MATERIALS WITH INTERNAL VARIABLES
AND
RELAXATION TO CONSERVATION LAWS

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Abstract. The theory of materials with internal state variables of Coleman and Gurtin [CG] provides a natural framework to investigate the structure of relaxation approximations of conservation laws from the viewpoint of continuum thermomechanics. After reviewing the requirements imposed on constitutive theories by the principle of consistency with the Clausius-Duhem inequality, we pursue two specific theories pertaining to stress relaxation and relaxation of internal energy, respectively. They each lead to a relaxation framework towards the theory of thermoelastic non-conductors of heat, equipped with globally defined "entropy" functions for the associated relaxation process. Next, we consider a semilinear model problem of stress relaxation. We discuss uniform stability and compactness for solutions of the relaxation system, in the zero-relaxation limit, and establish convergence to the system of isothermal elastodynamics, by using compensated compactness. Finally, we prove a strong dissipation estimate for the relaxation approximations proposed in Jin-Xin [JX], when the limit system is equipped with a strictly convex entropy.

1. Introduction

The presence of relaxation mechanisms is widespread in both the continuum mechanics as well as the kinetic theory contexts. The Broadwell model, the equations of chromatography, certain models for viscoelastic flow and models for traffic flow form an increasing list of examples involving relaxation mechanisms. The Chapman-Enskog expansion provides an effective equation for the relaxation process and reveals the stabilizing role of the subcharacteristic condition, Whitham [W], Liu [Li]. A framework for investigating relaxation to processes containing shocks is proposed in Chen-Levermore-Liu [CLL], and the mechanism motivates a class of nonoscillatory numerical schemes for conservation laws [JX]. Analytical investigations [CLL, N_1, TW, LM, KT, N_2, BCN] indicate that relaxation provides a subtle "dissipative" mechanism against the destabilizing effect of nonlinear response, as well as a damping effect on oscillations when assisted by nonlinear

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response. The theory of weak solutions for conservation laws with memory [Da_2, NRT, CD] provides another testing ground for dissipation induced by damping mechanisms.

The objective of the present study is (a) to investigate the mechanism of relaxation from the viewpoint of continuum thermomechanics, and (b) to study the strength of dissipation for relaxation processes. The dissipative structure for relaxation, as it emerges from general considerations of the second law of thermodynamics, is weaker than that of viscosity approximations. In general, under subcharacteristic type of conditions, it leads to what is called in [CLL] an "entropy" function and to control of the distance from equilibrium. Nevertheless, for several models of semilinear relaxation approximations a stronger dissipative structure occurs, matching the one of viscosity approximations. This is the case for a model pursued in [FM] and describing isothermal stress relaxation, see Section 3, as well as for the relaxation approximations of [JX] when the limit is a symmetric hyperbolic system, see Section 4.

It is well known that the subject of viscosity approximations for conservation laws is intimately tied to the mechanical issue of the passage from the theory of thermoviscoelasticity to the theory of thermoelasticity and, in turn, to the theory of thermoelastic non-conductors of heat, *e.g.* Dafermos [Da_1]. The relation among these theories is understood by monitoring the entropy production in the course of passing from one theory to the next. The natural framework, in the continuum thermomechanics context, to place relaxation approximations is offered by the theory of materials with internal variables of Coleman-Gurtin [CG]. This theory along with the (more general) theory of simple materials with fading memory, [Co], were developed to explore the dissipative structure of materials that exhibit memory effects. The requirement of consistency of constitutive relations with the second law of thermodynamics, in the form of the Clausius-Duhem inequality, reveals the dissipative structure of the theory of materials with internal variables. Remarkably, theories with internal variables that are consistent with the second law of thermodynamics are automatically equipped with what is called in the theory of relaxation an entropy function for the relaxation process.

The format of constitutive theories with internal variables can be complicated and is not generally given in closed-form relations, as is the case in the theory of thermoviscoelasticity. The problem of identifying specific constitutive theories and relating internal variable theories with the (limiting) equilibrium theories has been extensively studied in the mechanics literature, *c.f.* Coleman-Gurtin [CG], Gurtin-Williams-Suliciu [GWS], Faciu and Mihailescu-Suliciu [FM], Suliciu [Su] and references therein. The issue is important in the design of relaxation schemes for the equations of gas dynamics, Coquel-Perthame [CP].

