi \notin [a_k, b_k]} \varphi_k(\xi)\,, \qquad \alpha_k = \frac{1}{2} \min\{|a_k - \lambda_{k-}|^2,\, |b_k - \lambda_{k+}|^2\}\,. \tag{4.23}$$

Then (4.8) implies

$$\varphi_k(\xi) \leq s_k(\varepsilon) \leq d_k\, O\big(\frac{1}{\varepsilon}\, e^{-\frac{\alpha_k}{\varepsilon}}\big)\,, \qquad \xi \notin [a_k, b_k]\,, \tag{4.24}$$

and $s_k(\varepsilon)$ serves as a global bound outside the main support of the wave. The function $h_k = \frac{1}{\varepsilon}\, e^{-\frac{\alpha_k}{\varepsilon}}$ describing the decay rate behaves as follows: As ε increases h_k increases from 0 to its maximum value $1/e\,\alpha_k$, achieved at $\varepsilon = \alpha_k$, and then decreases down to 0 as $\varepsilon \to \infty$.

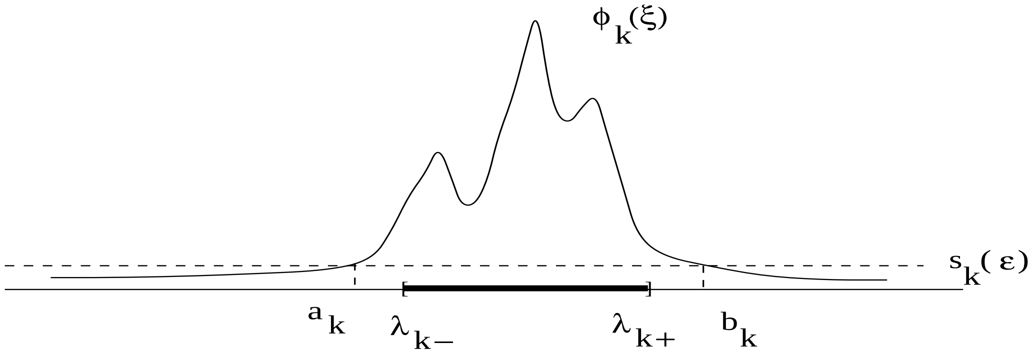

FIGURE 2.

Lemma 4.2. *Suppose the wave speeds $\lambda_k(V)$ satisfy (A_2) and*

$$(A_3) \qquad (1+\sqrt{3})\,(d_m + d_k) < d(\lambda_k,\, \lambda_m) = D_{km}$$

for $V \in \bar{\Omega}$. Then there exist constants $\alpha_{km} > 0$ depending on d_k, d_m, D_{km} but independent of ε, V such that

$$\left| e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k(\zeta)} \varphi_m(\zeta)\, d\zeta \right| \le \begin{cases} \frac{1}{D_{km}}\, \varepsilon\, \varphi_m + O(e^{-\frac{\alpha_{km}}{\varepsilon}})\, \frac{d_k d_m}{D_{km}}\, \varphi_k\,, & m \ne k, \\ |\xi - c_k|\, \varphi_k\,, & m = k. \end{cases} \tag{4.25}$$

Proof. When $m = k$, (4.25) follows from a direct calculation. So suppose that $m \ne k$. Using the notation $\lambda_k = \lambda_k(V(\xi))$ and (4.3) we obtain the chain of identities

$$\begin{aligned} e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k(\zeta)} \varphi_m(\zeta)\, d\zeta &= e^{-\frac{1}{\varepsilon}\int_{\rho_k}^{\xi} s-\lambda_k\, ds} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\rho_k}^{\zeta} s-\lambda_k\, ds}\, \frac{e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\zeta} s-\lambda_m\, ds}}{I_{m\varepsilon}}\, d\zeta \\ &= \frac{e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\xi} s-\lambda_m\, ds}}{I_{m\varepsilon}} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\xi}^{\zeta} s-\lambda_k\, ds}\, e^{-\frac{1}{\varepsilon}\int_{\xi}^{\zeta} s-\lambda_m\, ds}\, d\zeta \\ &= \varphi_m \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\xi}^{\zeta} \lambda_m-\lambda_k\, ds}\, d\zeta\,. \end{aligned} \tag{4.26}$$

In view of (A_2), we have

$$\begin{aligned} \left| \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\xi}^{\zeta} \lambda_m-\lambda_k\, ds}\, d\zeta \right| &\le \left| \pm \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\xi}^{\zeta} \lambda_m-\lambda_k\, ds}\, \frac{|\lambda_m - \lambda_k|}{d(\lambda_m, \lambda_k)}\, d\zeta \right| \\ &= \frac{1}{D_{mk}} \left| \varepsilon \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}\int_{\xi}^{\zeta} \lambda_m-\lambda_k\, ds}\, d\Big(\frac{1}{\varepsilon} \int_{\xi}^{\zeta} \lambda_m - \lambda_k\, ds \Big) \right| \\ &\le \frac{\varepsilon}{D_{mk}} \Big(1 + e^{-\frac{1}{\varepsilon}\int_{c_k}^{\xi} \lambda_m-\lambda_k\, ds} \Big)\,. \end{aligned} \tag{4.27}$$

Combining (4.26) with (4.27) and using (4.3), we arrive at the estimate

$$
\begin{aligned}
\Big| e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \varphi_m \, d\zeta \Big| &\leq \frac{\varepsilon}{D_{mk}} \left(\varphi_m + \frac{e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\xi} s-\lambda_m \, ds}}{I_{m\varepsilon}} e^{-\frac{1}{\varepsilon}\int_{\rho_k}^{\xi} \lambda_m-\lambda_k \, ds} e^{\frac{1}{\varepsilon}\int_{\rho_k}^{c_k} \lambda_m-\lambda_k \, ds} \right) \\
&= \frac{\varepsilon}{D_{mk}} \left(\varphi_m + \frac{e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\rho_k} s-\lambda_m \, ds}}{I_{m\varepsilon}} e^{-\frac{1}{\varepsilon}\int_{\rho_k}^{\xi} s-\lambda_k \, ds} e^{\frac{1}{\varepsilon}\int_{\rho_k}^{c_k} \lambda_m-\lambda_k \, ds} \right) \\
&= \frac{\varepsilon}{D_{mk}} \varphi_m + \frac{\varepsilon}{D_{mk}} \frac{1}{I_{m\varepsilon}} \left(e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\rho_k} s-\lambda_m \, ds + \frac{1}{\varepsilon}\int_{\rho_k}^{c_k} \lambda_m-\lambda_k \, ds} \right) I_{k\varepsilon} \, \varphi_k \, .
\end{aligned}
\tag{4.28}
$$

The goal is to show that under (A_3) the term in parentheses decays as $\varepsilon \to 0$. To this end, observe that

$$
- \int_{\rho_m}^{\rho_k} s - \lambda_m \, ds \leq -\frac{1}{2} D_{km}^2 \tag{4.29}
$$

$$
\int_{\rho_k}^{c_k} \lambda_m - \lambda_k \, ds \leq d_k \left(D_{km} + d_k + d_m \right) . \tag{4.30}
$$

It suffices to show

$$
-\alpha_{km} := -\frac{1}{2} D_{km}^2 + (d_k + d_m) \, D_{km} + (d_k + d_m)^2 < 0 \, . \tag{4.31}
$$

Since the roots of the quadratic $-\frac{1}{2}x^2 + x + 1$ are $1 \pm \sqrt{3}$, hypothesis (A_3) implies the inequality (4.31) and thus there exists a positive constant α_{km} such that

$$
e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\rho_k} s-\lambda_m \, ds + \frac{1}{\varepsilon}\int_{\rho_k}^{c_k} \lambda_m-\lambda_k \, ds} \leq O(e^{-\frac{\alpha_{km}}{\varepsilon}}) \, . \tag{4.32}
$$

The proof of the lemma follows from (4.28), (4.32) and (4.6). ∎

Our next objective is to use the facts that each φ_k is essentially supported on the interval $[\lambda_{k-}, \lambda_{k+}]$ and that such intervals are distinct in order to estimate the integrals

$$
F_{k,mn}(\xi) = e^{-\frac{1}{\varepsilon} g_k(\xi)} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k(\zeta)} \, \varphi_m(\zeta) \, \varphi_n(\zeta) \, d\zeta \, . \tag{4.33}
$$

We begin with

Lemma 4.3. *Suppose that* $\lambda_k(V)$, $k = 1, \ldots, N$, *satisfy* (A_2) *and* (A_3). *Then*

(i) *for* $m = 1, \ldots, N$,

$$
\Big| e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \, \varphi_m \, \varphi_k \, d\zeta \Big| \leq \varphi_k \, , \tag{4.34}
$$

(ii) *for* $m,\, n = 1,\, \dots,\, N$, *with* $m \neq n$, $m \neq k$ *and* $n \neq k$,

$$
\begin{aligned}
\Big|\, e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k}\, \varphi_m\, \varphi_n\, d\zeta \,\Big| \leq \frac{\varepsilon\, s_m(\varepsilon)}{D_{kn}}\, \varphi_n + \frac{\varepsilon\, s_n(\varepsilon)}{D_{km}}\, \varphi_m \qquad\qquad\qquad\qquad \\
+ \big[s_m(\varepsilon)\, O(e^{-\frac{\alpha_{kn}}{\varepsilon}}) \frac{d_k d_n}{D_{kn}} + s_n(\varepsilon)\, O(e^{-\frac{\alpha_{km}}{\varepsilon}}) \frac{d_k d_m}{D_{km}} \big]\, \varphi_k\,.
\end{aligned}
\tag{4.35}
$$

Proof. First we show (i). Since

$$
F_{k,mk} = e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k}\, \varphi_m\, \frac{e^{-\frac{1}{\varepsilon} g_k}}{I_{k\varepsilon}}\, d\zeta = \varphi_k \int_{c_k}^{\xi} \varphi_m\, d\zeta\,,
\tag{4.36}
$$

it follows that $|F_{k,mk}| \leq \varphi_k$ and (4.34) is proved. Observe next that because of (A_2)

$$
\varphi_m\, \varphi_n \leq s_m(\varepsilon)\, \varphi_n + s_n(\varepsilon)\, \varphi_m\,, \qquad \text{for } m \neq n\,,\ \xi \in \mathbb{R}\,.
\tag{4.37}
$$

Using (4.25) with $m \neq k$ and $n \neq k$, we obtain

$$
\begin{aligned}
|F_{k,mn}| &\leq s_m(\varepsilon)\, \big| e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k}\, \varphi_n\, d\zeta \,\big| + s_n(\varepsilon)\, \big| e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k}\, \varphi_m\, d\zeta \,\big| \\
&\leq \frac{s_m(\varepsilon)}{D_{kn}} \big(\varepsilon\, \varphi_n + O(e^{-\frac{\alpha_{kn}}{\varepsilon}})\, d_k d_n\, \varphi_k \big) + \frac{s_n(\varepsilon)}{D_{km}} \big(\varepsilon\, \varphi_m + O(e^{-\frac{\alpha_{km}}{\varepsilon}})\, d_k d_m\, \varphi_k \big)\,,
\end{aligned}
\tag{4.38}
$$

which in turn yields (4.35). ∎

It remains to estimate the integrals $F_{k,mm}$ with $m \neq k$, that account for the effect of self-interactions. Using (4.3), we write $F_{k,mm}$ in the form

$$
\begin{aligned}
F_{k,mm} &= e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k}\, \varphi_m^2\, d\zeta \\
&= e^{-\frac{1}{\varepsilon} \int_{\rho_k}^{\xi} s - \lambda_k\, ds} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} \int_{\rho_k}^{\zeta} s - \lambda_k\, ds}\, \frac{e^{-\frac{2}{\varepsilon} \int_{\rho_m}^{\zeta} s - \lambda_m\, ds}}{I_{m\varepsilon}^2}\, d\zeta \\
&= \frac{e^{-\frac{2}{\varepsilon} \int_{\rho_m}^{\xi} s - \lambda_m\, ds}}{I_{m\varepsilon}^2} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} \int_{\xi}^{\zeta} s - \lambda_k\, ds}\, e^{-\frac{2}{\varepsilon} \int_{\xi}^{\zeta} s - \lambda_m\, ds}\, d\zeta \\
&= \varphi_m^2 \int_{c_k}^{\xi} e^{-\frac{1}{\varepsilon} \int_{\xi}^{\zeta} s - \Lambda_{km}\, ds}\, d\zeta\,,
\end{aligned}
\tag{4.39}
$$

where we have set

$$
\Lambda_{km}(U) = 2\, \lambda_m(U) - \lambda_k(U) = \lambda_m(U) + \big(\lambda_m(U) - \lambda_k(U) \big)\,.
\tag{4.40}
$$

Note that the ordering goes $\lambda_k(U) < \lambda_m(U) < \Lambda_{km}(U)$ when $k < m$ and $\Lambda_{km}(U) < \lambda_m(U) < \lambda_k(U)$ when $k > m$. In order to estimate $F_{k,mm}$, it is necessary to study the ranges of the wave speeds $\lambda_k(V)$ and $\lambda_m(V)$ relative to the range of the composite speed $\Lambda_{km}(V)$, for $V \in \bar{\Omega}$, and to impose conditions that guarantee non-resonance between the wave speeds and the composite speed. Note that $\Lambda_{km}(V)$ is bounded by

$$\Lambda_{km-} \leq \Lambda_{km}(V(\xi)) \leq \Lambda_{km+} \tag{4.41}$$

where the constants Λ_{km-}, Λ_{km+} and d_{km}, the length of the range of $\Lambda_{km}(V)$, depend only on μ. We introduce the notation

$d(\xi, \Lambda_{km})$ = distance between the point ξ and the interval $[\Lambda_{km-}, \Lambda_{km+}]$,

$d(\lambda_m, \Lambda_{km})$ = distance between the intervals $[\lambda_{m-}, \lambda_{m+}]$, $[\Lambda_{km-}, \Lambda_{km+}]$,

and impose the strengthened version of Hypothesis (A_3) :

$$7\,(d_m + d_k) = 7\,[(\lambda_{m+} - \lambda_{m-}) + (\lambda_{k+} - \lambda_{k-})] < d(\lambda_k, \lambda_m) = D_{km}\,. \tag{A_4}$$

It is easy to calculate $d_{km} = \Lambda_{km+} - \Lambda_{km-} = 2d_m + d_k$, $d(\lambda_m, \Lambda_{km}) = D_{km} - d_m$, and to note that the ranges of $\lambda_k(V)$, $\lambda_m(V)$ and $\Lambda_{km}(V)$ are separated for $V \in \bar{\Omega}$ (see Figure 3).

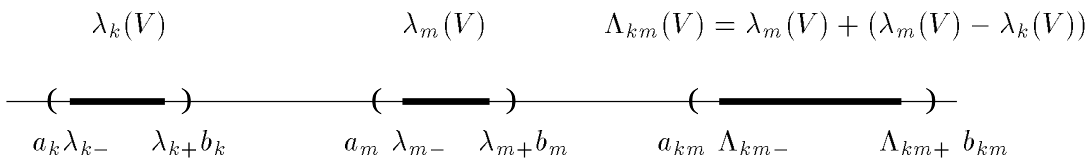

FIGURE 3.

The ranges of $\lambda_k(V)$, $\lambda_m(V)$ and $\Lambda_{km}(V)$ for $m > k$.

