RELATIVE ENTROPY IN HYPERBOLIC RELAXATION

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Abstract. We provide a framework so that hyperbolic relaxation systems are endowed with a relative entropy identity. This allows a direct proof of convergence in the regime that the limiting solution is smooth.

1. Introduction

Consider the system of hyperbolic equations with stiff relaxation terms

$$\partial_t U + \sum_\alpha \partial_\alpha F_\alpha(U) = \frac{1}{\varepsilon} R(U)\,, \tag{1.1}$$

where $R, F_\alpha : \mathbb{R}^N \to \mathbb{R}^N$, $\alpha = 1, \dots, d$, are smooth, defining the evolution of a state vector $U(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^N$. It is assumed that (1.1) is equipped with a set of n conservation laws,

$$\partial_t \mathbb{P}U + \sum_\alpha \partial_\alpha \mathbb{P}F_\alpha(U) = 0\,, \tag{1.2}$$

for the conserved quantities $u = \mathbb{P}U$. Here, $\mathbb{P} : \mathbb{R}^N \to \mathbb{R}^n$ is a projection matrix with $\operatorname{rank}\mathbb{P} = n$ which determines the conserved quantities and annihilates the vector field R, that is $\mathbb{P}R(U) = 0$. It is also assumed that the equilibrium solutions of $R(U) = 0$ are parametrized in terms of the conserved quantities $U_{eq} = M(u)$; these functions will be called Maxwellians. Under the above framework it is conceivable that the dynamics of $u(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^n$ in the hyperbolic limit $\varepsilon \to 0$ is described by the system of conservation laws

$$\partial_t u + \sum_\alpha \partial_\alpha \mathbb{P}F_\alpha(M(u)) = 0 \tag{1.3}$$

The structure of relaxation systems and the convergence of (1.1) to (1.3) has been an active field of research, both at the level of examples (*e.g.* [7, 14]) but also at the level of generality in (1.1) (see [8, 16, 5, 12, 17, 6]). Motivated by the structure of models in kinetic theory, it has been postulated in [8] that relaxation systems (1.1) be equipped with a globally defined, convex entropy $H(U)$ satisfying

$$\partial_t H(U) + \sum_\alpha \partial_\alpha Q_\alpha(U) - \frac{1}{\varepsilon}\frac{\partial H}{\partial U}(U) \cdot R(U) = 0 \tag{1.4}$$

with positive dissipation. This amounts to the conditions

$$\begin{aligned} \nabla^2 H \, \nabla F_\alpha &= (\nabla F_\alpha)^T \nabla^2 H \,, \quad \alpha = 1, \dots, d \,, \\ \frac{\partial H}{\partial U}(U) \cdot R(U) &\le 0 \,, \quad \forall U \in \mathbb{R}^N \,. \end{aligned} \tag{1.5}$$

Convex entropies play a stabilizing role in relaxation [8, 17] in accordance with kinetic theory [5] and thermodynamical considerations [15]. The stabilizing role of entropy dissipation has been extensively analyzed [12, 17, 6] and leads to global existence results (at least near equilibria) for relaxation systems.

The goal of this work is to produce a relative entropy identity for general relaxation systems. Our work is motivated by computations at the level of a specific relaxation systems [14, Thm 3.3] or kinetic BGK-system [2]. The usual convergence framework for relaxation limits proceeds through analysis of the linearized (collision or relaxation) operator [7, 17]. By contrast, a relative entropy identity provides a simple and direct convergence framework in the smooth regime. This notion, introduced in the theory of conservation laws in [9, 11], has recently been applied to a variety of models; see [3], [2], [14] and [4], [1], [14, Thm 4.4] for an approach applicable to relaxation systems with special structures.

The relative entropy computation hinges on entropy consistency, that is that the restriction of $H - Q_\alpha$ on Maxwellians induces an entropy - entropy flux pair for the equilibrium system (1.3) in the form

$$\eta(u) = H(M(u)) \,, \quad q_\alpha(u) = Q_\alpha(M(u)) \,. \tag{1.6}$$

This structure is natural for models that have a thermodynamic origin, it is directly motivated by the formal Hilbert expansion for the relaxation limit (1.1), and has an interpretation in terms of the Gibbs principle.

In equilibrium statistical mechanics, the Gibbs principle states that equilibrium configurations achieve the maximum entropy under the existing constraints. (In statistical mechanics the entropy is the negative of the quantity considered here, and thus maxima become minima and accordingly production becomes dissipation). It suggests to define the entropy of a subsystem by the minimization procedure $s(u) = \min_{\mathbb{P}U=u} H(U)$. For H convex the resulting s is also convex. Moreover the orthogonality condition $\frac{\partial H}{\partial U}(M(u)) \perp N(\mathbb{P})$, resulting from the relaxation framework, induces that the minimizers satisfy

$$s(u) = H(M(u)) = \min_{\mathbb{P}U=u} H(U)$$

(see section 2.4).