We begin in Section 2 with a review of the thermodynamics of materials with internal variables. Following that, we derive necessary conditions at the general level and pursue two specific constitutive theories, one pertaining to stress relaxation and one pertaining to relaxation of internal energy. These theories are completely solvable: it is possible to give simple necessary and sufficient conditions so that the relaxation theory is consistent with the Clausius-Duhem inequality, and the constitutive functions are explicitly identified. In turn, consistency with the second law of thermodynamics leads to "entropy" functions for the associated relaxation process.

In Section 3, we consider the semilinear system

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x \sigma &= 0 \\ \partial_t(\sigma - Eu) &= -\frac{1}{\varepsilon}(\sigma - g(u)) \,, \end{aligned} \tag{1.1}$$

describing isothermal motions of a viscoelastic material. This model is studied in [FM] and emerges as a special case of the isothermal theory of stress relaxation, developed in Section 2. It is achieved by a theory compatible with the Clausius-Duhem inequality, if and only if $g_u < E$, while, in the formal zero relaxation-time limit $\varepsilon \to 0$, it yields the equations of isothermal elastodynamics

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x g(u) &= 0 \,. \end{aligned} \tag{1.2}$$

In Section 3, we validate the $\varepsilon \to 0$ convergence of (1.1) to (1.2), under the subcharacteristic condition $0 < g_u < E$ and structure and growth hypotheses on $g(u)$, see Theorem 3.3. The main ingredients are the theory of compensated compactness, Tartar [Ta], Murat [M], the L^2 theory for the reduction of generalized Young measures for the equations of elastodynamics, Shearer [Sh], Serre-Shearer [SSh], and an estimate - valid under the hypothesis $0 < g_u < E$ - measuring the dissipative strength of the relaxation approximation (1.1) to (1.2), see Lemma 3.3.

In Section 4, we test the extent of validity of such strong dissipations estimates. We consider semilinear relaxation approximations of the type proposed in Jin-Xin [JX],

$$\begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + A\partial_x u &= -\frac{1}{\varepsilon}(v - F(u)) \,, \end{aligned} \tag{1.3}$$

where $u, v \in \mathbb{R}^n$, A is a positive definite symmetric $n \times n$ matrix, and the limit system

$$\partial_t u + \partial_x F(u) = 0 \tag{1.4}$$

is symmetric hyperbolic. Under the subcharacteristic condition, $A - F'(u)^2 \geq \nu I$ for some $\nu > 0$, these systems are equipped with the strong dissipation estimate (4.9). An analogous result holds in the multi-dimensional case, again for the limit system being symmetric hyperbolic, see (4.18).

Finally, in Section 5, we consider systems (1.4) that are equipped with a strictly convex entropy $\eta(u)$. It is well known that such systems are symmetrizable [FL]. We consider the relaxation approximation (1.3) and we prove that if $\eta_{uu} \le \alpha I$ and if

$$\frac{1}{2}\big(A^T \eta_{uu}(u) + \eta_{uu}(u) A\big) - \alpha F'^T(u) F'(u) \ge \nu I \tag{S$'$}$$

for some $\alpha, \nu > 0$, the relaxation process satisfies the strong dissipation estimate (5.4), see Proposition 5.1.

An earlier version of this material appears in the Lecture Notes [Tz].

2. Thermomechanical Theories for Materials with Internal State variables

We begin with a review of the restrictions imposed on thermomechanical theories with internal state variables by the principle of consistency with the second law of thermodynamics, and the ensuing structure of relaxation approximations. For simplicity, the presentation is done for one-dimensional thermomechanical theories. The general form of constitutive theories for materials with internal state variables is analogous in several space dimensions [CG].

2.a Thermomechanical theories in one-space dimension.