Since the lengths d_k are of order $O(\mu)$ while the distances D_{km} are of order $O(1)$ as $\mu \to 0$, hypotheses (A_3) and (A_4) are not particularly restrictive for solutions of small oscillation. $(A_3 - A_4)$ are imposed for all $k, m = 1, \ldots, N$, and points a_{km}, b_{km} are selected (near the support of $\Lambda_{km}(V)$) so that, upon rearranging a_m, b_m if necessary,

$$a_k < \lambda_{k-} \leq \lambda_{k+} < b_k < a_m < \lambda_{m-} \leq \lambda_{m+} < b_m < a_{km} < \Lambda_{km-} \leq \Lambda_{km+} < b_{km}\,, \tag{4.42a}$$

when $k < m$, and

$$a_{km} < \Lambda_{km-} \leq \Lambda_{km+} < b_{km} < a_m < \lambda_{m-} \leq \lambda_{m+} < b_m < a_k < \lambda_{k-} \leq \lambda_{k+} < b_k\,, \tag{4.42b}$$

when $k > m$. Such choices are clearly possible. The points a_k, b_k are now fixed, while the points a_{km}, b_{km} will be selected subject to (4.42) in the course of proving:

Lemma 4.4. *Suppose that $\lambda_k(V)$, $\lambda_m(V)$ satisfy $(A_2 - A_4)$. There are choices of a_{km}, b_{km}* and constants α_{km}, $\beta_{km} > 0$, *depending on d_k, d_m, D_{km} but not on ε, such that*

(a) *if $k < m$, then*

$$\text{(4.43a)} \qquad \left|e^{-\frac{1}{\varepsilon}g_k}\int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k}\,\varphi_m^2\,d\zeta\right| \le \begin{cases} \frac{1}{d(a_{km},\Lambda_{km})}\,\varepsilon\,\varphi_m^2 + \frac{d_m^2\,d_k}{d(\lambda_k,\Lambda_{km})}\,O(\frac{1}{\varepsilon}e^{-\frac{2\alpha_{km}}{\varepsilon}})\,\varphi_k\,, & \xi \le a_{km}, \\ d_{km}\,d_m O(\frac{1}{\varepsilon}\,e^{-\frac{\beta_{km}}{\varepsilon}})\,\varphi_m & \xi \ge a_{km}, \end{cases}$$

(b) *if $k > m$, then*

$$\text{(4.43b)} \qquad \left|e^{-\frac{1}{\varepsilon}g_k}\int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k}\,\varphi_m^2\,d\zeta\right| \le \begin{cases} d_{km}\,d_m O(\frac{1}{\varepsilon}\,e^{-\frac{\beta_{km}}{\varepsilon}})\,\varphi_m & \xi \le b_{km}, \\ \frac{1}{d(b_{km},\Lambda_{km})}\,\varepsilon\,\varphi_m^2 + \frac{d_m^2\,d_k}{d(\lambda_k,\Lambda_{km})}\,O(\frac{1}{\varepsilon}e^{-\frac{2\alpha_{km}}{\varepsilon}})\,\varphi_k\,, & \xi \ge b_{km}. \end{cases}$$

Proof. Let $k < m$ and proceed to prove (a). The ranges of $\lambda_k(V)$, $\lambda_m(V)$ and $\Lambda_{km}(V)$ for $V \in \bar{\Omega}$ are as in Figure 3, and a_{km} is any point compatible with (4.42). Let ρ_{km} be a point where the function $\int_\alpha^\xi s - \Lambda_{km}(V(s))ds$ achieves its global minimum. Then $\Lambda_{km}(V(\rho_{km})) = \rho_{km}$, $\Lambda_{km-} \le \rho_{km} \le \Lambda_{km+}$ and

$$\text{(4.44)} \qquad G_{km}(\xi) = \int_{\rho_{km}}^{\xi} s - \Lambda_{km}(V(s))\,ds \ge 0\,, \quad \text{for } \xi \in \mathbb{R}\,.$$

Consider first the region $\xi \le a_{km} < \Lambda_{km-}$. In this region $F_{k,mm}$ in (4.39) is decomposed into the integrals

$$\text{(4.45)} \quad F_{k,mm} = \varphi_m^2\,e^{\frac{1}{\varepsilon}\int_{\rho_{km}}^{\xi} s-\Lambda_{km}\,ds}\left(\int_{-\infty}^{\xi} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds}\,d\zeta - \int_{-\infty}^{c_k} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds}\,d\zeta\right).$$

The first integral is dominant when $\xi > c_k$ and the second is dominant when $\xi < c_k$. Since $\zeta < \xi \le a_{km} < \Lambda_{km-}$, the first integral is estimated by

$$\text{(4.46)} \qquad \begin{aligned} \int_{-\infty}^{\xi} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds}\,d\zeta &\le \int_{-\infty}^{\xi} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds}\,\Big(\frac{\Lambda_{km}-\zeta}{\Lambda_{km}-\xi}\Big)\,d\zeta \\ &\le \varepsilon\int_{-\infty}^{\xi}\frac{e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds}}{d(\xi,\Lambda_{km})}\,d\Big(-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds\Big) \\ &= \frac{\varepsilon}{d(a_{km},\Lambda_{km})}\,e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\xi} s-\Lambda_{km}\,ds}\,. \end{aligned}$$

In a similar fashion the second integral is estimated by

$$\text{(4.47)} \qquad \begin{aligned} \int_{-\infty}^{c_k} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\,ds}\,d\zeta &\le \frac{\varepsilon}{d(c_k,\Lambda_{km})}\,e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{c_k} s-\Lambda_{km}\,ds} \\ &\le \frac{\varepsilon}{d(\lambda_k,\Lambda_{km})}\,e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{c_k} s-\Lambda_{km}\,ds}\,. \end{aligned}$$

Let $D_1 = d(a_{km}, \Lambda_{km})$, $D_2 = d(\lambda_k, \Lambda_{km})$ and combine (4.45), (4.46), (4.47), and (4.3) to obtain

$$
\begin{aligned}
|F_{k,mm}| &\le \varphi_m^2 \left(\frac{\varepsilon}{D_1} + \frac{\varepsilon}{D_2} e^{\frac{1}{\varepsilon}\int_{c_k}^{\xi} s-\Lambda_{km}\, ds} \right) \\
&= \frac{\varepsilon}{D_1}\varphi_m^2 + \frac{\varepsilon}{D_2}\, \frac{e^{-\frac{2}{\varepsilon}\int_{\rho_m}^{\xi} s-\lambda_m\, ds}}{I_{m\varepsilon}^2}\, e^{\frac{1}{\varepsilon}\int_{c_k}^{\rho_k} s-\Lambda_{km}\, ds}\, e^{\frac{1}{\varepsilon}\int_{\rho_k}^{\xi} s-\Lambda_{km}\, ds} \\
&= \frac{\varepsilon}{D_1}\varphi_m^2 + \frac{\varepsilon}{D_2}\,\frac{1}{I_{m\varepsilon}^2}\, \left(e^{-\frac{2}{\varepsilon}\int_{\rho_m}^{\rho_k} s-\lambda_m\, ds + \frac{1}{\varepsilon}\int_{c_k}^{\rho_k} s-\Lambda_{km}\, ds}\right) I_{k\varepsilon}\,\varphi_k\,.
\end{aligned}
\tag{4.48}
$$

It suffices to show that the term in parentheses decays as $\varepsilon \to 0$. Using the estimations

$$
\begin{aligned}
&-2\int_{\rho_m}^{\rho_k} s-\lambda_m\, ds \le -D_{km}^2\,, \\
&\int_{c_k}^{\rho_k} s-\Lambda_{km}\, ds \le (\Lambda_{km+} - \lambda_{k-})d_k \le 2[D_{km} + (d_m + d_k)](d_m + d_k)\,,
\end{aligned}
\tag{4.49}
$$

together with the fact that (A_3) implies that (4.31) is satisfied, we conclude that

$$
e^{-\frac{2}{\varepsilon}\int_{\rho_m}^{\rho_k} s-\lambda_m\, ds + \frac{1}{\varepsilon}\int_{c_k}^{\rho_k} s-\Lambda_{km}\, ds} \le O(e^{-\frac{2\alpha_{km}}{\varepsilon}})\,.
\tag{4.50}
$$

In conjunction with (4.48) and (4.6), (4.50) shows (4.43) for $\xi \le a_{km}$, $k < m$.

Consider now the region $\xi \ge a_{km}$. An argument similar to the one leading to (4.20) shows that

$$
\left|\int_{c_k}^{\xi} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\, ds}\, d\zeta\right| \le \int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\zeta} s-\Lambda_{km}\, ds}\, d\zeta \le d_{km} + \sqrt{2\pi\varepsilon}\,.
\tag{4.51}
$$

Therefore, (4.39) and (4.40) give

$$
\begin{aligned}
|F_{k,mm}| &\le O(1)\, d_{km}\, \varphi_m^2\, e^{\frac{1}{\varepsilon}\int_{\rho_{km}}^{\xi} s-\Lambda_{km}\, ds} \\
&= O(1)\, d_{km}\, \varphi_m\, \frac{e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\xi} s-\lambda_m\, ds}}{I_{m\varepsilon}}\, e^{\frac{1}{\varepsilon}\int_{\rho_{km}}^{\xi} s-\lambda_m\, ds}\, e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\xi} \lambda_m-\lambda_k\, ds} \\
&\le O(1)\, d_{km}\, \varphi_m \left(\frac{d_m}{\varepsilon}\, e^{-\frac{1}{\varepsilon}\int_{\rho_m}^{\rho_{km}} s-\lambda_m\, ds}\, e^{-\frac{1}{\varepsilon}\int_{\rho_{km}}^{\xi} \lambda_m-\lambda_k\, ds} \right).
\end{aligned}
\tag{4.52}
$$

The goal is to choose a_{km} so that the term in parentheses decays as $\varepsilon \to 0$, for any $\xi \ge a_{km}$. Since $\rho_{km} \notin [\lambda_{m-}, \lambda_{m+}]$, the first term decays as $\varepsilon \to 0$ and its decay rate can be estimated by noting that

$$
\begin{aligned}
-\int_{\rho_m}^{\rho_{km}} s-\lambda_m\, ds &\le -\int_{\lambda_{m+}}^{\Lambda_{km-}} s-\lambda_{m+}\, ds \\
&= -\frac{1}{2}(\Lambda_{km-} - \lambda_{m+})^2 = -\frac{1}{2}(D_{km} - d_m)^2\,.
\end{aligned}
\tag{4.53}
$$

Since $\lambda_m(U) > \lambda_k(U)$, the second term decays for $\xi > \rho_{km}$ but grows for $\xi < \rho_{km}$. The fastest growth occurs for $\xi = a_{km}$ and the growth rate is estimated by

$$
\begin{aligned}
(4.54) \qquad -\int_{\rho_{km}}^{a_{km}} \lambda_m - \lambda_k \, ds &\le (\lambda_{m+} - \lambda_{k-})\,(\rho_{km} - a_{km}) \\
&\le (D_{km} + d_m + d_k)\,(d(a_{km}, \Lambda_{km}) + 2\,d_m + d_k)\,.
\end{aligned}
$$

It suffices to give conditions on d_k, d_m, D_{km} and to choose a_{km} so that

$$
(4.55) \qquad -\beta_{km} := -\frac{1}{2}(D_{km} - d_m)^2 + (D_{km} + d_m + d_k)\,(d(a_{km}, \Lambda_{km}) + 2\,d_m + d_k) < 0\,.
$$

For example, if we choose $a_{km} = \Lambda_{km-} - d_k$ and we require that

$$
(4.56) \qquad 4\,(D_{km} + d_m + d_k)\,(d_m + d_k) < [D_{km} - (d_m + d_k)]^2\,,
$$

then (4.55) is satisfied. By solving the inequality $y^2 - 6xy - 3x^2 > 0$ for y/x, we see that (A_4) implies (4.56). Therefore (4.52) yields the estimate

$$
(4.57) \qquad |F_{k,mm}| \le d_{km}\, d_m O(\frac{1}{\varepsilon}\, e^{-\frac{\beta_{km}}{\varepsilon}})\, \varphi_m
$$

for $\xi > a_{km}$, $k < m$, and completes the proof of part (a). The proof of part (b) is similar. ∎

Lemmas 4.3 and 4.4 provide estimates on the integrals $F_{k,mn}$, which calculate the effect of wave interactions induced by diffusion. The estimates are consequences of the separation hypotheses $(A_2 - A_4)$ on the wave speeds. Obviously (A_4) is the strongest hypothesis and implies the rest. In the sequel we make use of the following implication of (4.34), (4.35), (4.43) and (4.7).

Corollary 4.5. Suppose that $\lambda_k(V)$ *satisfy* (A_4), *for* k, m, $n = 1, \ldots, N$. There is $\varepsilon_0 > 0$ *and a constant* C, *depending on* d_k, D_{km}, D_{kn} *but not on* ε, *such that*

$$
(4.58) \qquad |F_{k,mn}| = \Big|e^{-\frac{1}{\varepsilon} g_k} \int_{\rho_k}^{\xi} e^{\frac{1}{\varepsilon} g_k}\, \varphi_m\, \varphi_n\, d\zeta\Big| \le C \sum_{j=1}^{N} \varphi_j
$$

for k, m, $n = 1, \ldots, N$ *and* $0 < \varepsilon \le \varepsilon_0$.

Remark. It is instructive to identify which of the integrals $F_{k,mn}$ have nonzero contributions in the limit $\varepsilon \to 0$. In view of (4.36) and (4.3), the terms $F_{k,mk}$ and $F_{k,km}$ $F_{k,kk}$ have nonzero limiting contributions supported on the k-th wave speed. On the other hand, (4.35) and (4.7) imply that $F_{k,mn} \to 0$ as $\varepsilon \to 0$ when $m \ne n$, $m \ne k$ and $n \ne k$, which suggests that diffusion induced interactions of two distinct families have no contribution as $\varepsilon \to 0$ on a third family. (Recall that we are dealing with Riemann data solutions). By contrast, (4.43) suggests that the terms $F_{k,mm}$, $m \ne k$, accounting for the effect of self-interactions of the m-th family on the k-th family, have a nonzero contribution in the $\varepsilon \to 0$ limit supported on the m-th wave speed.

5. Validation of the asymptotic expansion

The objective of this section is to solve the problem

$$\begin{aligned}
&\varepsilon\,\theta_k' + \left[\xi - \lambda_k(V(\xi))\right]\theta_k \\
&\qquad = \varepsilon \sum_{m=1}^{N}\sum_{n=1}^{N} \left[\nabla l_k(V(\xi))\, r_m(V(\xi)) \cdot r_n(V(\xi))\right] \left(\tau_m\,\varphi_m + \theta_m\right)\left(\tau_n\,\varphi_n + \theta_n\right) \qquad (5.1)\\
&\theta_k(c_k) = 0
\end{aligned}$$

where $V \in \bar{\Omega}$, defined in (4.1), and $\tau = (\tau_1, \ldots, \tau_N)$ is a vector-parameter in $\mathrm{I\!R}^N$. The aim is to construct solutions $\theta_k(\cdot\,;\tau)$ that are of order $O(|\tau|^2)$ in the wave strength $|\tau| = |\tau_1| + \ldots + |\tau_N|$ as $|\tau| \to 0$. This would validate the asymptotic expansion (3.29).

Throughout the section we use the notation

$$\begin{aligned}
\lambda_k &= \lambda_k(V(\xi))\,, \\
\beta_{k,mn} &= \beta_{k,mn}(V(\xi)) = \nabla l_k(V(\xi))\, r_m(V(\xi)) \cdot r_n(V(\xi))
\end{aligned} \qquad (5.2)$$

and assume that μ is small so that the hypotheses $(A_1 - A_4)$ on the wave speeds are fulfilled for $V \in \bar{\Omega}$. Moreover,

$$|\beta_{k,mn}| \le B \qquad (5.3)$$

with B depending only on μ. Recall that g_k is defined in (4.4) and that c_k is the middle point of the interval $[\lambda_{k-}, \lambda_{k+}]$. Using the variation of parameters formula, (5.1) is expressed as a system of integral equations

$$\theta_k(\xi) = e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \sum_{m,n=1}^{N} \beta_{k,mn}(V(\zeta))\,\left(\tau_m\,\varphi_m(\zeta) + \theta_m(\zeta)\right)\left(\tau_n\,\varphi_n(\zeta) + \theta_n(\zeta)\right) d\zeta\,. \qquad (5.4)$$

Our strategy is to formulate (5.4) as a fixed point problem, and to use the uniform contraction principle in order to construct solutions $\theta_k(\cdot\,;\tau)$, $k = 1, \ldots, N$.