Under such framework a relative entropy identity is computed, valid between *smooth* solutions U of the relaxation system (1.1) and *smooth* solutions $\widehat{u}$ of the associated equilibrium dynamics (1.3). It has the form

$$\begin{aligned}\partial_t H_r + \sum_\alpha \partial_\alpha Q_{\alpha,r} - \frac{1}{\varepsilon}\Big(\frac{\partial H}{\partial U}(U) - \frac{\partial H}{\partial U}(M(u))\Big)\cdot\big(R(U) - R(M(u))\big)\\ = -\sum_\alpha \nabla_u^2\eta(\widehat{u})\partial_\alpha\widehat{u} * \Big(g_\alpha(u) - g_\alpha(\widehat{u}) - \nabla g_\alpha(\widehat{u})(u-\widehat{u})\Big)\\ -\sum_\alpha \nabla_u^2\eta(\widehat{u})\partial_\alpha\widehat{u} * \mathbb{P}\big(F_\alpha(U) - F_\alpha(M(u))\big)\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}H_r &= H(U) - H(M(\widehat{u})) - \frac{\partial H}{\partial U}(M(\widehat{u}))\cdot(U - M(\widehat{u}))\\ Q_{\alpha,r} &= Q_\alpha(U) - Q_\alpha(M(\widehat{u})) - \frac{\partial H}{\partial U}(M(\widehat{u}))\cdot\big(F_\alpha(U) - F_\alpha(M(\widehat{u}))\big)\end{aligned}$$

are the relative entropy and associated fluxes respectively, while $g_\alpha(u) = \mathbb{P}F_\alpha(M(u))$ is the flux in (1.3). The identity then yields convergence of (1.1) to (1.3) in the smooth regime.

The paper is organized as follows: We start in section 2.1 by stating the relaxation framework and proceed in section 2.2 to examine its structural and geometric implications. In section 2.3 we review the Hilbert expansion in the context of relaxation problems (see also [7, 17]) and show that the entropy consistency is naturally suggested by applying the Hilbert expansion to the entropy dissipation identity (1.4). The connection between entropy consistency and Gibbs minimization is discussed in section 2.4. The structural properties derived in section 2 are used in section 3 to derive the relative entropy identity and prove the convergence Theorem 2.

2. Geometric picture of hyperbolic relaxation

We consider the relaxation system

$$\partial_t U + \sum_\alpha \partial_\alpha F_\alpha(U) = \frac{1}{\varepsilon} R(U) \tag{2.1}$$

that describes the evolution of the quantity U. We begin by reviewing the structural framework of relaxation systems (see [8] and also [5, 6, 12, 17]) and by developing those structural aspects that are useful for the relative entropy identity pursued in section 3.

2.1. **Hypotheses on the relaxation system.**

It is assumed that (1.1) is equipped with n conservation laws

$$\partial_t \mathbb{P} U + \sum_\alpha \partial_\alpha \mathbb{P} F_\alpha(U) = 0\,. \tag{2.2}$$

This means there exists a projection matrix

$$\begin{aligned} &\mathbb{P} : \mathbb{R}^N \to \mathbb{R}^n \quad \text{ with } \operatorname{rank} \mathbb{P} = n, \text{ and satisfying} \\ &\mathbb{P} R(U) = 0 \quad \forall U \in \mathbb{R}^N \end{aligned} \tag{h$_1$}$$

The rows p_i of $\mathbb{P}$ are linearly independent and generate the conserved quantities $u_i = p_i \cdot U$, $i = 1, \dots, n$.

The vector field $R : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (h$_1$). The equilibrium solutions of $R(U) = 0$ are called Maxwellians. It is assumed that the manifold $\mathcal{M}$ of Maxwellians is parametrized in terms of the n conserved quantities u,

$$U_{eq} = M(u) \quad \text{ with } \quad \mathbb{P} M(u) = u\,. \tag{h$_2$}$$

Furthermore, that the nondegeneracy conditions

$$\begin{aligned} \dim \mathcal{N}\big(\nabla R(M(u))\big) &= n \\ \dim \mathcal{R}\big(\nabla R(M(u))\big) &= N - n \end{aligned} \tag{h$_3$}$$

hold. For relaxation systems the flow of R is attracted towards the equilibrium manifold $\mathcal{M}$. This property is reflected in the entropy dissipation requirement below.

The system (1.1) is endowed with an entropy $H(U)$ with corresponding fluxes $Q_\alpha(U)$, $\alpha = 1, \dots, d$, such that

$$H : \mathbb{R}^N \to R \quad \text{is convex} \tag{h$_4$}$$

$$\begin{gathered} \frac{\partial H}{\partial U} \cdot \frac{\partial F_\alpha}{\partial U} = \frac{\partial Q_\alpha}{\partial U} \\ \frac{\partial H}{\partial U}(U) \cdot R(U) \le 0 \quad \forall U \in \mathbb{R}^N \end{gathered} \tag{h$_5$}$$

That is $H(U)$ is defined on the whole state space, it is convex, and dissipates along the relaxation process. Solutions of (1.1) satisfy the entropy equation

$$\partial_t H(U) + \sum_\alpha \partial_\alpha Q_\alpha(U) - \frac{1}{\varepsilon}\frac{\partial H}{\partial U}(U) \cdot R(U) = 0 \tag{2.3}$$

with the last term expressing dissipation.

The last hypothesis dictates consistency between the entropy structure of the relaxation system and that of the equilibrium system

$$\begin{gathered} \partial_t u + \sum_\alpha \partial_\alpha g_\alpha(u) = 0 \\ g_\alpha(u) := \mathbb{P} F_\alpha(M(u)) \end{gathered} \tag{2.4}$$

It is assumed that the restriction of the entropy pair $H - Q_\alpha$ on the equilibrium manifold $\mathcal{M}$,

$$\eta(u) := H(M(u)) \,, \quad q_\alpha(u) := Q_\alpha(M(u)) \,, \tag{h$_6$}$$

gives an entropy pair $\eta - q_\alpha$ for the system (1.3). That is smooth solutions of (1.3) satisfy the additional conservation law

$$\partial_t H(M(u)) + \sum_\alpha \partial_{x_\alpha} Q_\alpha(M(u)) = 0 \,. \tag{2.5}$$

2.2. **Structural and geometric implications.** Hypotheses (h_1)-(h_6) are a minimum set of hypotheses and as will be seen provide a rich structure for relaxation systems. We discuss the structural and geometric implications of (h_1)-(h_5) in this subsection and then separately (h_6) in the next two subsections.