Let the function $y(x,t)$ express the motion of a reference interval $[\alpha, \beta]$ and $\theta(x,t)$ express its temperature. The displacement $y(\cdot, t)$ is required, for each $t > 0$, to be a strictly increasing, bi-Lipschitz continuous map of the reference interval $[\alpha, \beta]$ onto the current configuration $[y(\alpha,t), y(\beta,t)]$. The list of quantities entering in a Lagrangean description of a thermomechanical process are: $\rho_0(x)$ the mass density in the reference configuration, $\rho(y,t)$ the mass density in the current configuration, y the motion, $u = \frac{\partial y}{\partial x}$ the strain, $v = \frac{\partial y}{\partial t}$ the velocity, τ the stress, f the body force per unit mass, θ the temperature ($\theta > 0$), e the specific internal energy ($e > 0$), q the heat flux, r the radiating heat density and η the specific entropy. The equations

$$\rho(y,t)\frac{\partial y}{\partial x} = \rho_0(x) \tag{2.1}$$

$$\partial_t u - \partial_x v = 0 \tag{2.2}$$

$$\partial_t(\rho_0 v) - \partial_x \tau = \rho_0 f \tag{2.3}$$

$$\partial_t\Big(\rho_0 \frac{1}{2} v^2 + \rho_0 e\Big) = \partial_x(\tau v) + \partial_x q + \rho_0 f v + \rho_0 r \tag{2.4}$$

express the balance of mass, the kinematic compatibility relation, the balance of linear momentum, and the balance of energy (the first law of thermodynamics), respectively. They are supplemented with the Clausius-Duhem inequality, which reads, in integral form,

$$\frac{d}{dt}\int_a^b \rho_0 \eta \, dx \ge \frac{q}{\theta}(x,t)\Big|_{x=a}^{x=b} + \int_a^b \frac{\rho_0 r}{\theta} dx \qquad \text{for } [a,b] \subset [\alpha,\beta] \text{ and } t > 0, \tag{2.5}$$

or, in local form,

$$\rho_0 \partial_t \eta \geq \partial_x \big(\frac{q}{\theta} \big) + \frac{\rho_0 r}{\theta} \,. \tag{2.6}$$

The Clausius-Duhem inequality expresses that the net production of entropy per unit time, in any control volume $[a, b]$, is positive, and manifests (a form of) the second law of thermodynamics.

The thermomechanical variables are connected through constitutive relations that characterize the material response. A constitutive theory is determined by assigning a class of independent (prime) variables and a class of dependent variables, derived from the prime variables via constitutive relations. In this separation, the set of thermodynamic variables is implicitly divided into "causes" and "effects". From the phenomenological standpoint of continuum thermomechanics, there is no a-priori reason why a cause in one constitutive relation should not be a cause in another. Therefore, in determining the general form of constitutive theories, one imposes Truesdell's *principle of equipresence*, which states that a quantity present as an independent variable in one constitutive relation should be present in all, except if its presence contradicts some law of physics or material symmetry [TN]. Severe restrictions result from the second law of thermodynamics and the invariance under change of observers, called respectively *principle of consistency with the Clausius-Duhem inequality* and *principle of material frame indifference.*

The list of constitutive variables (prime and dependent) does not include the reference density ρ_0, the body force f, and the radiating heat transfer r, which are viewed as externally prescribed fields. Given a constitutive theory, the kinematic compatibility relation, and the balance laws of momentum and energy form a system of equations whose solution determines the thermomechanical process. In the Lagrangean description, the role of the balance of mass is to determine the current density ρ, once the process is identified. The role of the Clausius-Duhem inequality is subtler: For *smooth processes*, the Clausius-Duhem inequality is viewed as restricting the form of constitutive relations. By contrast for *non-smooth processes*[1], it becomes an additional constraint that weak solutions must satisfy.

For smooth processes the balance of energy, balance of linear momentum and Clausius-Duhem inequality imply the energy dissipation inequality

$$\rho_0 \big(\partial_t e - \theta \partial_t \eta \big) - \tau u_t - \frac{q \theta_x}{\theta} \leq 0 \,. \tag{2.7}$$

Upon introducing the Helmholtz free energy $\psi = e - \theta \eta$, the latter takes the form

$$\rho_0 \partial_t \psi + \rho_0 \eta \partial_t \theta - \tau u_t - \frac{q \theta_x}{\theta} \leq 0 \,. \tag{2.8}$$

[1] The term non-smooth processes is used in a loose sense to signify processes containing shocks. It is a question of analysis to precise the smoothness class in each specific context.