Let $C_0(\mathrm{I\!R})$ stand for the continuous functions that decay to zero as $|\xi| \to \infty$ and define

$$E = \left\{ \chi = (\chi_1, \ldots, \chi_N) \in [C_0(\mathrm{I\!R})]^N \;:\; \sup_{\xi \in \mathrm{I\!R}} \frac{|\chi_j(\xi)|}{\sum_{i=1}^{N} \varphi_i(\xi)} < \infty\,, \quad j = 1, \ldots, N \right\}. \qquad (5.5)$$

E with the weighted sup-norm

$$\|\chi\| = \sum_{j=1}^{N} \sup_{\xi \in \mathrm{I\!R}} \frac{|\chi_j(\xi)|}{\sum_{i=1}^{N} \varphi_i(\xi)} \qquad (5.6)$$

with weight $\sum_{i=1}^N \varphi_i > 0$ is a Banach space. Let $B_\delta = \{\tau \in \mathbb{R}^N \,:\, |\tau| \le \delta\,\}$ and set

$$F = \Big\{\chi \in E \;:\quad |\chi_j(\xi)| \le A\,|\tau|^2 \sum_{i=1}^N \varphi_i(\xi)\,,\quad \xi \in \mathbb{R}\,,\, j = 1, \dots, N\,\Big\}\,, \tag{5.7}$$

where $\tau \in B_\delta$ and A is a constant to be determined later. F is a closed bounded subset of E in the weighted norm $\|\cdot\|$. Define the map T that takes $V \in \bar\Omega$, $\tau \in B_\delta$, $\chi \in F$ to the vector-valued function $T(\chi)$ with components

$$T_k(\chi) = e^{-\frac{1}{\varepsilon}g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \sum_{m,n=1}^N \beta_{k,mn}\,\big(\tau_m\,\varphi_m + \chi_m\big)\,\big(\tau_n\,\varphi_n + \chi_n\big)\,d\zeta\,, \tag{5.8}$$

$k = 1, \dots, N$. The map T has the following properties :

Proposition 5.1. *There exist positive constants A and δ_0 such that for $\delta < \delta_0$:*

(i) *$T : \bar\Omega \times B_\delta \times F \to F$ is well defined.*

(ii) *There exists α, $0 < \alpha < 1$, such that*

$$\|\,T(V,\tau,\chi) - T(V,\tau,\bar\chi)\,\| \le \alpha\,\|\chi - \bar\chi\|\,,\qquad \textit{for } \chi,\ \bar\chi \in F\,, \tag{5.9}$$

and for any $V \in \bar\Omega$, $\tau \in B_\delta$. Therefore $T(V,\tau,\cdot) : F \to F$ is a uniform contraction.

(iii) *There exists a positive constant C, depending on μ but independent of δ, such that*

$$\|\,T(V,\tau,\chi) - T(V,s,\chi)\,\| \le C\,\delta\,|\tau - s|\,,\qquad \textit{for } \tau,\ s \in B_\delta\,, \tag{5.10}$$

and for any $V \in \bar\Omega$, $\chi \in F$.

Proof. In the forthcoming estimates C, C' and C'' stand for generic constants that can be estimated in terms of B, the dimension of the system N, and the constant in the estimate (4.58). As a result, such constants ultimately depend on μ in (4.1), but are independent of δ. We proceed to establish (i). Let $V \in \bar\Omega$, $\tau \in B_\delta$ and $\chi \in F$ be fixed. Then (5.8), (5.7) and (4.58) imply

$$\begin{aligned}
|T_k(\chi)| &\le e^{-\frac{1}{\varepsilon}g_k}\Big|\int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \sum_{m,n=1}^N |\beta_{k,mn}|\,\big(|\tau_m|\,\varphi_m + |\chi_m|\big)\,\big(|\tau_n|\,\varphi_n + |\chi_n|\big)\,d\zeta\Big| \\
&\le B\,e^{-\frac{1}{\varepsilon}g_k}\Big|\int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \sum_{m,n=1}^N \big(|\tau_m|\,\varphi_m + A|\tau|^2 \sum_{i=1}^N \varphi_i\big)\,\big(|\tau_n|\,\varphi_n + A|\tau|^2 \sum_{j=1}^N \varphi_j\big)\,d\zeta\Big| \\
&\le C\,|\tau|^2\,(1 + 2A\delta + A^2\delta^2)\sum_{m,n=1}^N \Big|e^{-\frac{1}{\varepsilon}g_k}\int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k}\,\varphi_m\varphi_n\,d\zeta\Big| \\
&\le C\,(1 + A\delta)^2\,|\tau|^2 \sum_{j=1}^N \varphi_j\,.
\end{aligned} \tag{5.11}$$

Comparing the outcome with (5.7) we see that if

$$C\,(1+A\delta)^2 \le A\,, \tag{5.12}$$

then $T(V,\tau,\chi) \in F$ and (i) is established.

Next, we examine (ii). Let $V \in \bar{\Omega}$, $\tau \in B_\delta$ be fixed, and consider χ, $\bar{\chi} \in F$. Then

$$\begin{aligned} T_k(\chi) - T_k(\bar{\chi}) = e^{-\frac{1}{\varepsilon}g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \sum_{m,n=1}^{N} \beta_{k,mn} \big[\tau_m\, \varphi_m\,(\chi_n - \bar{\chi}_n) + \tau_n\, \varphi_n\,(\chi_m - \bar{\chi}_m) \\ + (\chi_m \chi_n - \bar{\chi}_m \bar{\chi}_n) \big]\, d\zeta\ . \end{aligned} \tag{5.13}$$

Using (5.6), (5.7), (5.11) and (4.58) we obtain

$$\begin{aligned} |T_k(\chi) - T_k(\bar{\chi})| &\le e^{-\frac{1}{\varepsilon}g_k} \Big| \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \sum_{m,n=1}^{N} |\beta_{k,mn}|\, \big[|\tau_m|\, \varphi_m\, |\chi_n - \bar{\chi}_n| + |\tau_n|\, \varphi_n\, |\chi_m - \bar{\chi}_m| \\ &\qquad + |\chi_m|\, |\chi_n - \bar{\chi}_n| + |\bar{\chi}_n|\, |\chi_m - \bar{\chi}_m| \big]\, d\zeta \Big| \\ &\le B\, e^{-\frac{1}{\varepsilon}g_k} \Big| \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \sum_{m,n=1}^{N} \big[2\, |\tau_m|\, \varphi_m\, \|\chi - \bar{\chi}\| \sum_{i=1}^{N} \varphi_i \\ &\qquad + 2\, A\, |\tau|^2\, (\sum_{j=1}^{N} \varphi_j)\, \|\chi - \bar{\chi}\| \sum_{i=1}^{N} \varphi_i \big]\, d\zeta \Big| \\ &\le C'\,(\delta + A\delta^2) \Big(\sum_{m,n=1}^{N} \Big| e^{-\frac{1}{\varepsilon}g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k} \varphi_m \varphi_n\, d\zeta\, \Big| \Big)\, \|\chi - \bar{\chi}\| \\ &\le C'\,\delta\,(1 + A\delta)\, (\sum_{j=1}^{N} \varphi_j)\, \|\chi - \bar{\chi}\|\,, \end{aligned} \tag{5.14}$$

which, on account of (5.6), in turn implies

$$\|T(\chi) - T(\bar{\chi})\| \le C'\,\delta\,(1 + A\delta)\, \|\chi - \bar{\chi}\|\,. \tag{5.15}$$

Therefore T will be a uniform contraction on F, provided that

$$C'\,\delta\,(1 + A\delta) =: \alpha < 1\,. \tag{5.16}$$

Note that (5.12) and (5.16) can be simultaneously satisfied for many choices of A and δ. In the sequel, we fix $A = 4C$ and $\delta < \delta_0 = \min\{\frac{1}{4C}, \frac{1}{2C'}\}$. For these choices, $1 + A\delta < 2$, both (5.12) and (5.16) are fulfilled, and the proof of (i) and (ii) is completed.

Last, we turn to (iii). Let $V \in \bar{\Omega}$, $\chi \in F$ be fixed and consider τ, $s \in B_\delta$. Then (5.8) yields (upon suppressing the χ and V dependence)

$$
\begin{aligned}
T_k(\tau) - T_k(s) = e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \sum_{m,n=1}^{N} \beta_{k,mn} \big[(\tau_m \tau_n - s_m s_n)\, \varphi_m\, \varphi_n \\
+ (\tau_m - s_m)\, \varphi_m\, \chi_n + (\tau_n - s_n)\, \varphi_n\, \chi_m \big]\, d\zeta\,.
\end{aligned}
\tag{5.17}
$$

Using (5.6), (5.7), (4.58) and (5.16) we deduce

$$
\begin{aligned}
|T_k(\tau) - T_k(s)| &\le e^{-\frac{1}{\varepsilon} g_k} \Big| \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \sum_{m,n=1}^{N} |\beta_{k,mn}| \Big(\big[|\tau_m - s_m|\, |\tau_n| + |\tau_n - s_n|\, |s_m| \big]\, \varphi_m\, \varphi_n \\
&\qquad + |\tau_m - s_m|\, \varphi_m\, |\chi_n| + |\tau_n - s_n|\, \varphi_n\, |\chi_m| \Big)\, d\zeta \Big| \\
&\le B\, e^{-\frac{1}{\varepsilon} g_k} \Big| \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \sum_{m,n=1}^{N} \Big(\delta\, \big[|\tau_m - s_m| + |\tau_n - s_n| \big]\, \varphi_m\, \varphi_n \\
&\qquad + A\delta^2\, \big(\sum_{j=1}^{N} \varphi_j\big) \big[|\tau_m - s_m|\, \varphi_m + |\tau_n - s_n|\, \varphi_n \big] \Big)\, d\zeta \Big| \\
&\le C''\, \delta\, (1 + A\delta)\, |\tau - s| \sum_{m,n=1}^{N} \big| e^{-\frac{1}{\varepsilon} g_k} \int_{c_k}^{\xi} e^{\frac{1}{\varepsilon} g_k} \varphi_m \varphi_n\, d\zeta \big| \\
&\le C''\, \delta\, |\tau - s| \sum_{j=1}^{N} \varphi_j
\end{aligned}
\tag{5.18}
$$

and, by virtue of (5.6),

$$
\| T(\tau) - T(s) \| \le C''\, \delta\, |\tau - s|
\tag{5.19}
$$

which completes the proof of (iii). ∎

The properties of the map T are useful both for solving (5.1) and for establishing properties of the constructed solution $\theta = (\theta_1, \ldots, \theta_N)$.

Corollary 5.2. *Let A and δ be as in Proposition 5.1. Given $V \in \bar{\Omega}$, $\tau \in B_\delta$, there exists a unique solution $\theta(\cdot\,;\tau)$ of (5.1) in the class of functions satisfying*

$$
|\theta_k(\cdot\,;\tau)| \le A\, |\tau|^2 \sum_{j=1}^{N} \varphi_j\,, \qquad |\tau| \le \delta\,, \;\; k = 1, \ldots, N\,.
\tag{5.20}
$$

Moreover, there exists a constant C independent of δ such that $\theta(\cdot\,;\tau)$ satisfies

$$
|\theta_k(\cdot\,;\tau) - \theta_k(\cdot\,;s)| \le C\, \delta\, |\tau - s| \sum_{j=1}^{N} \varphi_j\,, \qquad \textit{for } \tau,\, s \in B_\delta\,.
\tag{5.21}
$$

Proof. For each fixed $V \in \bar{\Omega}$, $\tau \in B_\delta$, the map $T(V, \tau, \cdot) : F \to F$ is a contraction with a uniform contraction constant $\alpha < 1$. The first part of the lemma is a direct consequence of the contraction mapping theorem.

The fixed point θ depends parametrically on V and τ. In the second part we are interested in regularity properties of θ in τ and need estimates that are uniform for $V \in \bar{\Omega}$, $\tau \in B_\delta$. Instead of using general versions of the implicit function theorem, we opt for a direct approach that gives precise information on the bounds. Let $V \in \bar{\Omega}$ be fixed and consider τ, $s \in B_\delta$ and the corresponding fixed points $\theta(\tau)$ and $\theta(s)$ of T. Then we have

$$\theta(\tau) - \theta(s) = \left[T(\tau, \theta(\tau)) - T(\tau, \theta(s))\right] + \left[T(\tau, \theta(s)) - T(s, \theta(s))\right] . \tag{5.22}$$

Using (ii) and (iii) in Proposition 5.1, we obtain

$$\begin{aligned} \|\theta(\tau) - \theta(s)\| &\leq \|T(\tau, \theta(\tau)) - T(\tau, \theta(s))\| + \|T(\tau, \theta(s)) - T(s, \theta(s))\| \\ &\leq \alpha \, \|\theta(\tau) - \theta(s)\| + C\, \delta \, |\tau - s| \, . \end{aligned} \tag{5.23}$$

Hence,

$$\|\theta(\tau) - \theta(s)\| \leq \frac{C}{1 - \alpha} \, \delta \, |\tau - s| \tag{5.24}$$

and (5.21) follows from (5.6). ∎

6. The map connecting the wave strengths to the Riemann data

For $V \in \bar{\Omega}$ the states $V(\xi)$ take values in the ball $B_\mu(U_-) = \{U \in \mathbb{R}^N : |U - U_-| \le \mu\}$. Because of the orthogonality relations (3.8) and the continuity properties of $l_i(U)$, $r_i(U)$, given $\eta > 0$ we can choose μ such that

$$
\begin{aligned}
& l_i(U_1) \cdot r_i(U_2) \ge 1 - \eta\,, && U_1, U_2 \in B_\mu(U_-)\,, \\
& |l_i(U_1) \cdot r_j(U_2)| \le \eta\,, && U_1, U_2 \in B_\mu(U_-)\,, \quad i \neq j\,.
\end{aligned}
\tag{6.1}
$$

Also for states in $B_\mu(U_-)$ the right and left eigenvectors are bounded,

$$
|r_i(U)| \le R\,, \quad |l_i(U)| \le R\,, \qquad U \in B_\mu(U_-)\,, \quad i = 1, \ldots, N\,,
\tag{6.2}
$$

by a constant R depending only on μ. For our future deliberations we place an additional hypothesis, which complements $(A_1 - A_4)$ and concerns the behavior of the right and left eigenvectors along functions in $\bar{\Omega}$: Namely, we fix $\eta < 1/N$ and require that

$$
\begin{aligned}
& l_i(U_-) \cdot r_i(V(\xi)) \ge 1 - \eta\,, \\
& |l_i(U_-) \cdot r_j(V(\xi))| \le \eta\,, \qquad i \neq j
\end{aligned}
\tag{A_5}
$$

for $V \in \bar{\Omega}$ and $\xi \in \mathbb{R}$. This is attained by restricting, if necessary, the size of μ.

Consider the system of differential equations

$$
\varepsilon\, a_k' + \big[\xi - \lambda_k(V(\xi))\big]\, a_k = \varepsilon \sum_{m=1}^{N} \sum_{n=1}^{N} \big[\nabla l_k(V(\xi))\, r_m(V(\xi)) \cdot r_n(V(\xi))\big]\, a_m\, a_n\,,
\tag{6.3}
$$

where $V \in \bar{\Omega}$. We saw in the previous section that (6.3) has solutions given by an asymptotic expansion in a parameter $\tau \in \mathbb{R}^N$ of the form

$$
a_k(\xi\,;\tau) = \tau_k\, \varphi_k(\xi) + \theta_k(\xi\,;\tau)\,.
\tag{6.4}
$$

The expansion is valid for $|\tau| \le \delta$ uniformly for $V \in \bar{\Omega}$, and $\theta_k(\cdot;\tau)$ satisfies (5.20) and is of order $O(|\tau|^2)$ as $|\tau| \to 0$. The parameter τ is associated with the data at c_k, as from (5.1) :

$$
a_k(c_k\,;\tau) = \tau_k \varphi_k(c_k)\,.
\tag{6.5}
$$

It is instructive to visualize $|\tau| = |\tau_1| + \ldots + |\tau_N|$ as measuring the wave strength of the solution of the Riemann problem associated to $a_k(\xi\,;\tau)$ (*cf.* (3.11)).

A comparison with the general outline in Section 3 shows that while the solvability of (3.15) is at this point well understood, it remains to select τ so that (3.13) is satisfied. The issue emerges of studying the connection between the parameter τ and the boundary data $U_\pm$. To this end let U_- be fixed and consider the map S that carries τ into the end-state vector

$$S(\tau) = U_- + \sum_{k=1}^{N} \int_{-\infty}^{\infty} \left[\tau_k\, \varphi_k(\zeta) + \theta_k(\zeta;\, \tau)\right] r_k(V(\zeta))\, d\zeta\,. \tag{6.6}$$

For $\tau \in B_\delta = \{\tau \in \mathrm{I\!R}^N \,:\, |\tau| \le \delta\}$ the map S is well defined, and depends explicitly on V and implicitly on ε. Our objective is to study the invertibility of S and to show that the inverse map is uniformly bounded in V and ε.

Proposition 6.1. *Assume that $(A_1 - A_5)$ are satisfied for $V \in \bar{\Omega}$. There exist positive constants r and δ such that :*

(i) *Given $U_+ \in B_r(U_-)$ there exists a unique solution of the equation $S(\tau) = U_+$ with $\tau \in B_\delta$.*

(ii) *For each $V \in \bar{\Omega}$ and $\varepsilon > 0$ the inverse map $S^{-1} : B_r(U_-) \to B_\delta$ is well defined and satisfies*

$$|S^{-1}(U_+)| \le 2\beta |U_+ - U_-|\,, \tag{6.7}$$

where β is a constant which depends on μ, but is independent of the particular $V \in \bar{\Omega}$ and ε.