Let $\mathbb{P} : \mathbb{R}^N \to \mathbb{R}^n$ be the projection operator with $\operatorname{rank} \mathbb{P} = n$ and define $\mathbb{P}^T : \mathbb{R}^n \to \mathbb{R}^N$ the adjoint of $\mathbb{P}$ which has $\operatorname{rank} \mathbb{P}^T = \operatorname{rank} \mathbb{P} = n$. Note that $\mathbb{R}^N$ is decomposed as a direct sum

$$\mathbb{R}^N = \mathcal{R}(P^T) \oplus \mathcal{N}(P)$$

where $\dim \mathcal{R}(\mathbb{P}^T) = \operatorname{rank} \mathbb{P}^T = n$ and $\dim \mathcal{N}(\mathbb{P}) = N - n$. Points $U \in \mathbb{R}^N$ admit the unique decomposition

$$U = \mathbb{P}^T w + V \,, \quad \text{with } w \in \mathbb{R}^n, \text{ and } \mathbb{P}V = 0.$$

In addition, $\mathbb{R}^n = \mathcal{R}(\mathbb{P})$ and points $u \in \mathbb{R}^n$ are decomposed as $u = \mathbb{P}U$ with $U \in \mathbb{R}^N$.

Let $M(u)$ be the Maxwellians of the system. From their defining property (h_2) we have that

$$\nabla R(M(u)) \frac{\partial M}{\partial u_k} = 0 \,, \quad k = 1, \ldots, n$$
$$\mathcal{N}(\nabla R(M(u))) \supset \operatorname{span} \left\{ \frac{\partial M}{\partial u_k},\, k = 1, \ldots, n \right\}$$

Since $\mathbb{P}\frac{\partial M}{\partial u_k} = e_k$ with e_k the standard basis in $\mathbb{R}^n$, we have that the vectors $\frac{\partial M}{\partial u_k}$, $k = 1, \ldots, n$, are linearly independent for all $u \in \mathbb{R}^n$. Then using (h_3) we conclude

$$\mathcal{N}(\nabla R(M(u))) = \operatorname{span} \left\{ \frac{\partial M}{\partial u_k},\, k = 1, \ldots, n \right\} = T\mathcal{M}_{M(u)} \,. \tag{2.6}$$

Next, hypothesis $\mathbb{P}R(U) = 0$ implies

$$\begin{aligned} \mathbb{P}\nabla R(U)A &= 0 \quad \forall A \in \mathbb{R}^N \\ \mathbb{P}\nabla^2 R(U)(A, B) &= 0 \quad \forall A\,, B \in \mathbb{R}^N \end{aligned} \tag{2.7}$$

and in turn $\mathcal{R}(\nabla R(U)) \subset \mathcal{N}(\mathbb{P})$ and $\mathcal{R}(\nabla^2 R(U)) \subset \mathcal{N}(\mathbb{P})$. Again, due to ($h_3$), we conclude

$$\mathcal{R}(\nabla R(M(u))) = \mathcal{N}(\mathbb{P}) \tag{2.8}$$

We now turn to (h_5). The existence of an entropy pair $H - Q_\alpha$ is equivalent to the property

$$\nabla^2 H \, \nabla F_\alpha = (\nabla F_\alpha)^T \nabla^2 H \,.$$

Alternatively, the latter may be expressed as

$$(\Lambda_{\alpha,i} - \Lambda_{\alpha,j}) R_{\alpha,i} \cdot \nabla^2 H R_{\alpha,j} = 0$$

for the eigenvalues and eigenvectors of the associated hyperbolic systems $\nabla F_\alpha R_{\alpha,i} = \Lambda_{\alpha,i} R_{\alpha,i}$.

We consider next the implications of

$$\begin{aligned} \frac{\partial H}{\partial U}(U) \cdot R(U) &\leq 0 \quad \forall U \in \mathbb{R}^N \\ R(M(u)) &= 0 \end{aligned} \tag{2.9}$$

The function φ defined by

$$\varphi(t) := \frac{\partial H}{\partial U}(M(u)+tA)\cdot R(M(u)+tA) \le 0 \quad \forall t\in\mathbb{R}\,,\forall A\in\mathbb{R}^N\,,$$

satisfies $\varphi(0)=0$, and thus

$$\begin{aligned}
\varphi'(0) &= \nabla^2 H(M(u))(A, R(M(u))) + \frac{\partial H}{\partial U}(M(u))\cdot \nabla R(M(u))A \\
&= 0 \\
\varphi''(0) &= \nabla^3 H(M(u)) : (A, A, R(M(u))) \\
&\quad + 2\nabla^2 H(M(u))(A, \nabla R(M(u))A) + \frac{\partial H}{\partial U}(M(u))\cdot \nabla^2 R(M(u))(A,A) \\
&\le 0
\end{aligned}$$

Using (2.8), the condition $\varphi'(0)=0$ implies

$$\frac{\partial H}{\partial U}(M(u))\cdot \nabla R(M(u))A = 0\,, \qquad \forall A\in\mathbb{R}^N$$

$$\frac{\partial H}{\partial U}(M(u)) \perp \mathcal{R}(\nabla R(M(u))) = \mathcal{N}(\mathbb{P}) \tag{2.10}$$

$$\frac{\partial H}{\partial U}(M(u))\cdot V = 0 \qquad \forall\, V\in\mathbb{R}^N \text{ with } \mathbb{P}V=0 \tag{2.11}$$

Property (2.10) suggests the entropy gradient is orthogonal to the relaxation flow as the latter approaches the equilibrium manifold, and it plays an important role in the relative entropy identity. By using once again (2.7) and (2.11), the condition $\varphi''(0)\le 0$ implies that

$$A\cdot\nabla^2 H(M(u))\nabla R(M(u))A \le 0 \quad \forall\, A\in\mathbb{R}^N\,, \tag{2.12}$$

that is $\left(\nabla R^T\nabla^2 H + \nabla^2 H\nabla R\right)(M(u))$ is negative semi-definite.