2.b Materials with internal variables.

Viscosity and heat conduction are one of the possible ways for prescribing dissipative mechanisms. Complementary descriptions of dissipation are supplied by the theory of simple materials with fading memory, [Co], and the theory of materials with internal state variables, [CG]. The class of materials with internal state variables is a subclass of the simple materials with fading memory, which is appealing in its simplicity and encompasses some interesting models (like the ideal gas with vibrational relaxation).

For materials with internal variables, the thermomechanical process is described by a vector function $(y(x,t), \theta(x,t), \alpha(x,t))$, where y is the motion, θ the temperature, and the internal vector-variable α evolves according to the differential law

$$\partial_t \alpha = F(u, \theta, \alpha)\,. \tag{2.9}$$

In rough terms, such models have fading memory when the differential system (2.9) is exponentially dissipative.

The independent variables of the constitutive theory are u, θ, the internal variable vector α and the temperature gradient g. The remaining thermomechanical variables are determined by constitutive relations of the general form

$$\psi = \Psi(u,\theta,g,\alpha)\,, \quad \eta = H(u,\theta,g,\alpha)\,, \quad \tau = S(u,\theta,g,\alpha)\,, \quad q = Q(u,\theta,g,\alpha)\,. \tag{2.10}$$

Note that while (2.10) satisfies the principle of equipresence, the differential constraint (2.9) does not. In fact, (2.9) is not viewed here as a constitutive relation but rather as defining the class of *admissible* processes. This simplifies somewhat the reduction process, while it is compatible with all the specific examples considered later. We refer to [CG] for the analysis of the case that F also depends on g.

Consistency with the Clausius-Duhem inequality is tested against all admissible processes, that is all smooth processes that are compatible with the differential constraint (2.9). A count of equations and unknowns indicates that all admissible processes can be realized, by externally regulating f and r so as to fulfill the balance of momentum and energy. Then (2.8), (2.9) and (2.10) imply that

$$(\rho_0 \Psi_u - S)\dot{u} + \rho_0(\Psi_\theta + H)\dot{\theta} + \rho_0 \Psi_g\, \dot{g} + \rho_0 \Psi_\alpha \cdot F(u,\theta,\alpha) - \frac{Qg}{\theta} \le 0 \tag{2.11}$$

holds for all admissible processes. Since the local values of u, θ, α, g, θ_t, u_t and g_t can be assigned independently, the constitutive relations have the reduced form

$$
\begin{aligned}
\psi &= \Psi(u,\theta,\alpha) \\
\tau &= S = \rho_0 \frac{\partial \Psi}{\partial u} \\
\eta &= H = -\frac{\partial \Psi}{\partial \theta} \\
q &= Q(u,\theta,g,\alpha)
\end{aligned}
\tag{2.12}
$$

subject to the constraint

$$
-\frac{\partial \Psi}{\partial \alpha} \cdot F(u,\theta,\alpha) + \frac{1}{\theta} Q(u,\theta,g,\alpha) g \geq 0 \quad \text{for all } u,\theta,g,\alpha. \tag{2.13}
$$

It follows from (2.13) that

$$
-\frac{\partial \Psi}{\partial \alpha} \cdot F(u,\theta,\alpha) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.14}
$$

If Q is given by a Fourier law for heat conduction, $Q = k(u,\theta,\alpha)g$, then (2.13) is equivalent to asserting (2.14) and $k \geq 0$.