Proof of Proposition 6.1. Let U_- be fixed. The equation $S(\tau) = U_+$ has the form

$$U_+ - U_- = \sum_{k=1}^{N} \tau_k \int_{-\infty}^{\infty} \varphi_k\, r_k(V(\zeta))\, d\zeta + \sum_{k=1}^{N} \int_{-\infty}^{\infty} \theta_k(\zeta;\, \tau)\, r_k(V(\zeta))\, d\zeta\,. \tag{6.8}$$

If $A(V)$ is the matrix whose k-th column is given by

$$a_k(V) = \int_{-\infty}^{\infty} \varphi_k\, r_k(V(\zeta))\, d\zeta\,, \qquad k = 1, \ldots, N\,, \tag{6.9}$$

then (6.8) reduces to

$$U_+ - U_- = A(V)\, \tau + \sum_{k=1}^{N} \int_{-\infty}^{\infty} \theta_k(\zeta;\, \tau)\, r_k(V(\zeta))\, d\zeta \tag{6.10}$$

and the issue becomes to study the solvability of (6.10) in τ.

First we show that Hypothesis (A_5) implies that $A(V)$ is invertible.

Lemma 6.2. *Assume that (A_5) holds (with $\eta < 1/N$). The matrix $A(V)$ is invertible for any $V \in \bar{\Omega}$, and the inverse matrix $A^{-1}(V)$ is uniformly bounded,*

$$|A^{-1}(V)| \le \beta\,, \qquad V \in \bar{\Omega}\,, \tag{6.11}$$

by a constant β independent of ε.

Proof of Lemma 6.2. Since φ_k are averaging measures, the mean value theorem implies

$$a_k(V) = \int_{-\infty}^{\infty} \varphi_k\, r_k(V(\zeta))\, d\zeta = r_k(V_k^*) \tag{6.12}$$

for some $V_k^* \in B_\mu(U_-)$. Since $\{r_i(U_-)\}$ are linearly independent, by choosing μ sufficiently small it is guaranteed that the vectors $r_1(V_1^*)$, ..., $r_N(V_N^*)$ are linearly independent and thus $A(V)$ is invertible.

We now show (6.11) and in the process provide an alternative way of showing that $A(V)$ is nonsingular. For τ, $y \in \mathbb{R}^N$ consider the equation $A(V)\tau = y$ and write it in the form

$$\sum_{k=1}^{N} \tau_k \int_{-\infty}^{\infty} \varphi_k\, r_k(V(\zeta))\, d\zeta = y\,. \tag{6.13}$$

Taking the inner product of (6.13) with $l_i(U_-)$ and rearranging the terms we obtain

$$\tau_i \int_{-\infty}^{\infty} \varphi_i \left[l_i(U_-)\cdot r_i(V(\zeta))\right] d\zeta = l_i(U_-)\cdot y - \sum_{k\neq i} \tau_k \int_{-\infty}^{\infty} \varphi_k \left[l_i(U_-)\cdot r_k(V(\zeta))\right] d\zeta\,. \tag{6.14}$$

Then (A_5), (4.3) and (6.14) yield

$$|\tau_i|\,(1-\eta) \le |l_i(U_-)\cdot y| + \eta \sum_{k\neq i} |\tau_k|\,. \tag{6.15}$$

Adding the resulting equations for $i = 1, \dots, N$ and using the fact that $\eta < 1/N$, we obtain the estimate

$$|\tau| \le \frac{1}{1-N\eta} \sum_{i=1}^{N} |l_i(U_-)\cdot y| \le \beta\,|y| = \beta\,|A(V)\,\tau|\,. \tag{6.16}$$

The first implication of (6.16) is that the only possible solution of $A(V)\,\tau = 0$ is the trivial solution $\tau = 0$. Therefore $a_1(V), \dots, a_N(V)$ are linearly independent and $A(V)$ is invertible. In addition, (6.16) implies that

$$|A^{-1}(V)\,y| \le \beta\,|y|\,, \qquad y \in \mathbb{R}^N\,, \tag{6.17}$$

which proves (6.11). ∎

Next, we formulate solving the equation $S(\tau) = U_+$ as a fixed point problem. Let $B_r(U_-)$ be the ball centered at U_- of radius r and consider the map P that takes $U_+ \in B_r(U_-)$, $V \in \bar{\Omega}$, $\tau \in B_\delta$ into the vector

$$P(U_+, V, \tau) = A^{-1}(V)\,(U_+ - U_-) - A^{-1}(V) \sum_{k=1}^{N} \int_{-\infty}^{\infty} \theta_k(\zeta;\, \tau)\, r_k(V(\zeta))\, d\zeta\,. \tag{6.18}$$

Since $A(V)$ is invertible, solutions of (6.10) are fixed points of the map $P(U_+, V, \cdot)$.

Lemma 6.3. *There exist positive constants δ and r such that $P : B_r(U_-) \times \bar{\Omega} \times B_\delta \to B_\delta$ and has the property that there exists a constant α with $0 < \alpha < 1$ such that*

$$|P(U_+, V, \tau) - P(U_+, V, s)| \leq \alpha |\tau - s| \,, \qquad \tau,\, s \in B_\delta \,, \tag{6.19}$$

for any $U_+ \in B_r(U_-)$, $V \in \bar{\Omega}$; that is, $P(U_+, V, \cdot)$ is a uniform contraction on B_δ.

Proof of Lemma 6.3. Let $U_+ \in B_r(U_-)$, $V \in \bar{\Omega}$ and $\tau \in B_\delta$. Using (6.18), (6.11), (6.2) and (5.20), we obtain

$$\begin{aligned} |P(U_+, V, \tau)| &\leq |A^{-1}(V)| \left(|U_+ - U_-| + \sum_{k=1}^{N} \int_{-\infty}^{\infty} |\theta_k(\zeta; \tau)| \, |r_k(V(\zeta))| \, d\zeta \right) \\ &\leq \beta \left(r + R\, A\, |\tau|^2\, N \sum_{j=1}^{N} \int_{-\infty}^{\infty} \varphi_j d\zeta \right) \\ &\leq \beta \, (r + R\, A\, N^2\, \delta^2) \,. \end{aligned} \tag{6.20}$$

The first assertion of the lemma is true, provided that r and δ satisfy

$$\beta\, r + \beta\, R\, A\, N^2\, \delta^2 \leq \delta \,. \tag{6.21}$$

Let now τ, $s \in B_\delta$ and observe that

$$P(U_+, V, \tau) - P(U_+, V, s) = -A^{-1}(V) \sum_{k=1}^{N} \int_{-\infty}^{\infty} \left[\theta_k(\zeta; \tau) - \theta_k(\zeta; s) \right] r_k(V(\zeta))\, d\zeta \,. \tag{6.22}$$

On account of (6.11), (6.2) and (5.21), (6.22) gives

$$\begin{aligned} |P(U_+, V, \tau) - P(U_+, V, s)| &\leq \beta \sum_{k=1}^{N} \int_{-\infty}^{\infty} |\theta_k(\zeta; \tau) - \theta_k(\zeta; s)| \, |r_k(V(\zeta))| \, d\zeta \\ &\leq \beta\, R\, N\, C\, \delta\, |\tau - s| \sum_{j=1}^{N} \int_{-\infty}^{\infty} \varphi_j d\zeta \\ &\leq \beta\, R\, N^2\, C\, \delta\, |\tau - s| \,. \end{aligned} \tag{6.23}$$

Therefore, if

$$\alpha = \beta\, R\, N^2\, C\, \delta < 1 \tag{6.24}$$

then $P(U_+, V, \cdot) : B_\delta \to B_\delta$ is a uniform contraction.

Note that if $\delta \leq \frac{1}{2}\min\{(\beta R N^2 C)^{-1}, (\beta R N^2 A)^{-1}\}$ and $r \leq \frac{1}{2\beta}\delta$ then both (6.21) and (6.24) are simultaneously satisfied, and the proof of the lemma is complete. ∎

We return to the proof of Proposition 6.1. Lemma 6.3 implies that given $U_+ \in B_r(U_-)$ there exists a unique fixed point of $P(U_+, V, \cdot)$ in the ball B_δ and thus a unique solution of $S(\tau) = U_+$. Hence, S^{-1} is well defined. Let U_+ and $\tau = S^{-1}(U_+)$ be two corresponding points related through (6.10). Using (5.20), (6.2) and (4.3), we obtain

$$
\begin{aligned}
|A(V)\,\tau| &\leq |U_+ - U_-| + \sum_{k=1}^{N} \int_{-\infty}^{\infty} |\theta_k(\zeta;\,\tau)|\,|r_k(V(\zeta))|\,d\zeta \\
&\leq |U_+ - U_-| + R\,A\,N\,|\tau|^2 \sum_{j=1}^{N} \int_{-\infty}^{\infty} \varphi_j\,d\zeta \\
&= |U_+ - U_-| + R\,A\,N^2\,|\tau|^2 \,.
\end{aligned}
\tag{6.25}
$$

Using Lemma 6.2, in conjunction with (6.21) and the choice of δ, we deduce from (6.25)

$$
|\tau| \leq \beta\,|U_+ - U_-| + \beta\,R\,A\,N^2\,\delta\,|\tau| \leq \beta\,|U_+ - U_-| + \frac{1}{2}|\tau|\,, \tag{6.26}
$$

which implies (6.7) and completes the proof of the proposition. ∎

7. Proof of Theorem 3.1

This is the concluding section of the derivation of a priori estimates for $(\mathcal{P}_\varepsilon)$. The analysis of Sections 3 to 6 is combined in order to prove the main theorem.

Let U_- be fixed and define $\bar{\Omega}$ by (4.1). $\bar{\Omega}$ is a closed, convex and bounded subset of the Banach space $C^0(-\infty,\infty)$ of continuous, bounded functions. Fix $\varepsilon > 0$ and consider the map T carrying $V \in \bar{\Omega}$ to the continuous function W defined by the following procedure :

(a) Let φ_k be as in (4.3). We obtain the solution $\theta_k(\cdot\,;\tau)$ of (5.1), for $\tau \in \mathbb{R}^N$ small, and define $a_k(\cdot\,;\tau) = \tau_k \varphi_k + \theta_k(\cdot\,;\tau)$. The resulting a_k form a solution of the system of equations (6.3).

(b) Let S be the map defined in (6.6). Let t be the solution of the equation $S(\tau) = U_+$, that is $t = S^{-1}(U_+)$.

(c) W is then defined by setting

$$W(\xi) = U_- + \int_{-\infty}^{\xi} \sum_{k=1}^{N} [t_k\, \varphi_k(\zeta) + \theta_k(\zeta;t)]\, r_k\big(V(\zeta)\big)\, d\zeta\,. \tag{7.1}$$

The construction is feasible for the following reasons : The parameter μ in the definition of $\bar{\Omega}$ is fixed so that Hypotheses $(A_1 - A_5)$ are satisfied for $V \in \bar{\Omega}$. Also, we fix the parameters A and δ_0 as in Proposition 5.1 and let $\delta < \delta_0$. Then Corollary 5.2 states that for $\tau \in B_\delta$ the problem (5.1) has a unique solution satisfying the estimate

$$|\theta_k(\cdot\,;\tau)| \le A\,|\tau|^2 \sum_{j=1}^{N} \varphi_j\,, \qquad \tau \in B_\delta\,. \tag{7.2}$$

According to Proposition 6.1, for r and δ sufficiently small the map $S : B_\delta \to B_r(U_-)$ is invertible, $S(\tau) = U_+$ is uniquely solvable in B_δ, and the inverse $t = S^{-1}(U_+)$ satisfies for some fixed β (independent of V and ε) the estimate

$$|t| = |S^{-1}(U_+)| \le 2\,\beta\,|U_+ - U_-|\,, \qquad U_+ \in B_r(U_-)\,. \tag{7.3}$$

As a result $W(-\infty) = U_-$ and $W(+\infty) = S(t) = U_+$. From (7.1) we obtain

$$\frac{dW}{d\xi} = \sum_{k=1}^{N} \big[t_k \varphi_k + \theta_k(\cdot\,;t)\big]\, r_k\big(V(\cdot)\big) \tag{7.4}$$

which, in conjunction with (7.2) and (6.2), yields

$$\begin{aligned} \Big|\frac{dW}{d\xi}\Big| &\le \sum_{k=1}^{N} \Big[\, |t_k|\, \varphi_k + A|t|^2 \sum_{j=1}^{N} \varphi_j \,\Big]\, \big|r_k(V)\big| \\ &\le R|t|\,(1 + AN\,|t|) \sum_{j=1}^{N} \varphi_j\,. \end{aligned} \tag{7.5}$$

In turn, (7.1), (7.3) and (7.5) imply

$$
\begin{aligned}
(7.6)\qquad |W(\xi)-U_-| &\leq \left| \int_{-\infty}^{\xi} \Big| \sum_{k=1}^{N} \big[t_k\varphi_k(\zeta)+\theta_k(\zeta;t)\big]\, r_k\big(V(\zeta)\big)\Big|\, d\zeta \right| \\
&\leq 2\beta NR\left(1+2\beta AN\,|U_+-U_-|\right)|U_+-U_-|\,.
\end{aligned}
$$

It follows that, if $U_+ \in B_r(U_-)$ and r is restricted by

$$
(7.7)\qquad 2\beta NRr(1+2\beta ANr)\leq \mu\,,
$$

the function W defined in steps (a - c) satisfies

$$
(7.8)\qquad |W(\xi)-U_-|\leq \mu, \qquad \xi\in \mathbb{R}\,.
$$

In the sequel we fix r and δ to simultaneously satisfy (7.7), (6.21), (6.24) and (5.16). All the stated constructions and estimations are then feasible, and the map $T:\bar\Omega\to\bar\Omega$ is well defined. In addition, (7.5), (7.3) and Lemma 4.1 dictate that there is a constant C such that

$$
(7.9)\qquad
\begin{aligned}
\big|W(\xi)-U_-\big| &\leq |U_+-U_-|\,\frac{C}{\varepsilon}\int_{-\infty}^{\xi} e^{-\frac{1}{2\varepsilon}(\zeta-\lambda_{1-})^2}\,d\zeta\,, &&\text{for } \xi<\lambda_{1-}\,,\\
\big|W(\xi)-U_+\big| &\leq |U_+-U_-|\,\frac{C}{\varepsilon}\int_{\xi}^{\infty} e^{-\frac{1}{2\varepsilon}(\zeta-\lambda_{N+})^2}\,d\zeta\,, &&\text{for } \xi>\lambda_{N+}\,.
\end{aligned}
$$

Our next task is to apply the Schauder fixed point theorem to the map T.

(i) $T(\bar\Omega)$ *is precompact in* $C^0(-\infty,\infty)$.

Consider a sequence $\{V^n\}\subset\bar\Omega$ and let $W^n=T(V^n)$. Estimates (7.5), (7.3), (7.8) and (4.7) imply that $\{W^n\}$ is uniformly bounded and uniformly equicontinuous on the reals. It follows from the Ascoli-Arzela theorem and a diagonal argument that there is a subsequence $\{W^{n_j}\}$ and a continuous function W such that $W^{n_j}\to W$ uniformly on compact subsets of $\mathbb{R}$. But then the decay estimates (7.9) imply that the convergence is in fact uniform, and thus $T(\bar\Omega)$ is precompact in $C^0(-\infty,\infty)$.

(ii) $T:\bar\Omega\to\bar\Omega$ *is continuous.*

Let $\{V^n\}\subset\bar\Omega$ be a convergent sequence in $C^0(-\infty,\infty)$, with $V^n\to V^0$, and set $W^n=T(V^n)$, $W^0=T(V^0)$. We proceed to show $T(V^n)\to T(V^0)$. Recall that ε is held fixed, and that W^n and W^0 are defined in terms of the intermediate quantities φ_k^n, $\theta_k^n(\cdot\,;\tau)$, $a_k^n(\cdot\,;\tau)$, S^n, t^n and φ_k^0, $\theta_k^0(\cdot\,;\tau)$ $a_k^0(\cdot\,;\tau)$, S^0, t^0 in steps (a - c) for $V=V^n$ and $V=V^0$, respectively.

First, we show $\varphi_k^n\to\varphi_k^0$ in $C^0(-\infty,\infty)$. One first uses (4.7), (4.2) and (4.8) to show that $\{\varphi_k^n\}$ is a uniformly bounded and equicontinuous sequence of functions that satisfies the decay estimates (4.8) as $|\xi| \to \infty$. An argument as in (i) implies that there exists a subsequence $\{\varphi_k^{n_j}\}$ and a function φ_k^∞ such that $\varphi_k^{n_j} \to \varphi_k^\infty$ uniformly in $\mathbb{R}$. Passing to the limit in (4.3) along the subsequence n_j,

$$\varphi_k^{n_j} = \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_\xi^\zeta s-\lambda_k(V^{n_j}(s))\,ds}\,d\zeta} \to \frac{1}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon}\int_\xi^\zeta s-\lambda_k(V^{0}(s))\,ds}\,d\zeta} = \varphi_k^0 \,, \tag{7.10}$$

we deduce $\varphi_k^\infty = \varphi_k^0$. The sequence $\{\varphi_k^n\}$ has limit points, and any limit point is equal to φ_k^0. Hence, the whole sequence $\{\varphi_k^n\}$ converges to φ_k^0.