In summary, the problem (2.9) and condition (h_3) imply that the entropy H satisfies (2.10) and (2.12) as necessary conditions. As $\mathcal{R}(P^T)=\mathcal{N}(P)^\perp$ the orthogonality property $\frac{\partial H}{\partial U}(M(u))\perp\mathcal{N}(\mathbb{P})$ is equivalent to stating that, for some $a_k\in\mathbb{R}$,

$$\frac{\partial H}{\partial U}(M(u)) = \sum_{k=1}^{n} a_k p_k^T\,, \tag{2.13}$$

where p_k are the rows of $\mathbb{P}$ and generators of the conserved quantities. The term $D = -\frac{\partial H}{\partial U}\cdot R$ captures the entropy dissipation and may, using (2.11), be re-expressed in the suggestive form

$$
\begin{aligned}
D &= -\frac{\partial H}{\partial U}(U) \cdot R(U) \\
&= -\Big(\frac{\partial H}{\partial U}(U) - \frac{\partial H}{\partial U}(M(u))\Big) \cdot \Big(R(U) - R(M(u))\Big) \\
&\geq 0
\end{aligned}
\tag{2.14}
$$

While the relaxation system lives in $\mathbb{R}^N$ the equilibrium dynamics occurs on the n-dimensional manifold $\mathcal{M}$ which is parametrized in $\mathbb{R}^n$. The relation of the two canonical coordinate systems is induced via property (h_2)

$$
\mathbb{P} M(u) = u
$$

which implies

$$
\mathbb{P} \frac{\partial M}{\partial u_\alpha} = e_\alpha = (0 \dots 1 \dots 0)^T
$$

where e_α are the canonical basis in $\mathbb{R}^n$. Let now $a = \sum_\beta a_\beta e_\beta \in \mathbb{R}^n$, $A \in \mathbb{R}^N$ be such that $a = \mathbb{P} A$. We then have

$$
\mathbb{P} \left(A - \sum_\beta a_\beta \frac{\partial M}{\partial u_\beta} \right) = a - \sum_\beta a_\beta \mathbb{P} \frac{\partial M}{\partial u_\beta} = 0
$$

and thus

$$
A = \sum_\beta a_\beta \frac{\partial M}{\partial u_\beta} + V \,, \qquad V \in \mathcal{N}(\mathbb{P}) \,.
$$

As an implication of the entropy consistency hypothesis (h_6) and the orthogonality property (2.10), we obtain the following relation on the differential works performed by the equilibrium and the relaxing entropies:

$$
\begin{aligned}
\nabla_u \eta(u) * a : &= \sum_\beta \frac{\partial \eta}{\partial u_\beta} a_\beta \\
&= \frac{\partial H}{\partial U}(M(u)) \cdot \sum_\beta \frac{\partial M}{\partial u_\beta} a_\beta \\
&= \frac{\partial H}{\partial U}(M(u)) \cdot A \qquad \forall a \in \mathbb{R}^n \,, A \in \mathbb{R}^N \text{ with } a = \mathbb{P} A
\end{aligned}
\tag{2.15}
$$

where we have denoted by $*$ the inner product in $\mathbb{R}^n$.

2.3. **The Hilbert expansion.** We next show that (h_6) is naturally induced by applying the Hilbert expansion to the entropy identity (1.4). Consider the system (1.1), apply the Hilbert expansion

$$
U^\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots
$$

and let

$$u^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

be the associated expansion of the moments $u^\varepsilon = \mathbb{P}U^\varepsilon$. The Hilbert expansion produces the equations

$$\begin{aligned}
R(U_0) &= 0 && O(\tfrac{1}{\varepsilon})\\
\partial_t U_0 + \sum_\alpha \partial_{x_\alpha} F_\alpha(U_0) &= \nabla R(U_0) U_1 && O(1)\\
\partial_t U_1 + \sum_\alpha \partial_{x_\alpha} \left(\nabla F_\alpha(U_0) U_1\right) &= \nabla R(U_0) U_2 + \frac{1}{2}\nabla^2 R(U_0)(U_1, U_1) && O(\varepsilon)
\end{aligned}$$

and so on.

We can reconstruct the terms of the expansion as follows. First, $U_0 = M(u_0)$ where $u_0 = \mathbb{P}U_0$ is determined by solving the system of conservation laws

$$\partial_t u_0 + \sum_\alpha \partial_{x_\alpha} \mathbb{P} F_\alpha(M(u_0)) = 0\,.$$

Having identified U_0, the term U_1 is determined by solving the problem:

$$\begin{aligned}
\nabla R(U_0) U_1 &= \partial_t U_0 + \sum_\alpha \partial_{x_\alpha} F_\alpha(U_0)\\
\partial_t \mathbb{P} U_1 + \sum_\alpha \partial_{x_\alpha} (\mathbb{P} \nabla F_\alpha(U_0) U_1) &= 0
\end{aligned}$$

where the last identity is derived from projecting the $O(\varepsilon)$ term and using (2.7). Using (2.6) and (2.8), the first equation is inverted

$$\begin{aligned}
U_1 &= (\nabla R(U_0))^{-1} \left(\partial_t U_0 + \sum_\alpha \partial_{x_\alpha} F_\alpha(U_0) \right) + \sum_k \phi_k \frac{\partial M}{\partial u_k}\\
&=: g_0 + \sum_k \phi_k \frac{\partial M}{\partial u_k}
\end{aligned}$$