The thermomechanical process $(y(x,t), \theta(x,t), \alpha(x,t))$ is described by (2.2-2.4) supplemented with (2.9) and the constitutive relations (2.12-2.14). For Fourier heat conduction, this reads

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\partial_t(\rho_0 v) - \partial_x S(u,\theta,\alpha) &= \rho_0 f \\
\partial_t(\rho_0 \frac{1}{2} v^2 + \rho_0 E(u,\theta,\alpha)) - \partial_x (S(u,\theta,\alpha) v) &= \partial_x(k\theta_x) + \rho_0 f v + \rho_0 \\
\partial_t \alpha &= F(u,\theta,\alpha)
\end{aligned}
\tag{2.15}
$$

where $E = \Psi + \theta H$. For smooth processes, a direct computation yields the identity

$$
\rho_0 \partial_t H(u,\theta,\alpha) - \Big(\frac{k\theta_x}{\theta}\Big)_x = -\rho_0 \frac{1}{\theta} \Psi_\alpha \cdot F(u,\theta,\alpha) + \frac{k\theta_x^2}{\theta^2} + \frac{\rho_0 r}{\theta} \tag{2.16}
$$

which captures the dissipative structure of a heat conducting thermoelastic material with internal variables.

2.c The connection with relaxation approximations.

Theories with internal variables provide a natural framework to consider the structure of relaxation approximations to conservation laws, from the continuum thermomechanics perspective. Consider a theory with one scalar internal variable α evolving according to the differential law

$$
\partial_t \alpha = -\lambda(\alpha - h(u,\theta)), \tag{2.17}
$$

where $\lambda > 0$ is a parameter. The law is of exponential dissipative type with relaxation time $\frac{1}{\lambda}$ and the equilibrium states are described by $\alpha_{eq} = h(u, \theta)$. The internal variable theory is completed with constitutive relations for the free energy, stress and entropy, and (for simplicity) with a Fourier law for the heat flux,

$$\psi = \Psi(u, \theta, \alpha)\,, \quad \eta = H(u, \theta, \alpha)\,, \quad \tau = S(u, \theta, \alpha)\,, \quad Q = k(u, \theta, \alpha) g\,. \tag{2.18}$$

They are required to comply with (2.12-2.14), with $F = -\lambda(\alpha - h(u, \theta))$, so that the internal variable theory is consistent with the second law of thermodynamics. Accordingly, smooth processes satisfy the dissipation estimate (2.16) and the function $-H(u, \theta, \alpha)$ provides, in the terminology of [CLL], a (possibly not convex) "entropy" function for the emerging relaxation process. In the sequel, we explore the relations between the thermomechanical model corresponding to $\lambda > 0$ with the model emerging in the small-relaxation time limit $\lambda \to \infty$.

In the limit $\lambda \to \infty$, it is expected that the internal variable α will tend to its equilibrium value $\alpha_{eq} = h(u, \theta)$ and the corresponding constitutive relations, (2.18) with $\alpha = \alpha_{eq} = h(u, \theta)$, become the constitutive relations for thermoelastic conductors of heat. The correspondence between equilibrium and relaxation system has been extensively studied in the mechanics literature, *c.f.* [GWS, FM, Su] and references therein; the issue is also relevant in the design of relaxation schemes for the equations of gas dynamics [CP].

The general question is: Given a differential law like (2.17), derive conditions on the constitutive functions Ψ, S and H so that they are achieved from a theory of materials with internal variables consistent with the second law of thermodynamics. Here, we break down the question as follows. Assume that one of the constitutive variables of the internal variable theory is completely specified. The goal is to derive conditions on the functional form so that it is achieved from an internal variable theory consistent with the second law of thermodynamics, and also to derive the form of the remaining constitutive functions. (For instance, suppose that experimental measurements on one variable are available and ask whether these measurements fit under a theory with internal variables.) We pursue two such instances, prescribed distribution of stress and prescribed distribution of internal energy. Then, in Sections 2d and 2e, we derive constitutive relations for two special theories, pertaining to relaxation of stress and relaxation of internal energy respectively. The formal limit of both is the theory of thermoelastic conductors of heat and they are equipped with "entropy" functions for the relaxation process.

i. Prescribed distribution of stress

Suppose we are given a stress distribution $S(u, \theta, \alpha)$ and ask if the distribution can be achieved from a theory with internal variables. The question becomes to investigate if there is a free energy function $\Psi(u,\theta,\alpha)$ such that

$$\begin{aligned} &\frac{\partial \Psi}{\partial u} = \frac{1}{\rho_0} S(u,\theta,\alpha) \\ \text{subject to} \quad &\frac{\partial \Psi}{\partial \alpha}(\alpha - h(u,\theta)) \geq 0 \quad \text{for all } u,\theta,\alpha. \end{aligned} \tag{2.19}$$