Second, we show that for τ fixed $\theta_k^n(\cdot\,;\tau) \to \theta_k^0(\cdot\,;\tau)$ in $C^0(-\infty,\infty)$. This follows by a similar in spirit argument that we only sketch: Using (5.20), (5.1), (4.7) and (4.8), we show that $\{\theta_k^n\}$ possesses a subsequence $\{\theta_k^{n_j}\}$ and a limit point θ_k^∞ so that $\theta_k^{n_j} \to \theta_k^\infty$ uniformly in $\mathbb{R}$. Passing to the limit in (5.4) along the subsequence n_j and using the convergence of V^n and φ_k^n, we obtain

$$\theta_k^\infty(\xi) = e^{-\frac{1}{\varepsilon}g_k^0}\int_{c_k}^{\xi} e^{\frac{1}{\varepsilon}g_k^0} \sum_{m,n=1}^{N} \beta_{k,mn}(V^0(\zeta))\,\big(\tau_m\,\varphi_m^0(\zeta)+\theta_m^\infty(\zeta)\big)\,\big(\tau_n\,\varphi_n^0(\zeta)+\theta_n^\infty(\zeta)\big)\,d\zeta\,. \tag{7.11}$$

Since the limiting θ_k^∞ inherits the estimate (5.20), the uniqueness part of Corollary 5.2 implies that any limit point of $\{\theta_k^n\}$ is of the form $\theta_k^\infty(\cdot\,;\tau) = \theta_k^0(\cdot\,;\tau)$. Consequently $\theta_k^n(\cdot\,;\tau) \to \theta_k^0(\cdot\,;\tau)$.

The third step is to show that $t^n \to t^0$ in $\mathbb{R}^N$. Let S^n and S^0 be the maps associated with V^n and V^0 respectively and define t^n and t^0 satisfying $S^n(t^n) = S^0(t^0) = U_+$. Since $\{t^n\}$ is bounded, there is a subsequence $\{t^{n_j}\}$ and a vector t^∞ such that $t^{n_j} \to t^\infty$. We use (5.20), (5.21) to pass to the limit in $S^{n_j}(t^{n_j}) = U_+$ and to obtain $S^0(t^\infty) = U_+$. Because of the unique invertibility of the map S^0 it is $t^0 = t^\infty$ and thus the sequence $\{t^n\}$ converges to t^0.

The precompactness of T implies that the sequence $\{W^n\}$ has a subsequence $\{W^{n_j}\}$ and a limit function W^∞ such that $W^{n_j} \to W^\infty$ in $C^0(-\infty,\infty)$. Using the established convergences and (5.21), we pass to the limit in (7.1) along n_j and obtain

$$W^\infty(\xi) = U_- + \int_{-\infty}^{\xi} \sum_{k=1}^{N} [t_k^0\,\varphi_k^0(\zeta) + \theta_k^0(\zeta;t^0)]\,r_k\big(V^0(\zeta)\big)\,d\zeta = T(V^0)(\xi)\,. \tag{7.12}$$

Therefore any limit point of $\{W^n\}$ is equal to $T(V^0)$ and thus $T(V^n) \to T(V^0)$ in $C^0(-\infty,\infty)$. Hence, T is continuous.

The Schauder fixed point theorem implies that there exists a fixed point U_ε of the map T in $\bar{\Omega}$. By construction U_ε satisfies

$$U_\varepsilon(\xi) = U_- + \int_{-\infty}^{\xi} \sum_{k=1}^{N} a_{k\varepsilon}(\zeta;\tau_\varepsilon)\,r_k\big(U_\varepsilon(\zeta)\big)\,d\zeta\,, \tag{7.13}$$

where

$$a_{k\varepsilon}(\xi;\,\tau_\varepsilon) = \tau_{k,\varepsilon}\,\varphi_{k\varepsilon}(\xi) + \theta_{k\varepsilon}(\xi;\,\tau_\varepsilon) \tag{7.14}$$

solves (3.15). The functions $\varphi_{k\varepsilon}$, $\theta_{k\varepsilon}$ and $a_{k\varepsilon}$ depend implicitly on ε, and the quantities τ_ε satisfy

$$S_\varepsilon(\tau_\varepsilon) = U_- + \sum_{k=1}^{N} \int_{-\infty}^{\infty} a_{k\varepsilon}(\zeta;\,\tau_\varepsilon)\, r_k(U_\varepsilon(\zeta))\, d\zeta = U_+\,. \tag{7.15}$$

As a result, $U_\varepsilon(\pm\infty) = U_\pm$ and

$$\begin{aligned} U_\varepsilon'(\xi) &= \sum_{k=1}^{N} a_{k\varepsilon}(\xi;\,\tau_\varepsilon)\, r_k\big(U_\varepsilon(\xi)\big)\,, \\ a_{k\varepsilon}(\xi;\,\tau_\varepsilon) &= l_k(U_\varepsilon(\xi)) \cdot U_\varepsilon'(\xi)\,. \end{aligned} \tag{7.16}$$

Using (7.16) and $(3.5-3.8)$, we can rewrite (3.15) in the form

$$l_k(U_\varepsilon)\cdot\big[-\xi + \nabla F(U_\varepsilon)\big]U_\varepsilon' = l_k(U_\varepsilon)\cdot U_\varepsilon'' \tag{7.17}$$

which implies that U_ε is a solution of $(\mathcal{P}_\varepsilon)$.

Consider a family $\{U_\varepsilon\}_{\varepsilon>0}$ of such solutions to $(\mathcal{P}_\varepsilon)$. By construction, U_ε are of uniformly bounded (and small) oscillation (C_o) and satisfy the representation formula (3.33). Relations (7.2) and (7.3) imply that there exist constants C, independent of ε, so that $|\tau_\varepsilon| \le C|U_+ - U_-|$ and

$$\begin{aligned} |a_{k\varepsilon}(\xi;\,\tau_\varepsilon)| &\le |\tau_{k,\varepsilon}|\varphi_{k\varepsilon} + C|\tau_\varepsilon|^2 \sum_{j=1}^{N}\varphi_{j\varepsilon} \\ &\le C\,|U_+ - U_-|\big(\varphi_{k\varepsilon} + |U_+ - U_-|\sum_{j=1}^{N}\varphi_{j\varepsilon}\big)\,. \end{aligned} \tag{7.18}$$

As a result,

$$|U_\varepsilon'(\xi)| \le K \sum_{j=1}^{N}\varphi_{j\varepsilon} \tag{7.19}$$

with K a constant of order $O(|U_+ - U_-|)$ and independent of ε. As $\{\varphi_{j\varepsilon}\}$ are uniformly bounded in $L^1(\mathbb{R})$, it follows that $\{U_\varepsilon'\}$ are uniformly bounded in $L^1(\mathbb{R})$ and $\{U_\varepsilon\}$ is of uniformly bounded variation. The total variation of the family is controlled by $|U_+ - U_-|$ and is thus small. The proof of Theorem 3.1 is complete.

8. Solution of the Riemann problem

Our next objective is to construct solutions of the Riemann problem $(\mathcal{P})$ by taking $\varepsilon \to 0$ limits of solutions of $(\mathcal{P}_\varepsilon)$ and to identify the structure of the emerging solutions. The analysis is patterned within the framework developed in the previous sections. Nevertheless, it is instructive to single out the set of hypotheses used in performing the $\varepsilon \to 0$ limit and to provide an independent presentation. Let $\{U_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to $(\mathcal{P}_\varepsilon)$ that connect U_- to U_+ and enjoy the properties :

$$
(A_s) \qquad
\begin{aligned}
U_\varepsilon &\ \text{ satisfy the uniform bounds } \ (C_o),\ (S) \ \text{ for } \varepsilon > 0 \ , \\
\lambda_k(U_\varepsilon) &\ \text{ satisfy the uniform bounds } \ (3.9),\ (3.10) \ \text{ for } \varepsilon > 0 \ , \\
U'_\varepsilon &\ \text{ satisfies (7.19) where } \varphi_{k\varepsilon} \text{ is given by (3.25)} \ .
\end{aligned}
$$

Solutions satisfying (A_s) were constructed in Theorem 3.1, and the resulting families are of small oscillation and variation. The results of this section remain valid for families of large oscillation and variation, provided the global separation of the eigenvalues and, most important, estimate (7.19) hold. Helly's selection principle implies there exists a subsequence of the original family, denoted again by $\{U_\varepsilon\}$, with $\varepsilon \to 0$, and a function U of bounded variation such that

$$
U_\varepsilon(\xi) \to U(\xi) \qquad \text{pointwise on } \ (-\infty, \infty) \ . \tag{8.1}
$$

Since U is of bounded variation its domain can be decomposed into two disjoint subsets $\mathcal{C}$ and $\mathcal{S}$: $\mathcal{C}$ consists of the points of continuity of U and $\mathcal{S}$ of the points of jump discontinuity. $\mathcal{S}$ is at most countable, and the right and left limits of U exist at any $\xi \in \mathcal{S}$ and are denoted $U(\xi\pm)$.

We proceed to show U satisfies $(\mathcal{P})$. In the sequel C denotes a generic constant that can be estimated in terms of the bounds in (A_s) and the Riemann data and which is independent of ε.

Theorem 8.1. *Suppose that* (1.1) *is strictly hyperbolic and let* $\{U_\varepsilon\}_{\varepsilon>0}$ *be a family of solutions of* $(\mathcal{P}_\varepsilon)$ *corresponding to data* $U_\pm$ *and satisfying* (A_s). *There exists a subsequence* $\{U_{\varepsilon_n}\}$ *with* $\varepsilon_n \to 0$ *and a function of bounded variation* U *such that* $U_{\varepsilon_n} \to U$ *pointwise on the reals.* U *satisfies*

$$
-\xi\, U' + F(U)' = 0 \tag{8.2}
$$

in the sense of measures, the Rankine-Hugoniot conditions

$$
-\xi\,[U(\xi+) - U(\xi-)] + [F(U(\xi+)) - F(U(\xi-)] = 0 \tag{8.3}
$$

hold at any point $\xi \in \mathcal{S}$, *and there exist constant vectors* $U_0, \ldots, U_N \in \mathbb{R}^N$ *with* $U_0 = U_-$, $U_N = U_+$ *such that*

$$
(8.4) \qquad U(\xi) = \begin{cases} U_0 = U_- & -\infty < \xi < \lambda_{1-}\,, \\ U_k & \lambda_{k+} < \xi < \lambda_{(k+1)-}\,, \quad k = 1, \ldots, (N-1)\,, \\ U_N = U_+ & \lambda_{N+} < \xi < +\infty\,. \end{cases}
$$

Proof. Let $\{U_\varepsilon\}$ be a convergent subsequence as in (8.1), satisfying the uniform bounds (C_o), (S), and $\psi \in [C_c^\infty(\mathbb{R})]^N$ be a test function with compact support. Then $(\mathcal{P}_\varepsilon)$ gives

$$
(8.5) \qquad \int_{\mathbb{R}} U_\varepsilon \cdot (\xi\psi)' - F(U_\varepsilon) \cdot \psi' \, d\xi = \varepsilon \int_{\mathbb{R}} U_\varepsilon \cdot \psi'' \, d\xi\,.
$$

Passing to the limit $\varepsilon \to 0$ we deduce

$$
(8.6) \qquad \int_{\mathbb{R}} U \cdot (\xi\psi)' - F(U) \cdot \psi' \, d\xi = 0\,,
$$

that is U satisfies (8.2) in the sense of distributions. Since U is of bounded variation, it also satisfies (8.2) in the sense of measures.

Let $\mathcal{L} = [\lambda_{1-}, \lambda_{1+}] \cup \ldots \cup [\lambda_{N-}, \lambda_{N+}]$ stand for the range of variation of the wave speeds $\lambda_k(U_\varepsilon)$. Then (4.8) and (7.19) imply

$$
(8.7) \qquad \begin{aligned} &\varphi_{k\varepsilon} \le \frac{C}{\varepsilon} \exp\{-\frac{1}{2\varepsilon} d(\xi, \lambda_k)^2\} && \xi \in (-\infty, \infty) - [\lambda_{k-}, \lambda_{k+}]\,, \\ &|U_\varepsilon'| \le K \sum_{j=1}^N \varphi_{j\varepsilon} \le \frac{C}{\varepsilon} \exp\{-\frac{1}{2\varepsilon} d(\xi, \mathcal{L})^2\} && \xi \in (-\infty, \infty) - \mathcal{L}\,, \end{aligned}
$$

where $d(\xi, \lambda_k)$ and $d(\xi, \mathcal{L})$ are the distances between the point ξ and the sets $[\lambda_{k-}, \lambda_{k+}]$ and $\mathcal{L}$ respectively. Therefore the limiting function U stays constant on each connected component of $(-\infty, \infty) - \mathcal{L}$ and (8.4) follows. In addition, $U_\varepsilon(\pm\infty) = U_\pm$ implies $U_0 = U_-$ and $U_N = U_+$.

The Rankine-Hugoniot conditions (8.3) are a consequence of the fact that U of bounded variation solves (8.2). We outline a different proof, in the spirit of self-similar viscous limits. Integrating the equation $(\mathcal{P}_\varepsilon)$ on an interval (a, b) we obtain the weak form

$$
(8.8) \qquad [-b\, U_\varepsilon(b) + F(U_\varepsilon(b))] - [-a\, U_\varepsilon(a) + F(U_\varepsilon(a))] + \int_a^b U_\varepsilon(\zeta)\, d\zeta = \varepsilon U_\varepsilon'(b) - \varepsilon U_\varepsilon'(a)\,.
$$

For $\xi \in \mathcal{S}$ and $\delta > 0$, we evaluate (8.8) between the points θ and τ, with $\tau < \lambda_{1-}$, and integrate the resulting equation in θ over $[\xi, \xi + \delta]$ to arrive at the identity

$$
(8.9) \qquad \begin{aligned} &\int_\xi^{\xi+\delta} -\theta U_\varepsilon(\theta) + F(U_\varepsilon(\theta))\, d\theta + \int_\xi^{\xi+\delta} \int_\tau^\theta U_\varepsilon(\zeta) d\zeta d\theta \\ &\qquad = \varepsilon \int_\xi^{\xi+\delta} U_\varepsilon'(\theta)\, d\theta - \varepsilon\delta U_\varepsilon'(\tau) + \delta[-\tau\, U_\varepsilon(\tau) + F(U_\varepsilon(\tau))]\,. \end{aligned}
$$

Using the consequence of (7.19) and (3.25)

$$\int_{-\infty}^{\infty} |U_\varepsilon'|\, d\zeta \le KN\ , \tag{8.10}$$

in conjunction with (A_s), (8.1) and (8.7), we take first $\varepsilon \to 0$ in (8.9) and then divide the resulting equation by δ and take $\delta \to 0+$ to obtain

$$-\xi U(\xi+) + F(U(\xi+)) + \int_\tau^\xi U(\zeta) d\zeta = -\tau\, U(\tau) + F(U(\tau))\ . \tag{8.11}$$

In a similar manner, given any θ and $\tau < \lambda_{1-}$, we establish

$$-\theta U(\theta-) + F(U(\theta-)) + \int_\tau^\theta U(\zeta) d\zeta = -\tau\, U(\tau) + F(U(\tau))\ . \tag{8.12}$$

Then (8.3) follows from (8.11) and (8.12) for $\xi = \theta$. ∎

With $U(\xi)$ as above, define

$$V(x,t) = U(\frac{x}{t})\,, \qquad (x,t) \in (-\infty,\infty) \times (0,\infty)\ . \tag{8.13}$$

Clearly $\lim_{t\to 0} V(x,t) = U_-$ for $x < 0$, U_+ for $x > 0$. Furthermore, a solution V of the form (8.13) is a weak solution of (1.1) on $(-\infty,\infty)\times(0,\infty)$ if and only if U is a weak solution of (8.2) on $(-\infty,\infty)$. The equivalence follows from an argument due to Dafermos [D$_3$]. Let $\chi(x,t)$ be a C^∞ $\mathbb{R}^N$-valued function with compact support in $(-\infty,\infty)\times(0,\infty)$ and define

$$\psi(\xi) = \int_0^\infty \chi(\xi t, t)\, dt\,. \tag{8.14}$$

The resulting function $\psi \in [C_c^\infty(-\infty,\infty)]^N$. Conversely, any test function ψ may be represented in the form (8.14) by choosing $\chi = \psi(x/t)a(t)$, with $a(t) \in C_c^\infty(0,\infty)$ a fixed function such that $\int_0^\infty a(t)\,dt = 1$. For solutions of the type (8.13) the weak form of (1.1) may be written as

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty V(x,t)\cdot\chi_t(x,t) + F(V(x,t))\cdot\chi_x(x,t)\, dxdt \\
&= \int_{-\infty}^\infty U(\xi)\cdot\Big(\int_0^\infty \chi_t(\xi t,t)\, t\, dt\Big) + F(U(\xi))\cdot\Big(\int_0^\infty \chi_x(\xi t,t)\, t\, dt\Big)\, d\xi \\
&= \int_{-\infty}^\infty U(\xi)\cdot\big(-\xi\,\psi(\xi)\big)' + F(U(\xi))\cdot\psi'(\xi)\, d\xi\,,
\end{aligned} \tag{8.15}$$

and the equivalence follows from the chain of identities. Theorem 8.1 in conjunction with Theorem 3.1 lead to an existence theorem for the Riemann problem.