The coefficients ϕ_k are determined from the second equation, using (2.7) and the property $\mathbb{P}\frac{\partial M}{\partial u_k} = e_k$, which gives

$$\begin{aligned}
\partial_t \sum_k \phi_k e_k + \sum_\alpha \partial_{x_\alpha} \sum_k \phi_k \left(\mathbb{P} \nabla F_\alpha(U_0) \frac{\partial M}{\partial u_k}(u_0) \right)\\
= -\mathbb{P} \left(\partial_t g_0 + \sum_\alpha \partial_{x_\alpha} \nabla F_\alpha(U_0) g_0 \right)
\end{aligned}$$

Solving the linear hyperbolic equation provides ϕ_k and in turn determines U_1. To proceed to the next order and reconstruct U_2 we have to solve the problem

$$\nabla R(U_0)U_2 = \partial_t U_1 + \sum_\alpha \partial_{x_\alpha}(\nabla F_\alpha(U_0)U_1) - \frac{1}{2}\nabla^2 R(U_0)(U_1, U_1)$$

$$\partial_t \mathbb{P}U_2 + \sum_\alpha \partial_{x_\alpha}\big(\mathbb{P}\nabla F_\alpha(U_0)U_2\big) = -\sum_\alpha \partial_{x_\alpha}\big(\mathbb{P}\frac{1}{2}\nabla^2 F_\alpha(U_0)(U_1, U_1)\big)$$

where the last equality is obtained by projecting the $O(\varepsilon^2)$ term of the Hilbert expansion. This is accomplished by a similar argument as for the U_1 case, and we may continue to all orders.

It is instructive to also expand the entropy identity in terms of the Hilbert expansion. Introducing the expansion in (1.4), we obtain

$$\begin{aligned}
O(\tfrac{1}{\varepsilon}) \ : \qquad & \frac{\partial H}{\partial U}(U_0) \cdot R(U_0) = 0 \\
O(1) \ : \qquad & \partial_t H(U_0) + \sum_\alpha \partial_{x_\alpha} Q_\alpha(U_0) = \frac{\partial H}{\partial U}(U_0) \cdot \nabla R(U_0)U_1 \\
O(\varepsilon) \ : \qquad & \partial_t \nabla H(U_0)U_1 + \sum_\alpha \partial_{x_\alpha}\left(\nabla Q_\alpha(U_0)U_1\right) \\
& \quad = \frac{\partial H}{\partial U}(U_0) \cdot \Big(\nabla R(U_0)U_2 + \frac{1}{2}\nabla^2 R(U_0)(U_1, U_1)\Big) \\
& \qquad + \nabla^2 H(U_0)U_1 \cdot \nabla R(U_0)U_1
\end{aligned}$$

Using $U_0 = M(u_0)$ and (2.10), the $O(1)$ term produces

$$\partial_t H(U_0) + \sum_\alpha \partial_{x_\alpha} Q_\alpha(U_0) = 0 \tag{2.16}$$

that is $H(M(u_0))$-$Q_\alpha(M(u_0))$ is an entropy pair for the limit conservation law (1.3). The $O(\varepsilon)$ term is the first one contributing to the entropy dissipation. Its contribution is computed using (2.11), (2.12) and reads

$$\partial_t \nabla H(U_0)U_1 + \sum_\alpha \partial_{x_\alpha}\left(\nabla Q_\alpha(U_0)U_1\right) = \nabla^2 H(U_0)U_1 \cdot \nabla R(U_0)U_1 \le 0\,.$$

Of course, the Hilbert expansion fails near shocks and the above calculations are only valid away from shocks. Nevertheless, they indicate that the hypothesis of entropy consistency (h_6) is very natural for the relaxation model.

2.4. **Gibbs principle in relaxation.** We discuss here the role of (h_6) from the perspective of the Gibbs principle. Consider the problem of minimizing

$H(U)$ subject to the constraint $\mathbb{P}U = u$, that is

$$s(u) = \min_{\mathbb{P}U=u} H(U) \tag{2.17}$$

Under the hypothesis that H is convex and satisfies $H(U) \to \infty$ as $|U| \to \infty$, the minimizer in (2.17) is achieved and thus $s(u)$ is well defined. Moreover, s has the following properties:

Proposition 1. *If H is convex and $\lim_{|U|\to\infty} H(U) = \infty$ then the function s defined by $s(u) = \min_{\mathbb{P}U=u} H(U)$ is convex. Moreover,*

$$\frac{\partial H}{\partial U}(M(u)) \perp \mathcal{N}(\mathbb{P}) \quad \textit{if and only if} \quad s(u) = H(M(u)) \tag{2.18}$$

Proof. One easily checks that convexity of H implies convexity for s. Indeed, consider the sets

$$A = \{U \in \mathbb{R}^N : \mathbb{P}U = u\}\,, \quad B = \{V \in \mathbb{R}^N : \mathbb{P}V = u\}$$

and note that for $0 < \theta < 1$ we may write

$$C := \{W \in \mathbb{R}^N : \mathbb{P}W = \theta u + (1-\theta)v\} = \theta A + (1-\theta)B\,.$$

This implies

$$\begin{aligned} s(\theta u + (1-\theta)v) &= \min_{W \in \theta A + (1-\theta)B} H(W) \\ &= \min_{U\in A, V\in B} H(\theta U + (1-\theta)V) \\ &\leq \theta \min_{U\in A, V\in B} H(U) + (1-\theta) \min_{U\in A, V\in B} H(V) \\ &= \theta s(u) + (1-\theta)s(v) \end{aligned}$$