Theorem 8.2. *Assume that* (1.1) *is strictly hyperbolic. Given any data* U_-, U_+ *with* $|U_+ - U_-|$ *sufficiently small, there exists a function of bounded variation* $U(\xi)$ *defined on* $(-\infty, \infty)$ *such that* $U(\frac{x}{t})$ *is a weak solution of the Riemann problem for* (1.1).

Next we investigate the structure of the emerging solution U. It is instructive to use the correspondence between functions of bounded variation and finite signed Borel measures on $\mathbb{R}$ (Folland [F, Sec 3.5, Sec 7.3]). Let μ be the (vector valued) measure generated by the right continuous function of (normalized) bounded variation $\big(U(\xi+) - U_-\big)$. Consider now the functions

$$\Phi_{k\varepsilon}(\xi) = \int_{-\infty}^{\xi} \varphi_{k\varepsilon}(\zeta)\, d\zeta\,. \tag{8.16}$$

In view of (3.25) the family $\{\Phi_{k\varepsilon}\}$ consists of increasing uniformly bounded functions. Therefore $\Phi_{k\varepsilon}$ converge along a subsequence to an increasing function Φ_k pointwise on the reals. The measures generated by $\Phi_k(\xi+)$ are denoted by ϕ_k; they are positive measures with total mass one.

Introduce the measures associated with the functions U'_ε and $\varphi_{k\varepsilon}$ defined by

$$\begin{aligned} <\mu_\varepsilon, \psi> &= \int_{\mathbb{R}} U'_\varepsilon(\xi)\cdot\psi(\xi)\, d\xi\,, \\ <\phi_{k\varepsilon}, \chi> &= \int_{\mathbb{R}} \varphi_{k\varepsilon}(\xi)\,\chi(\xi)\, d\xi\,, \end{aligned} \tag{8.17}$$

where $\psi \in [C_c(\mathbb{R})]^N$, $\chi \in C_c(\mathbb{R})$ are continuous functions with compact support. Then (3.25), (7.19), (8.10) and Helly's convergence theorem imply

$$\begin{aligned} \int_{\mathbb{R}} U'_\varepsilon\cdot\psi\, d\xi &\to \int_{\mathbb{R}} \psi\cdot dU =<\mu,\psi>\,, \quad \text{for } \psi \in [C_c(\mathbb{R})]^N\,, \\ \int_{\mathbb{R}} \varphi_{k\varepsilon}\,\chi\, d\xi &\to \int_{\mathbb{R}} \chi\, d\Phi_k =<\phi_k,\chi>, \quad \text{for } \chi \in C_c(\mathbb{R})\,. \end{aligned} \tag{8.18}$$

In the language of functional analysis $\mu_\varepsilon \rightharpoonup \mu$ and $\phi_{k\varepsilon} \rightharpoonup \phi_k$ weak-$\star$ in measures.

Using (8.18) we can express $<\mu,\psi>= -\int_{\mathbb{R}} U\cdot\psi' d\xi$ for test functions $\psi \in [C^1_c(\mathbb{R})]^N$. Note that $\xi \notin \operatorname{supp}\mu$ if and only if there is an open interval $I \ni \xi$ such that $<\mu,\psi>= -\int_{\mathbb{R}} U\cdot\psi' d\xi = 0$ for $\psi \in [C^1_c(I)]^N$. This is in turn equivalent to the function U being a.e. equal to a constant vector on I. Consequently $\operatorname{supp}\mu$ coincides with the region in the ξ-domain where U is not a constant state. From (8.7) it follows that μ is absolutely continuous with respect to $\sum_{k=1}^N \phi_k$ and that

$$\begin{aligned} &\operatorname{supp}\phi_{\mathrm{k}} \subset [\lambda_{k-},\, \lambda_{k+}] \\ &\operatorname{supp}\mu \subset \bigcup_{k=1}^{N} \operatorname{supp}\phi_{\mathrm{k}} \subset \mathcal{L} = \bigcup_{\mathrm{k}=1}^{\mathrm{N}} [\lambda_{\mathrm{k}-},\, \lambda_{\mathrm{k}+}]\,. \end{aligned} \tag{8.19}$$

The following proposition states an important property of ϕ_k, that incorporates admissibility restrictions induced by the self-similar viscosity. In preparation, recall that

$$\varphi_{k\varepsilon} = \frac{e^{-\frac{1}{\varepsilon} g_{k\varepsilon}}}{\int_{-\infty}^{\infty} e^{-\frac{1}{\varepsilon} g_{k\varepsilon}} \, d\zeta} \tag{8.20}$$

where

$$g_{k\varepsilon}(\xi) = \int_{\rho_{k\varepsilon}}^{\xi} s - \lambda_k(U_\varepsilon(s)) \, ds \,, \tag{8.21}$$

and assume (by restricting to a further subsequence) that $\rho_{k\varepsilon} \to \rho_k$ as $\varepsilon \to 0$. Using (8.1), (C_o) and the Ascoli-Arzela theorem we deduce that

$$g_{k\varepsilon}(\xi) = \int_{\rho_{k\varepsilon}}^{\xi} s - \lambda_k(U_\varepsilon(s)) \, ds \to \int_{\rho_k}^{\xi} s - \lambda_k(U(s)) \, ds =: g_k(\xi) \tag{8.22}$$

uniformly on compact subsets of $(-\infty, \infty)$. We show that points in the support of ϕ_k are global minima for the function g_k.

Proposition 8.3. *If* $\xi \in \operatorname{supp} \phi_k$ *then* $g_k(\zeta) \geq g_k(\xi)$ *for* $\zeta \in (-\infty, \infty)$.

Proof. The proof has two steps. First, fix any $\xi \in \mathbb{R}$ and $\alpha > 0$ and consider the set

$$\mathcal{A} = \{\zeta \in \mathbb{R} \; : \quad g_k(\zeta) - g_k(\xi) < -\alpha < 0\} \, . \tag{8.23}$$

Since g_k is continuous either $\mathcal{A}$ is empty or it has positive Lebesgue measure $m(\mathcal{A})$. We will prove that if $m(\mathcal{A}) > 0$ there exists an open interval $I \ni \xi$ such that $< \phi_k, \chi >= 0$ for any $\chi \in C_c(I)$. As a result, if $m(\mathcal{A}) > 0$ then $\xi \notin \operatorname{supp} \phi_k$.

To establish the claim, observe first that

$$g_k(\zeta) - g_k(\xi) \geq \frac{1}{2}(\zeta^2 - \xi^2) - \max\{|\lambda_{k-}|, \, |\lambda_{k+}|\} \, |\zeta - \xi| \tag{8.24}$$

implies $g_k(\zeta) \to \infty$ as $|\zeta| \to \infty$ and $\mathcal{A}$ is contained in some compact interval $[a, b]$. Fix $\delta > 0$ such that for $\theta \in (\xi - \delta, \xi + \delta)$ we have $|g_k(\xi) - g_k(\theta)| < \frac{\alpha}{6}$. By virtue of (8.22), there is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then

$$|g_{k\varepsilon}(\theta) - g_k(\theta)| \leq \frac{\alpha}{6} \qquad \text{for } \theta \in \mathcal{A} \cup (\xi - \delta, \, \xi + \delta) \, . \tag{8.25}$$

From (8.23) and (8.25) we deduce that if $\theta \in (\xi - \delta, \xi + \delta)$, $\varepsilon < \varepsilon_0$ and $\zeta \in \mathcal{A}$ then

$$\begin{aligned} g_{k\varepsilon}(\zeta) - g_{k\varepsilon}(\theta) &\leq g_k(\zeta) - g_k(\xi) + |g_k(\xi) - g_k(\theta)| \\ &\quad + |g_{k\varepsilon}(\theta) - g_k(\theta)| + |g_{k\varepsilon}(\zeta) - g_k(\zeta)| < -\frac{\alpha}{2} \, . \end{aligned} \tag{8.26}$$

In turn (8.20) and (8.21) yield for $\theta \in I := (\xi - \delta, \xi + \delta)$

$$0 < \varphi_{k\varepsilon}(\theta) \leq \frac{1}{\int_{\mathcal{A}} \exp\{-\frac{1}{\varepsilon}\big(g_{k\varepsilon}(\zeta) - g_{k\varepsilon}(\theta)\big)\}\, d\zeta} \leq \frac{e^{-\frac{\alpha}{2\varepsilon}}}{m(\mathcal{A})} \,. \tag{8.27}$$

Let $\chi \in C_c(I)$. Then (8.18) and (8.27) imply

$$< \phi_{k\varepsilon}, \chi >= \int_{(\xi-\delta,\xi+\delta)} \varphi_{k\varepsilon}(\theta)\, \chi(\theta)\, d\theta \to 0\ , \quad \text{as} \ \ \varepsilon \to 0\,. \tag{8.28}$$

Hence, $< \phi_k, \chi >= 0$ for $\chi \in C_c(I)$.

Suppose next that $\xi \in \operatorname{supp} \phi_{\mathrm{k}}$. Then $\mathcal{A}$ is empty for any $\alpha > 0$ and $g_k(\zeta) \geq g_k(\xi)$ for any $\zeta \in (-\infty, \infty)$. ∎

The minimization properties for the g_k yield information on the structure of U. In particular, a weak form of the Lax shock conditions is induced at points of discontinuity.

Proposition 8.4. *Let* ξ, $\xi' \in \operatorname{supp} \mu \cap [\lambda_{\mathrm{k}-}, \lambda_{\mathrm{k}+}]$ *with* $\xi < \xi'$.

(a) *If* $\xi \in \mathcal{C}$ *then*

$$\xi = \lambda_k(U(\xi))\,. \tag{8.29}$$

(b) *If* $\xi \in \mathcal{S}$ *then* U *satisfies at* ξ *the jump conditions* (8.3) *and the inequalities*

$$\lambda_k(U(\xi+)) \leq \xi \leq \lambda_k(U(\xi-))\,. \tag{8.30}$$

(c) *If* ξ, $\xi' \in \operatorname{supp}\, \mu \cap [\lambda_{\mathrm{k}-}, \lambda_{\mathrm{k}+}]$ *then* $\lambda_k(U(\xi+)) = \xi$, $\lambda_k(U(\xi'-)) = \xi'$. *Moreover, for any point* $\theta \in (\xi, \xi')$

$$\begin{aligned} \theta = \lambda_k(U(\theta)) \quad &\mathit{if} \quad \theta \in \mathcal{C}\ , \\ \lambda_k(U(\theta+)) = \theta = \lambda_k(U(\theta-)) \quad &\mathit{if} \quad \theta \in \mathcal{S}\,. \end{aligned} \tag{8.31}$$

Proof. The function g_k in (8.22) is continuous and has the behavior $g_k(\xi) \to \infty$ as $|\xi| \to \infty$. Since U is of bounded variation, the limits

$$\lim_{\zeta \to \xi\pm} \frac{g_k(\zeta) - g_k(\xi)}{\zeta - \xi} = \lim_{\zeta \to \xi\pm} \frac{1}{\zeta - \xi} \int_{\xi}^{\zeta} s - \lambda_k(U(s))\, ds = \xi - \lambda_k(U(\xi\pm)) \tag{8.32}$$

exist and imply that the derivative $\frac{dg_k}{d\xi}$ exists and is continuous for $\xi \in \mathcal{C}$, while only the right and left derivatives exist for $\xi \in \mathcal{S}$. Fix a point $\xi \in \operatorname{supp}\, \mu \cap [\lambda_{\mathrm{k}-}, \lambda_{\mathrm{k}+}]$. It follows from (8.19) and Proposition 8.3 that $\xi \in \operatorname{supp} \phi_{\mathrm{k}}$ and that $g_k(\zeta) \geq g_k(\xi)$ for $\zeta \in \mathbb{R}$. In turn, (8.32) yields

$$\xi - \lambda_k(U(\xi+)) \geq 0 \quad , \quad \xi - \lambda_k(U(\xi-)) \leq 0\,, \tag{8.33}$$

which leads to (8.29) for $\xi \in \mathcal{C}$ and to (8.30) for $\xi \in \mathcal{S}$.

It remains to show (c). Let ξ, $\xi' \in \operatorname{supp}\mu \cap \operatorname{supp}\phi_{\mathrm{k}}$ with $\xi < \xi'$. Then ξ, ξ' are both global minima for g_k with $g_k(\xi) = g_k(\xi')$. We claim

$$g_k(\theta) = g_k(\xi) \quad \text{for any} \quad \theta \in (\xi, \xi') \,. \tag{8.34}$$

If (8.34) is violated at some point, there exist a, b with $\xi \le a < b \le \xi'$ such that

$$g_k(a) = g_k(b) = g_k(\xi) \quad , \quad g_k(\theta) > g_k(\xi) \quad \text{for} \quad a < \theta < b \,. \tag{8.35}$$

At the points a, b we have

$$\begin{aligned} &\lambda_k(U(a+)) \le a \le \lambda_k(U(a-)) \\ &\lambda_k(U(b+)) \le b \le \lambda_k(U(b-)) \,. \end{aligned} \tag{8.36}$$

On the other hand, at any $\theta \in (a,b)$ the set $\mathcal{A} = \{\zeta \in \mathrm{I\!R} : g_k(\zeta) - g_k(\theta) < -\alpha\}$ is nonempty for some $\alpha > 0$. Proposition 8.3 and (8.19) then imply $\theta \notin \operatorname{supp}\phi_{\mathrm{k}}$ and the function $U(\xi)$ remains constant on the interval (a,b). Hence $\lambda_k(U(a+)) = \lambda_k(U(b-))$ and the inequalities (8.36) yield $b \le a$. This contradicts $a < b$ and (8.35) follows. ∎

In summary, the region where U is nonconstant consists of (at most) N disjoint closed intervals $I_{\lambda_k} = [a_k, b_k]$, $k = 1, \ldots, N$. Each I_{λ_k} is associated with one characteristic family $\lambda_k(U)$ and could be empty or consist of just a single point. The function U takes constant values on the complement of $\bigcup_{k=1}^N I_{\lambda_k}$ and has the properties listed in Proposition 8.4 at points of I_{λ_k}. The emerging solution consists of N wave fans separated by constant states. Next we use the weak form of (8.2),

$$-\xi\, U(\xi+) + \theta\, U(\theta-) + F(U((\xi+)) - F(U(\theta-)) + \int_\theta^\xi U(s)\, ds = 0 \qquad \xi,\, \theta \in \mathrm{I\!R}\,, \tag{8.37}$$

in conjunction with relations (8.29 – 8.31) to obtain a fuller description of the behavior of U on the wave fans.

Proposition 8.5 Suppose that $I_{\lambda_k} = [a_k, b_k]$ is a full interval, $a_k < b_k$.

(i) For each $\xi \in [a_k, b_k)$ such that $\nabla\lambda_k(U(\xi+)) \cdot r_k(U(\xi+)) \neq 0$,

$$\lim_{h\to 0\,,\, h>0} \frac{1}{h}\big(U(\xi + h-) - U(\xi+)\big) = \frac{1}{\nabla\lambda_k(U(\xi+)) \cdot r_k(U(\xi+))}\, r_k(U(\xi+)) \,. \tag{8.38}$$

(ii) For each $\xi \in (a_k, b_k]$ such that $\nabla\lambda_k(U(\xi-)) \cdot r_k(U(\xi-)) \neq 0$,

$$\lim_{h\to 0\,,\, h<0} \frac{1}{h}\big(U(\xi + h+) - U(\xi-)\big) = \frac{1}{\nabla\lambda_k(U(\xi-)) \cdot r_k(U(\xi-))}\, r_k(U(\xi-)) \,. \tag{8.39}$$

Proof. We show (i). Fix $\xi \in [a_k, b_k)$ and let $h > 0$ such that $\xi + h \in I_{\lambda_k}$. The weak form (8.37) taken between the points $\xi+$ and $\xi + h-$ gives

$$\begin{aligned}
&\big[-\xi\, I + \nabla F(U(\xi+)) \big] \big(U(\xi+h-) - U(\xi+)\big) \\
&\quad = - \big[F(U(\xi+h-)) - F(U(\xi+)) - \nabla F(U(\xi+)) \big(U(\xi+h-) - U(\xi+)\big) \big] \\
&\quad\quad - \int_{\xi}^{\xi+h} \big[U(s) - U(\xi+) \big]\, ds + h\, \big(U(\xi+h-) - U(\xi+)\big)\,.
\end{aligned} \tag{8.40}$$

The increment $(U(\xi+h-) - U(\xi+))$ is expanded in the basis of right eigenvectors :

$$\omega(h) := U(\xi+h-) - U(\xi+) = \sum_i \omega_i(h)\, r_i(U(\xi+))\,. \tag{8.41}$$

Note that for a function U of bounded variation $\omega(h) \to 0$ as $h \to 0+$, and that by (3.8)

$$\omega_i(h) = l_i(U(\xi+)) \cdot \omega(h)\,. \tag{8.42}$$

Taking the inner product of (8.40) with $l_i(U(\xi+))$ and using (3.6), (8.42) and the Taylor expansion, we obtain

$$[-\xi + \lambda_i(U(\xi+))]\, \omega_i(h) = O\big(|\omega(h)|^2\big) + O\Big(\int_0^h |\omega(s)|\, ds \Big) + O\big(h\, |\omega(h)|\big)\,, \tag{8.43}$$

On account of Proposition 8.4 and the strict hyperbolicity of (1.1), the coefficient $[-\xi + \lambda_i(U(\xi+))]$ is nonzero for $i \neq k$ but vanishes for $i = k$.

Next, using (8.29 – 8.31) and the Taylor expansion of λ_k, we see that

$$\begin{aligned}
&\lambda_k(U(\xi+h-)) - \lambda_k(U(\xi+)) = h \\
&\quad = \nabla \lambda_k(U(\xi+)) \cdot \big(U(\xi+h-) - U(\xi+)\big) + O\big(|\omega(h)|^2\big)\,.
\end{aligned} \tag{8.44}$$

If we set $j_k = \nabla \lambda_k(U(\xi+)) \cdot r_k(U(\xi+))$, $j_k \neq 0$ by hypothesis, and use (8.44), (8.41) and relations (8.43) for $i \neq k$, we arrive at the estimate

$$\begin{aligned}
&j_k\, \omega_k(h) - h = O\Big(\sum_{i \neq k} |\omega_i(h)| \Big) + O\big(|\omega(h)|^2\big) \\
&\quad = O\big(|\omega(h)|^2\big) + O\Big(\int_0^h |\omega(s)|\, ds \Big) + O\big(h\, |\omega(h)|\big)\,.
\end{aligned} \tag{8.45}$$

Adding (8.43) for $i \neq k$ with (8.45) gives

$$\begin{aligned}
&\varphi(h) : = |j_k\, \omega_k(h) - h| + \sum_{i \neq k} |\omega_i(h)| \\
&\quad = O\big((|\omega(h)| + h)\, |\omega(h)| \big) + O\Big(\int_0^h |\omega(s)|\, ds \Big) \\
&\quad = O\big((|\omega(h)| + h)\, \varphi(h) \big) + O\Big(\int_0^h \varphi(s)\, ds \Big) + O(h^2)\,.
\end{aligned} \tag{8.46}$$

Since $\omega(h) \to 0$ as $h \to 0+$, we can choose δ sufficiently small so that for $0 < h \leq \delta$

$$\varphi(h) \leq C\, h^2 + C \int_0^h \varphi(s)\, ds\,. \tag{8.47}$$

The integral inequality, in turn, yields

$$0 \leq \varphi(h) \leq C'\, h^2 \qquad \text{for } 0 < h \leq \delta \tag{8.48}$$

and thus

$$\lim_{h\to 0+} \frac{\omega_i(h)}{h} = 0 \quad \text{for } i \neq k\,, \qquad \lim_{h\to 0+} \frac{\omega_k(h)}{h} = \frac{1}{j_k}\,. \tag{8.49}$$

This shows (8.38). The proof of part (ii) is virtually identical. ∎

Proposition 8.5 implies that U has right and left derivatives at any point ξ which is not an accumulation point of $\mathcal{S}$. If such a point ξ belongs to $\mathcal{C}$ then U is Lipshitz there, and if, in addition, it is an interior point of I_{λ_k} then f is differentiable there. It also completes the picture regarding the structure of the wave fans. We distinguish the following cases:

(i) If I_{λ_k} consists of a single point then the solution is a shock wave satisfying the weak form of the Lax shock conditions (8.30).

(ii) If I_{λ_k} is a full interval of points in $\mathcal{C}$ the solution is a k-rarefaction wave (provided that $\nabla\lambda_k \cdot r_k \neq 0$ on I_{λ_k} which is anyway necessary for rarefactions).

(iii) In general I_{λ_k} consists of an alternating sequence of shock waves and k-rarefaction waves such that each shock adjacent to a rarefaction from one side is a contact discontinuity on that side.

9. Self-similar viscous limits and shock profiles

In this section we discuss the relation between self-similar viscous limits and shock profiles for strictly hyperbolic systems. It was conjectured by Dafermos [D_2] and Tupciev [Tu_2], and proved for systems of two equations [D_2] that self-similar viscous limits have the internal structure of traveling wave solutions. We pursue here the question in the context of general systems.

Let ξ be a point of discontinuity of U and note that $U(\xi\pm)$ satisfy the Rankine-Hugoniot conditions (8.3). Consider a sequence of points $\{\xi_\varepsilon\}$ with the property $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$. Define the function

$$V_\varepsilon(\zeta) = U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta), \qquad -\infty < \zeta < \infty\,. \tag{9.1}$$

This introduces a stretching of the independent variable centered around the point ξ_ε, a shift of the shock speed ξ. The uniform estimates (C_b), (S) imply that V_ε is uniformly bounded and that

$$TV_\zeta\, V_\varepsilon(\cdot) = TV_\zeta\, U_\varepsilon(\xi_\varepsilon + \varepsilon\,\cdot) = TV_\xi\, U_\varepsilon(\cdot) \le C\,. \tag{9.2}$$

Using Helly's theorem and a diagonal argument we establish the existence of a subsequence and a function V such that

$$U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \to V(\zeta) \quad \text{pointwise for } -\infty < \zeta < \infty\,. \tag{9.3}$$

Proposition 9.1. *Let $\xi \in \mathcal{S}$ and suppose that $\{\xi_\varepsilon\}$ is a sequence of points with $\xi_\varepsilon \to \xi$. Then the function $V(\zeta)$ defined in* (9.3) *is continuously differentiable and satisfies on $(-\infty,\infty)$ the traveling wave equations*

$$-\xi\,[\,V - U(\xi-)\,] + [\,F(V) - F(U(\xi-))\,] = \frac{dV}{d\zeta} \tag{9.4}$$

with initial conditions

$$V(0) = \lim_{\varepsilon\to 0} U_\varepsilon(\xi_\varepsilon)\,. \tag{9.5}$$

The limits $\lim\limits_{\zeta\to\pm\infty} V(\zeta) =: V_\pm$ exist, are finite, and V_+, V_- satisfy the equations

$$-\xi\,[\,V - U(\xi-)\,] + [\,F(V) - F(U(\xi-))\,] = 0\,. \tag{9.6}$$

Proof. We evaluate (8.8) between the points $\xi_\varepsilon + \varepsilon\zeta$ and θ and then integrate the resulting equation in θ between ξ and $\xi+\delta$, for some $\delta \ne 0$, to arrive at

$$\begin{aligned} &[\,-(\xi_\varepsilon + \varepsilon\,\zeta)U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) + F(U_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta))\,] - \frac{1}{\delta}\int_\xi^{\xi+\delta} [\,-\theta U_\varepsilon(\theta) + F(U_\varepsilon(\theta))\,]\,d\theta \\ &\quad + \frac{1}{\delta}\int_\xi^{\xi+\delta}\int_\theta^{\xi_\varepsilon+\varepsilon\zeta} U_\varepsilon(\tau)\,d\tau d\theta = \frac{d}{d\zeta}\big(U_\varepsilon(\xi_\varepsilon + \varepsilon\zeta)\big) - \varepsilon\frac{1}{\delta}\int_\xi^{\xi+\delta} U_\varepsilon'(\theta)\,d\theta\,. \end{aligned} \tag{9.7}$$

After an integration in ζ we get

$$
(9.8)\qquad \begin{aligned}
&\int_0^\zeta [\,-(\xi_\varepsilon+\varepsilon s)U_\varepsilon(\xi_\varepsilon+\varepsilon s)+F(U_\varepsilon(\xi_\varepsilon+\varepsilon s))\,]\,ds-\zeta\,\frac{1}{\delta}\int_\xi^{\xi+\delta}[-\theta U_\varepsilon(\theta)+F(U_\varepsilon(\theta))]\,d\theta\\
&\quad+\frac{1}{\delta}\int_0^\zeta\int_\xi^{\xi+\delta}\int_\theta^{\xi_\varepsilon+\varepsilon s}U_\varepsilon(\tau)\,d\tau d\theta ds=U_\varepsilon(\xi_\varepsilon+\varepsilon\zeta)-U_\varepsilon(\xi_\varepsilon)-\frac{\varepsilon\zeta}{\delta}\int_\xi^{\xi+\delta}U_\varepsilon'(\theta)\,d\theta\,.
\end{aligned}
$$

Letting $\varepsilon\to 0$ and using (9.3), (C_b), (8.1) and (8.10) we deduce

$$
(9.9)\qquad \begin{aligned}
&\int_0^\zeta [\,-\xi\,V(s)+F(V(s))\,]\,ds-\zeta\,\frac{1}{\delta}\int_\xi^{\xi+\delta}[\,-\theta U(\theta)+F(U(\theta))\,]\,d\theta\\
&\quad+\zeta\frac{1}{\delta}\int_\xi^{\xi+\delta}\int_\theta^{\xi}U(\tau)\,d\tau d\theta=V(\zeta)-V(0)\,.
\end{aligned}
$$

From (9.9), by letting consecutively $\delta\to 0+$ and $\delta\to 0-$, we obtain

$$
(9.10)\qquad \int_0^\zeta\Big[\,-\xi\big(V(s)-U(\xi\pm)\big)+F(V(s))-F(U(\xi\pm))\Big]\,ds=V(\zeta)-V(0)\,.
$$

It follows from (9.10) that $V(\zeta)$ is a continuously differentiable function that satisfies the traveling wave equations (9.4) and the initial conditions (9.5). Since V is of bounded variation on $\mathbb{R}$, the limits $\lim\limits_{\zeta\to\pm\infty}V(\zeta)=:V_\pm$ exist and are finite. Also, for any integer n

$$
(9.11)\qquad \int_n^{n+1}\Big[\,-\xi\big(V(s)-U(\xi-)\big)+F(V(s))-F(U(\xi-))\Big]\,ds=V(n+1)-V(n)\,.
$$

Taking the i-th component of (9.11) and using the mean value theorem, we deduce that there are t_n^i with $n\le t_n^i\le n+1$ such that

$$
(9.12)\quad -\xi\big(V^i(t_n^i)-U^i(\xi-)\big)+F^i(V(t_n^i))-F^i(U(\xi-))=V^i(n+1)-V^i(n)\,,\qquad i=1\,,\,\dots\,,\,N\,.
$$

Letting $n\to\infty$ shows that V_+ is an equilibrium for (9.4). Similarly, V_- satisfies (9.6). ∎

The function V as well as the limiting values $V_\pm$ depend on the choice of the sequence $\{\xi_\varepsilon\}$. For several choices of $\{\xi_\varepsilon\}$ it may happen that the traveling wave disintegrates to a constant solution. Two questions arise: (i) Is it always possible to choose $\{\xi_\varepsilon\}$ so that the resulting V does not disintegrate to a constant solution of (9.4). (ii) What is the relation of $U(\xi-)$, $U(\xi+)$ and nontrivial heteroclinic orbits.

Proposition 9.2. *Let $\xi\in\mathcal{S}$ be fixed and suppose the set of solutions to* (9.6) *is not connected. There exists a sequence of shock shifts $\{\xi_\varepsilon\}$ such that the resulting V in* (9.3) *is a nontrivial heteroclinic* (*or homoclinic*) *orbit.*

Proof. Suppose the solution set of (9.6) with ξ fixed is contained in two open sets $\mathcal{O}_- \ni U(\xi-)$ and $\mathcal{O}_+ \ni U(\xi+)$ with $\mathcal{O}_- \cap \mathcal{O}_+ = \emptyset$. Because of (C_b) we may restrict attention to a ball B_M containing U_ε. For a large integer n it is $U(\xi - \frac{1}{n}) \in \mathcal{O}_-$ and $U(\xi + \frac{1}{n}) \in \mathcal{O}_+$. Choose ε_n such that $U_{\varepsilon_n}(\xi - \frac{1}{n}) \in \mathcal{O}_-$, $U_{\varepsilon_n}(\xi + \frac{1}{n}) \in \mathcal{O}_+$. There exists $\{\xi_{\varepsilon_n}\}$ satisfying $\xi - \frac{1}{n} \leq \xi_{\varepsilon_n} \leq \xi + \frac{1}{n}$ and $U_{\varepsilon_n}(\xi_{\varepsilon_n}) \in B_M - \big(\mathcal{O}_- \cup \mathcal{O}_+\big)$. Along a subsequence $\xi_{\varepsilon_n} \to \xi$ and $U_{\varepsilon_n}(\xi_{\varepsilon_n}) \to V(0)$ with $V(0) \notin \mathcal{O}_- \cup \mathcal{O}_+$. The resulting V is a nonconstant solution of (9.4) connecting two equilibria V_- and V_+. ∎

The hypothesis in Proposition 9.2 is violated only for shocks associated with a linearly degenerate characteristic field : $\nabla\lambda_k(U) \cdot r_k(U) = 0$ for all U (*cf.* Section 10) . Addressing (ii) is quite complicated at the full level of generality. We give one result indicating what can happen if there is a finite number of equilibria in B_M, the range of variation of U_ε.

Proposition 9.3. *Let $\xi \in \mathcal{S}$ and suppose that* (9.6) *has a finite number of solutions in B_M. There exists a subsequence $\varepsilon_n \to 0$ and choices $\{\xi_{1\varepsilon_n}\}$, $\{\xi_{2\varepsilon_n}\}$ of the shock shifts such that $\xi_{1\varepsilon_n} \leq \xi_{2\varepsilon_n}$, $\xi_{1\varepsilon_n} \to \xi$, $\xi_{2\varepsilon_n} \to \xi$,*

$$(9.14) \qquad U_{\varepsilon_n}(\xi_{1\varepsilon_n} + \varepsilon_n\zeta) \to V_1(\zeta)\,, \quad U_{\varepsilon_n}(\xi_{2\varepsilon_n} + \varepsilon_n\zeta) \to V_2(\zeta)\,, \quad \text{pointwise for } -\infty < \zeta < \infty\,,$$

and the resulting V_1, V_2 are solutions of (9.4) *that satisfy $V_1(-\infty) = U(\xi-)$, $V_2(+\infty) = U(\xi+)$.*

Proof. Let B_M be the ball where the solutions U_ε range, and suppose (9.6) has a finite number of solutions $U(\xi-)$, $U(\xi+)$ and $U_1, \dots, U_J$. Fix two open balls B_-, B_+ and an open set $\mathcal{O}$ with the properties that B_-, B_+ and $\mathcal{O}$ lie inside B_M, B_- is centered at $U(\xi-)$, B_+ is centered at $U(\xi+)$, $\mathcal{O}$ contains $U_1, \dots, U_J$, and the distances among any two of the sets B_-, B_+ and $\mathcal{O}$ are strictly positive. Since U is of bounded variation, we can fix $\delta > 0$ such that $U(\theta) \in B_-$ for $\theta \in [\xi - \delta, \xi)$ and $U(\theta) \in B_+$ for $\theta \in (\xi, \xi + \delta]$.