Suppose now that $\frac{\partial H}{\partial U}(M(u)) \perp \mathcal{N}(\mathbb{P})$. By the convexity of H,

$$H(U) \geq H(M(u)) + \frac{\partial H}{\partial U}(M(u)) \cdot (U - M(u))$$

and thus

$$H(U) \geq H(M(u)) \qquad \forall U \text{ such that } \mathbb{P}U = u\,,$$

and $s(u) = H(M(u))$. Conversely, if $\min_{\mathbb{P}U=u} H(U) = H(M(u))$ then

$$H(M(u) + tV) \geq H(M(u)) \qquad \forall\, t \in \mathbb{R}\,,\ \forall\, V \text{ such that } \mathbb{P}V = 0$$

and thus $\frac{\partial H}{\partial U}(M(u)) \cdot V = 0$ for $V \in \mathcal{N}(\mathbb{P})$. $\square$

We see from the proposition that the entropy $\eta(u) = H(M(u))$ in (h_6) is for one convex and also can be realized via the minimization $\eta(u) = \min_{\mathbb{P}U=u} H(U)$.

3. Relative entropy

Let $U(x,t)$ be a smooth solution of (1.1), let $u(x,t) = \mathbb{P}U(x,t)$ be the conserved quantities associated to U, and let $\widehat{u}(x,t)$ be a solution of the equilibrium system (1.3). With the objective to compare U and $\widehat{u}$, we define the relative entropy

$$H_r := H(U) - H(M(\widehat{u})) - \frac{\partial H}{\partial U}(M(\widehat{u})) \cdot (U - M(\widehat{u})) \tag{3.1}$$

and proceed to calculate an identity for H_r.

Note that $H(U)$ satisfies the entropy dissipation identity (1.4), $\eta(\widehat{u}) = H(M(\widehat{u}))$ satisfies entropy conservation

$$\partial_t H(M(\widehat{u})) + \sum_\alpha \partial_\alpha Q_\alpha(M(\widehat{u})) = 0\,, \tag{3.2}$$

u satisfies the conservation law (1.2), while $\widehat{u}$ satisfies

$$\partial_t \widehat{u} + \sum_\alpha \partial_\alpha g_\alpha(\widehat{u}) = 0\,,$$

with $g_\alpha(\widehat{u}) = \mathbb{P}F_\alpha(M(\widehat{u}))$. From (1.2) and (1.3) we have

$$\partial_t(u - \widehat{u}) + \sum_\alpha \partial_\alpha \left(\mathbb{P}F_\alpha(U) - \mathbb{P}F(M(\widehat{u})) \right) = 0$$

Taking the inner product ($*$ in $\mathbb{R}^n$) with $\frac{\partial \eta}{\partial u}(\widehat{u})$ we obtain

$$\begin{aligned}
I :&= \partial_t\Big(\frac{\partial \eta}{\partial u}(\widehat{u}) * (u - \widehat{u})\Big) + \sum_\alpha \partial_\alpha \Big(\frac{\partial \eta}{\partial u}(\widehat{u}) * (\mathbb{P}F_\alpha(U) - \mathbb{P}F_\alpha(M(\widehat{u})))\Big) \\
&= \partial_t\Big(\frac{\partial \eta}{\partial u}(\widehat{u})\Big) * (u - \widehat{u}) + \sum_\alpha \partial_\alpha \Big(\frac{\partial \eta}{\partial u}(\widehat{u})\Big) * (\mathbb{P}F_\alpha(U) - \mathbb{P}F_\alpha(M(\widehat{u}))) \\
&= -\nabla_u^2 \eta(\widehat{u}) \Big(\sum_\alpha \nabla g_\alpha(\widehat{u}) \partial_\alpha \widehat{u}\Big) * (u - \widehat{u}) \\
&\qquad + \sum_\alpha \nabla_u^2 \eta(\widehat{u}) \partial_\alpha \widehat{u} * (\mathbb{P}F_\alpha(U) - \mathbb{P}F_\alpha(M(\widehat{u}))) \\
&= \sum_\alpha \nabla_u^2 \eta(\widehat{u}) \partial_\alpha \widehat{u} * \Big(g_\alpha(u) - g_\alpha(\widehat{u}) - \nabla g_\alpha(\widehat{u})(u - \widehat{u}) \Big) \\
&\qquad + \sum_\alpha \nabla_u^2 \eta(\widehat{u}) \partial_\alpha \widehat{u} * \mathbb{P}\big(F_\alpha(U) - F_\alpha(M(u))\big)
\end{aligned} \tag{3.3}$$

where we used the identity $g_\alpha(u) = \mathbb{P}F_\alpha(M(u))$ and entropy compatibility for (1.3)

$$\nabla^2 \eta \nabla g_\alpha = (\nabla g_\alpha)^T \nabla^2 \eta\,.$$

On account of the property (2.15),

$$\frac{\partial\eta}{\partial u}(\widehat{u}) * (u-\widehat{u}) = \frac{\partial H}{\partial U}(M(u))\cdot(U-M(\widehat{u}))$$

$$\frac{\partial\eta}{\partial u}(\widehat{u}) * \mathbb{P}\big(F_\alpha(U)-F_\alpha(M(\widehat{u}))\big) = \frac{\partial H}{\partial U}(M(u))\cdot\big(F_\alpha(U)-F_\alpha(M(\widehat{u}))\big)\,,$$