Consider a convergent sequence $U_{\varepsilon_n} \to U$ pointwise on $\mathbb{R}$. In the sequel we will be extracting appropriate subsequences that are denoted again by U_{ε_n}. Choose n_0 such that $U_{\varepsilon_n}(\xi - \delta) \in B_-$, $U_{\varepsilon_n}(\xi + \delta) \in B_+$ for $n \geq n_0$. For each $n \geq n_0$ choose points $a_n^\imath$, $A_n^\imath$, $b_n^\imath$, $B_n^\imath$, $\imath = 1, \dots, K(n)$, in the interval $I_\delta = [\xi - \delta, \xi + \delta]$ to be respectively: $a_n^1 = \xi - \delta$, b_n^1 the first point where U_{ε_n} enters the ball B_+, A_n^1 the last point in (a_n^1, b_n^1) that U_{ε_n} exits B_-, a_n^2 the first point after b_n^1 that U_{ε_n} enters the ball B_- (if applicable), B_n^1 the last point after b_n^1 that U_{ε_n} exits B_+, and so on until finally $B_n^{K(n)} = \xi + \delta$. These are defined as follows:

$$b_n^\imath = \inf\{\theta > a_n^\imath : U_{\varepsilon_n}(\theta) \in B_+\}\,, \quad A_n^\imath = \sup\{\theta \in (a_n^\imath, b_n^\imath) : U_{\varepsilon_n}(\theta) \in B_-\}\,,$$

$$a_n^{\imath+1} = \inf\{\theta > b_n^\imath : U_{\varepsilon_n}(\theta) \in B_-\}\,, \quad B_n^\imath = \sup\{\theta \in (b_n^\imath, a_n^{\imath+1}) : U_{\varepsilon_n}(\theta) \in B_+\}\,;$$

if in the $\imath$-th step $a_n^{\imath+1}$ is not well defined then $\imath = K(n)$ and $B_n^\imath = \xi + \delta$. Since U_{ε_n} is of uniformly bounded variation, it can go back and forth between B_- to B_+ at most a finite number of times, thus $K(n)$ is bounded. By restricting to subsequences we may assume $K(n)$ is some positive integer K for large n, and that $a_n^\imath \to a^\imath$, $A_n^\imath \to A^\imath$, $b_n^\imath \to b^\imath$ and $B_n^\imath \to B^\imath$, $\imath = 1, \dots, K$, as $n \to \infty$.

By construction $a_n^\imath < A_n^\imath < b_n^\imath < B_n^\imath < a_n^{\imath+1}$ and U_{ε_n} has the behavior $U_{\varepsilon_n}(\theta) \in B_M - B_+$ on $(a_n^\imath, A_n^\imath)$, $U_{\varepsilon_n}(\theta) \in B_M - \big(B_- \cup B_+\big)$ on $(A_n^\imath, b_n^\imath)$ and $U_{\varepsilon_n}(\theta) \in B_M - B_-$ on $(b_n^\imath, B_n^\imath)$. As a result the limits $a^\imath$, $A^\imath$, $b^\imath$ $B^\imath$ have the following properties: (i) $a^\imath \le A^\imath = b^\imath \le B^\imath = a^{\imath+1}$, (ii) if $B^\imath < \xi$ then $A^\imath = b^\imath = B^\imath = a^{\imath+1}$, and (iii) if $a^\imath > \xi$ then $B^{\imath-1} = a^\imath = A^\imath = b^\imath$. To see (ii) suppose that $B^\imath < \xi$; if $A^\imath < a^{\imath+1}$ there is $\theta < \xi$ such that $U_{\varepsilon_n}(\theta) \notin B_-$ for large n. Passing to the limit we see that $U(\theta) \notin B_-$ a contradiction. The rest are proved by similar arguments.

In what follows we fix $\jmath$ to be the first index that $B^\jmath = \xi$ and k to be the last index that $a^k = \xi$. Then we have the ordering

$$a^\jmath < A^\jmath = b^\jmath = B^\jmath = \dots = \xi = \dots = a^k = A^k = b^k < B^k \tag{9.15}$$

for any index between $\jmath$ and k.

Consider first the case that (9.6) has precisely two solutions $U(\xi-)$ and $U(\xi+)$. Set

$$\begin{aligned} \xi_{1\varepsilon_n} &= A_n^\jmath\,, \quad V_{1\varepsilon_n}(\zeta) = U_{\varepsilon_n}(A_n^\jmath + \varepsilon_n \zeta)\,, \\ \xi_{2\varepsilon_n} &= b_n^k\,, \quad V_{2\varepsilon_n}(\zeta) = U_{\varepsilon_n}(b_n^k + \varepsilon_n \zeta)\,. \end{aligned} \tag{9.16}$$

Then $V_{1\varepsilon_n}(0)$ lies on ∂B_- and $V_{2\varepsilon_n}(0)$ lies on ∂B_+. Along a subsequence $V_{1\varepsilon_n}$ and $V_{2\varepsilon_n}$ converge pointwise to a solution of (9.4) and the limits $V_i(\pm\infty) = V_{i\pm}$ exist and are finite. Since no solutions of (9.6) lie on the boundaries of B_- and B_+ the resulting traveling waves are nontrivial. From the definition of $V_{1\varepsilon_n}$ and $V_{2\varepsilon_n}$ follows that

$$V_{1\varepsilon_n}(\zeta) \notin B_+ \quad \text{for} \quad \frac{a_n^\jmath - A_n^\jmath}{\varepsilon_n} \le \zeta < 0\,, \qquad V_{2\varepsilon_n}(\zeta) \notin B_- \quad \text{for} \quad 0 < \zeta \le \frac{b_n^k - B_n^k}{\varepsilon_n}\,. \tag{9.17}$$

Since $\lim a_n^\jmath = a^\jmath < \xi = \lim A_n^\jmath$ and $\lim b_n^k = \xi < B^k = \lim B_n^k$,

$$V_1(\zeta) \notin \bar{B}_+ \quad \text{for} \quad -\infty < \zeta < 0\,, \qquad V_2(\zeta) \notin \bar{B}_- \quad \text{for} \quad 0 < \zeta < \infty\,. \tag{9.18}$$

As $U(\xi-)$ is the only equilibrium in $B_M - \bar{B}_+$ and $U(\xi+)$ is the only equilibrium in $B_M - \bar{B}_-$ it follows that $V_1(-\infty) = U(\xi-)$ and $V_2(+\infty) = U(\xi+)$.

Suppose next that (9.4) has more than two equilibria. If U_{ε_n} never enters $\mathcal{O}$ the previous proof shows the desired result. If U_{ε_n} enters $\mathcal{O}$, we restrict attention to the interval $[a_n^\jmath, b_n^\jmath]$ and note that $U_{\varepsilon_n}(\theta) \in B_M - \bar{B}_+$ on $[a_n^\jmath, b_n^\jmath)$ and $U_{\varepsilon_n}(b_n^\jmath) \in \partial B_+$. As in the previous step we choose points $c_n^\imath \le C_n^\imath \le d_n^\imath \le D_n^\imath$, $\imath = 1, \dots, K(n)$, with the properties: $c_n^1 = a_n^\jmath$, d_n^1 the first point after c_n^1 that U_{ε_n} enters $\mathcal{O}$, C_n^1 the last point before d_n^1 that U_{ε_n} exits B_-; if U_{ε_n} reenters B_- then we define c_n^2 to be the first point after d_n^1 that U_{ε_n} enters the ball B_-, define D_n^1 to be the last point before c_n^2 that U_{ε_n} exits $\mathcal{O}$, and reiterate the above procedure; if U_{ε_n} does not reenter B_- then we set D_n^1 to be the last point of exit from $\mathcal{O}$ before touching ∂B_+ and stop at this step. As the sequence $\{U_{\varepsilon_n}\}$ is of uniformly bounded variation the process will conclude in a finite number of steps. By restricting to subsequences we may assume $K(n) = K < \infty$, $c_n^\imath \to c^\imath$, $C_n^\imath \to C^\imath$, $d_n^\imath \to d^\imath$, $D_n^\imath \to D^\imath$. Again if $d^\imath < \xi$ for some $\imath$ then $C^\imath = d^\imath = D^\imath = c^{\imath+1}$. Let l be the first index such that $D^l = \xi$. Then $D^{l-1} = c^l < C^l = d^l = D^l = \xi$. If we set

$$\xi_{1\varepsilon_n} = C_n^l, \quad V_{1\varepsilon_n}(\zeta) = U_{\varepsilon_n}(C_n^l + \varepsilon_n \zeta) \tag{9.19}$$

then $V_{1\varepsilon_n}$ satisfies

$$V_{1\varepsilon_n}(\zeta) \notin \mathcal{O} \cup B_+ \quad \text{for} \quad \frac{c_n^l - C_n^l}{\varepsilon_n} < \zeta < 0 \tag{9.20}$$

and the resulting traveling wave V_1 has the property $V_1(-\infty) = U(\xi-)$. A similar construction shows the second part of the proposition. ∎

Proposition 9.3 shows that if ξ is a point of discontinuity of a solution U arising via self-similar viscous limits then there exists one heteroclinic orbit of (9.4) that emanates from $U(\xi-)$ and one that concludes at $U(\xi+)$. It is expected that in general this will be the same heteroclinic orbit. However, if more than two states in B_M satisfy the Rankine-Hugoniot conditions (9.6) at a given $\xi \in \mathcal{S}$, or if multiple heteroclinic connections between two equilibria are possible, then the precise relation between self-similar limits and shock profiles requires a detailed analysis of the heteroclinic orbits (the proof is suggestive as to what possibilities must be excluded). In specific examples it usually happens that there is a single shock profile connecting $U(\xi-)$ to $U(\xi+)$. It is however possible that there are intermediate states V_j, $j = 1, \dots, J$, satisfying (9.6) and a chain of shock profiles with the same shock speed ξ that connect successively $U(\xi-)$ to V_1, each of the points V_j to the next, and V_J to $U(\xi+)$. The latter situation occurs for the equations of elasticity in the presence of multiple inflection points in the stress-strain relation, for specific positions of the Riemann data relative to the stress-strain curve [Tz$_2$].

10. Comparisons with the classical solution of the Riemann problem

In this section we compare the classical solution of the Riemann problem with the solution obtained via self-similar viscous limits. For systems of strictly hyperbolic conservation laws the classical solution of the Riemann problem is based on a detailed study of elementary solutions of rarefaction waves and shock waves, and was established, for $|U_+ - U_-|$ small, by Lax [La$_1$] in the genuinely nonlinear case and by Liu [Li$_1$, Li$_2$] in the general case.

Fix U_0. Let $\mathcal{R}_k = \mathcal{R}_k(U_0)$ be the integral curves of the vector field r_k emanating from U_0. Rarefaction wave solutions take values on the curves $\mathcal{R}_k$. Shock waves emerge by solving the Rankine Hugoniot conditions

$$s\big(U - U_0\big) = F(U) - F(U_0)\,. \tag{10.1}$$

For U near U_0, the set of solutions of (10.1) consists of N smooth curves $\mathcal{S}_k = \mathcal{S}_k(U_0)$ tangent to $\mathcal{R}_k(U_0)$ at U_0, $k = 1, \ldots, N$. Each $\mathcal{S}_k$ is associated with the k-th characteristic field, it is defined by parametric equations $U = U_k(\tau)$ and $s = s_k(\tau)$ for $|\tau|$ small, and the parametrization may be arranged so that

$$\begin{aligned}
&U_k(0) = U_0\,, \qquad \dot U_k(0) = r_k(U_0)\,,\\
&s_k(0) = \lambda_k(U_0)\,, \qquad \dot s_k(0) = \frac{1}{2}\,\nabla\lambda_k(U_0)\cdot r_k(U_0)\,,\\
&\lambda_{k-1}(U_k(\tau)) < s_k(\tau) < \lambda_{k+1}(U_k(\tau))\,.
\end{aligned} \tag{10.2}$$

A state $U_k(\tau) \in \mathcal{S}_k(U_0)$ gives rise to a shock wave solution with speed $s_k(\tau)$, left state U_0, and right state $U_k(\tau)$. Liu [Li$_2$] performed a detailed study of the shock curves and proposed the following shock admissibility criterion. A shock $(U_0, U_k(\tau), s_k(\tau))$ is admissible if it satisfies

$$s_k(\tau) \le s_k(t) \quad \text{for } t \text{ between } 0 \text{ and } \tau\,. \tag{E}$$

Using (E) and imposing some mild geometric conditions, Liu obtained a unique solution of the Riemann problem.

Consider the solution U constructed via self-similar viscous limits in the previous sections. $U(\xi)$ takes values in a small ball $B_\mu(U_-)$, the wave speeds $\lambda_k(U(\xi))$ are separated, and $U(\xi)$ has the properties indicated at the end of Section 8. Each wave fan is studied separately and we distinguish three cases:

(i) λ_k *is genuinely nonlinear* : $\nabla\lambda_k(U)\cdot r_k(U) \neq 0$ for all U.

For a genuinely nonlinear characteristic field the shock speed $s_k(\tau)$ is increasing in one direction of the shock curve $\mathcal{S}_k(U_0)$ and decreasing in the opposite direction. Contact discontinuities are excluded for weak shocks. The behavior of U on I_{λ_k} simplifies considerably: Either I_{λ_k} is empty, or I_{λ_k} consists of a single point of jump discontinuity ξ with U satisfying at ξ the Lax shock conditions

$$\lambda_k(U(\xi+)) < \xi < \lambda_k(U(\xi-)) \,, \tag{10.3}$$

or I_{λ_k} is a full interval of points of continuity and the solution is a k-rarefaction wave on I_{λ_k}. Therefore, for genuinely nonlinear and strictly hyperbolic systems, the emerging structure of U is identical to that determined by Lax [La$_1$].

(ii) λ_k *is linearly degenerate* : $\nabla\lambda_k(U)\cdot r_k(U) = 0$ for all U.

For a linearly degenerate characteristic field the k-th shock curve emanating from U_0 is given by $U = U_k(\tau)$, $s = s_k(\tau)$ where

$$s_k(\tau) = \lambda_k(U_0) \,, \qquad \frac{dU_k}{d\tau}(\tau) = r_k(U_k(\tau)) \,, \quad U_k(0) = U_0 \,. \tag{10.4}$$

A version of the converse is also true: If (10.1) has a curve of solutions $U(\tau)$ corresponding to $s(\tau) = s_0$ fixed then $\dot{U}(\tau) = r_k(U(\tau))$, $s_0 = \lambda_k(U(\tau))$ for some k, and the k-th field is linearly degenerate. Since λ_k remains constant on the curves $\mathcal{R}_k$, rarefaction wave solutions are not possible for linearly degenerate characteristic fields. A close look in the proof of Propositions 8.4 and 8.5 shows that it is not possible that I_{λ_k} is a full interval. Therefore, either I_{λ_k} is empty, or it consists of a single point of jump discontinuity and U is a contact discontinuity.

(iii) The curves $\mathcal{R}_k$ intersect the set $\{U : \nabla\lambda_k(U)\cdot r_k(U) = 0\}$ at discrete points.

The solution U cannot be further simplified in this case. The relation with the Liu shock admissibility criterion (E) is established indirectly, using Proposition 9.3 on the relation between self-similar limits and shock profiles, in conjunction with results of Liu [Li$_4$], Majda and Pego [MP] on the relation between shock profiles and (a strict inequality version of) condition (E). Majda and Pego [MP, Theorem 3.1] prove that, given two states $U(\xi-)$ and $U(\xi+)$ in a small ball $B_\mu(U_-)$ satisfying the Rankine-Hugoniot conditions for some speed ξ, a shock profile connecting $U(\xi-)$ to $U(\xi+)$ and lying in $B_\mu(U_-)$ exists if and only if condition (E) is satisfied as a strict inequality. Moreover, there exists at most one trajectory $V(\zeta)$ of (9.4) connecting $U(\xi-)$ and $U(\xi+)$ which remains in $B_\mu(U_-)$ for all ζ.

Fix $\xi \in \mathcal{S} \cap I_{\lambda_k}$ and consider the set of all solutions to the Rankine-Hugoniot conditions that are compatible with (8.30). If $U(\xi-)$ and $U(\xi+)$ are the only states with this property then there is a shock profile connecting them and the shock speed ξ satisfies the strict condition (E). If there are more than two such solutions of (9.6) then there is a shock profile in $B_\mu(U_-)$ connecting $U(\xi-)$ to some state V_j and another shock profile connecting a state V_i to $U(\xi+)$. It is expected that in this case there will be a chain of shock profiles that connect $U(\xi-)$ through intermediate states with (eventually) $U(\xi+)$.

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