I may be re-expressed in terms of H as

$$\begin{aligned} I = \partial_t\Big(&\frac{\partial H}{\partial U}(M(\widehat{u}))\cdot(U-M(\widehat{u}))\Big) \\ &+\sum_\alpha \partial_\alpha\Big(\frac{\partial H}{\partial U}(M(\widehat{u}))\cdot\big(F_\alpha(U)-F_\alpha(M(\widehat{u}))\big)\Big) \end{aligned} \tag{3.4}$$

We next derive the relative entropy identity. Let H_r be as in (3.1) and set

$$Q_{\alpha,r} := Q_\alpha(U) - Q_\alpha(M(\widehat{u})) - \frac{\partial H}{\partial U}(M(\widehat{u}))\cdot\big(F_\alpha(U)-F_\alpha(M(\widehat{u}))\big) \tag{3.5}$$

Combining (1.4), (3.2), (3.3) and (3.4), we deduce

$$\begin{aligned} \partial_t H_r &+ \sum_\alpha \partial_\alpha Q_{\alpha,r} + \frac{1}{\varepsilon}D \\ &= -\sum_\alpha \nabla_u^2\eta(\widehat{u})\partial_\alpha\widehat{u} * \Big(g_\alpha(u)-g_\alpha(\widehat{u})-\nabla g_\alpha(\widehat{u})(u-\widehat{u})\Big) \\ &\qquad -\sum_\alpha \nabla_u^2\eta(\widehat{u})\partial_\alpha\widehat{u} * \mathbb{P}\big(F_\alpha(U)-F_\alpha(M(u))\big) \\ &=: J_1 + J_2 \end{aligned} \tag{3.6}$$

where the term $\frac{1}{\varepsilon}D$ expressing the entropy dissipation may be written in any of the forms

$$\begin{aligned} D &= -\frac{\partial H}{\partial U}(U)\cdot R(U) \\ &= -\left(\frac{\partial H}{\partial U}(U) - \frac{\partial H}{\partial U}(M(u))\right)\cdot\big(R(U)-R(M(u))\big) \end{aligned}$$

with $u=\mathbb{P}U$.

Identity (3.6) provides a yardstick for measuring distance between the relaxation dynamics and “statistical equilibrium” response manifested in (1.3). From a mathematical perspective it may be used to obtain stability and convergence of the relaxation system (1.1) to the conservation laws (1.3), so long as the solutions of both systems remain smooth.

Theorem 2. *Under hypotheses* (h_1)-(h_6) *the relative entropy identity* (3.6) *holds. Suppose H is uniformly convex on compacts of $\mathbb{R}^N$. Let $\{U^\varepsilon\}$ be a family of smooth solutions of* (1.1) *and $\widehat{u}$ a smooth solution of* (1.3) *defined* *on $\mathbb{R}^d \times [0,T]$ and emanating from smooth data U_0^ε and $\widehat{u}_0$. Suppose that U^ε, $M(u^\varepsilon)$ and $M(\widehat{u})$ take values in a ball $B_M \subset \mathbb{R}^N$, and that for some $\nu = \nu(M)$ we have*

$$(\mathrm{h}_7) \qquad -\Big(\frac{\partial H}{\partial U}(U) - \frac{\partial H}{\partial U}(M(u))\Big)\cdot\big(R(U) - R(M(u))\big) \geq \nu |U - M(u)|^2$$

for $U, M(u) \in B_M$, where $u = \mathbb{P}U$. Then, for $R > 0$ there exist constants $C = C(R,T,M,\nabla\widehat{u}) > 0$ and s independent of ε such that

$$\int_{|x|<R} H_r(x,t)dx \leq C\left(\int_{|x|<R+st} H_r(x,0)dx + \varepsilon\right).$$

In particular, if the initial data satisfy

$$\int_{|x|<R+sT} H_r(x,0)dx \longrightarrow 0\,, \quad \text{as } \varepsilon \downarrow 0\,,$$

then

$$\sup_{t\in[0,T]} \int |U^\varepsilon - M(\widehat{u})|^2(x,t)dx \to 0$$

Proof. Fix $R > 0$, $t \in [0,T)$ and let $\mathcal{C}_t$ denote the cone

$$\mathcal{C}_t = \{(x,\tau) : 0 < \tau < t, \ \ |x| < R + s(t-\tau)\}$$

where s is a constant to be selected. The aim is to monitor the quantity

$$\varphi(\tau) = \int_{|x|<R+s(t-\tau)} H_r(x,\tau)dx\,, \quad 0 \leq \tau \leq t\,.$$

Consider the relative entropy identity

$$\partial_t H_r + \sum_\alpha \partial_\alpha Q_{\alpha,r} + \frac{1}{\varepsilon}D = J_1 + J_2$$

in its weak form

$$\begin{aligned}(3.7) \qquad &\iint\left(-H_r\partial_t\phi - \sum_\alpha Q_{\alpha,r}\partial_\alpha\phi + \frac{1}{\varepsilon}\phi D\right)dxd\tau \\ &\quad - \int H_r(x,0)\phi(x,0)dx = \iint \phi(J_1+J_2)dxd\tau\end{aligned}$$

where ϕ is Lipschitz continuous function compactly supported in $\mathbb{R}^d\times[0,T)$. The argument proceeds along the lines of [10, Thm 5.2.1]. Let $R > 0$, $t \in [0,T)$ be fixed, $\delta > 0$ such that $t+\delta < T$, and s to be precised later. We select the test function $\phi(x,\tau) = \theta(\tau)\psi(x,\tau)$ with

$$\theta(\tau) = \begin{cases} 1 & 0 \leq \tau < t \\ 1 - \frac{1}{\delta}(\tau - t) & t \leq \tau \leq t+\delta \\ 0 & t+\delta \leq \tau\,, \end{cases}$$

$$\psi(x,\tau) = \begin{cases} 1 & \tau > 0\,,\ |x| - R - s(t-\tau) < 0 \\ 1 - \frac{1}{\delta}\big(|x| - R - s(t-\tau)\big) & \tau > 0\,,\ 0 < |x| - s(t-\tau) - R < \delta \\ 0 & \tau > 0\,,\ \delta < |x| - R - s(t-\tau)\,. \end{cases}$$

and introduce it to (3.7). This gives

$$\begin{aligned} &\frac{1}{\delta}\int_t^{t+\delta}\int_{|x|<R} H_r dx d\tau \\ &\qquad + \frac{1}{\delta}\int_0^t \int_{0<|x|-R-s(t-\tau)<\delta} (sH_r + \sum_\alpha \frac{x_\alpha}{|x|} Q_{\alpha,r}) dx d\tau \\ &\qquad + \frac{1}{\varepsilon}\int_0^t \int_{|x|<R+s(t-\tau)} D dx d\tau + O(\delta) \\ &= \int_{|x|<R+st} H_r(x,0)dx + \int_0^t \int_{|x|<R+s(t-\tau)} (J_1 + J_2) dx d\tau\,. \end{aligned} \tag{3.8}$$

In what follows all state functions are controlled by taking into account that the states U, $M(u)$, $M(\widehat{u}) \in B_M$. The uniform convexity of H on compacts implies that for some $c = c(M)$

$$H_r \geq c\,|U - M(\widehat{u})|^2$$

Using (h_5), we obtain $|Q_{\alpha,r}|^2 \leq C|U - M(\widehat{u})|^2$ for some $C = C(M)$. We then select s so that

$$sH_r + \sum_\alpha \frac{x_\alpha}{|x|} Q_{\alpha,r} > 0$$

Letting $\delta \to 0$ and using (h_7), we have

$$\begin{aligned} &\int_{|x|<R} H_r(x,t)dx + \frac{\nu}{\varepsilon}\iint_{\mathcal{C}_t} |U - M(u)|^2 dx d\tau \\ &\qquad \leq \int_{|x|<R+st} H_r(x,0)dx + \iint_{\mathcal{C}_t} |J_1| + |J_2| dx d\tau\,. \end{aligned} \tag{3.9}$$

To handle the right hand sides of (3.9) we use the bounds

$$\begin{aligned} \iint_{\mathcal{C}_t} |J_1| dx d\tau &\leq C \iint_{\mathcal{C}_t} |u - \widehat{u}|^2 dx d\tau \\ &= C \iint_{\mathcal{C}_t} |\mathbb{P}U - \mathbb{P}M(\widehat{u})|^2 dx d\tau \\ &\leq C \iint_{\mathcal{C}_t} |U - M(\widehat{u})|^2 dx d\tau \\ \iint_{\mathcal{C}_t} |J_2| dx d\tau &\leq \frac{\nu}{\varepsilon} \iint_{\mathcal{C}_t} |U - M(u)|^2 dx + C\varepsilon\,, \end{aligned}$$

where C is a positive constant depending on the $\big(L^\infty \cap L^2\big)(\mathcal{C}_t)$-norm of $\nabla \widehat{u}$ on the cone $\mathcal{C}_t = \{0 < \tau < t, |x| < |x| < R + s(t-\tau)\}$ and M. We obtain

$$\varphi(t) \le \varphi(0) + C\left(\varepsilon + \int_0^t \varphi(\tau)\right)$$

and conclude via the Gronwall lemma. □

Most examples of relaxation systems satisfy the structural hypotheses (h_1)-(h_6) and are thus endowed with a relative entropy identity. This includes the discrete velocity Boltzmann equations, and the stress-relaxation models in [14]. The relative entropy is of course based on the relaxation system being equipped with a globally defined convex entropy. Perhaps, the most difficult hypothesis to justify is (h_3); however, for the relative entropy calculation, it suffices to show (2.11) and the latter may be directly justified if the Maxwellians are explicitly known. This is for instance the case for discrete velocity Boltzmann models.

Theorem 2 is based on (i) the availability of uniform bounds for the relaxation system (1.1), and (ii) on hypothesis (h_7). Hypothesis (i) is difficult to justify in practice, but for some special systems it is possible to replace it by global growth assumptions for the underlying constitutive functions, see [14, Thm 3.3] and [2] for specific examples.

Hypothesis (h_7) is the key to the convergence theorem allowing that the relative entropy provides a Lyapunov functional. Examples of relaxation systems that verify (h_7) are given in [13, 15, 14]. The discrete velocity BGK-models in [17, Thm 5.2] provide yet another example verifying (h_7). Note that (h_7) implies the necessary condition

$$A \cdot \left[\big(\nabla R^T \nabla^2 H + \nabla^2 H \nabla R\big)\,(M(u))\right]\, A \le -\nu |A|^2 \quad \forall A \in \mathcal{N}(\mathbb{P})\,, \tag{3.10}$$

and that conversely the condition

$$\begin{aligned} A \cdot \left[\nabla R^T(V) \nabla^2 H(U) + \nabla^2 H(U) \nabla R(V)\right] A &\le -\nu |A|^2 \\ \forall\, U\,,\, V \in B_M\,,\ \forall A &\in \mathcal{N}(\mathbb{P})\,, \end{aligned} \tag{3.11}$$

implies (h_7). Such conditions should be contrasted to the convexity of H, which is also required to apply Theorem 2. We note that (3.10) plays the key stabilizing role in the convergence analysis of [16] based on the linearized operator, and we refer to [17] for further properties of such inequalities.

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