Note that (2.19) implies in particular that Ψ satisfies

$$\begin{cases} \frac{\partial \Psi}{\partial \alpha} \geq 0 & \text{for } \alpha > h(u,\theta) \\ \frac{\partial \Psi}{\partial \alpha} = 0 & \text{for } \alpha = \alpha_{eq} = h(u,\theta) \\ \frac{\partial \Psi}{\partial \alpha} \leq 0 & \text{for } \alpha < h(u,\theta), \end{cases} \tag{2.20}$$

and that, since solutions of $(2.19)_1$ are given by

$$\rho_0 \Psi(u,\theta,\alpha) = G(\theta,\alpha) + \int_0^u S(\xi,\theta,\alpha)\, d\xi\ , \tag{2.21}$$

the inequality $(2.19)_2$ is satisfied if and only if there is a function $G(\theta,\alpha)$ such that

$$\Big(G_\alpha(\theta,\alpha) + \int_0^u S_\alpha(\xi,\theta,\alpha)\, d\xi\Big)\Big(\alpha - h(u,\theta)\Big) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.22}$$

We emphasize that solving (2.22) is equivalent to deciding whether the given model with internal variables is consistent with the second law of thermodynamics, and that, for (2.22) to admit solutions, conditions must be imposed on the functions S and h. For instance, (2.20) implies the necessary condition

$$G_\alpha(\theta, h(u,\theta)) = -\int_0^u S_\alpha(\xi,\theta,h(u,\theta))\, d\xi \tag{2.23}$$

If a solution G of (2.22) can be found, the associated free energy function is given by (2.22). Note that the free energy is only achieved within adding an arbitrary function of θ. This is not surprising; the requirement of achieving a prescribed distribution of stress does not constraint the purely thermal aspects of the constitutive theory even at equilibrium.

ii. Prescribed distribution of internal energy

Consider next the case that the internal energy distribution $E(u,\theta,\alpha)$ is prescribed. Then we ask whether there is a free energy function $\Psi(u,\theta,\alpha)$ such that

$$\begin{aligned} &\Psi - \theta\frac{\partial \Psi}{\partial \theta} = E(u,\theta,\alpha) \\ \text{subject to} \quad &\frac{\partial \Psi}{\partial \alpha}(\alpha - h(u,\theta) \geq 0 \quad \text{for all } u,\theta,\alpha. \end{aligned} \tag{2.24}$$

The solution of $(2.24)_1$ is calculated by

$$\Psi(u,\theta,\alpha) = \theta F(u,\alpha) - \theta \int_1^\theta \frac{1}{\zeta^2} E(u,\zeta,\alpha)\, d\zeta\,, \tag{2.25}$$

where F is an arbitrary function. Then inequality $(2.24)_2$ is satisfied if and only if there is a function $F(u,\alpha)$ such that

$$\theta\Big(F_\alpha(u,\alpha) - \int_1^\theta \frac{1}{\zeta^2} E_\alpha(u,\zeta,\alpha)\, d\zeta\Big)\Big(\alpha - h(u,\theta)\Big) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.26}$$

Again solving (2.26) is equivalent to deciding whether the given model with internal variables is consistent with the second law of thermodynamics, and (2.26) implies the necessary condition

$$F_\alpha(u,\alpha) = \int_1^\theta \frac{1}{\zeta^2} E_\alpha(u,\zeta,\alpha)\, d\zeta \qquad \text{for } \alpha = \alpha_{eq} = h(u,\theta) \tag{2.27}$$

Once again, if a solution F of (2.26) can be found the free energy function is given by (2.25), and the procedure will select F within adding an arbitrary function of u.

2.d A hierarchy of models with stress relaxation.

Consider next a special case, where the given stress distribution is

$$S(u,\theta,\alpha) = f(u,\theta) + \alpha\,. \tag{2.28}$$

This case is completely solvable. Indeed, (2.22) becomes to find a function $G(\theta,\alpha)$ such that $j(\theta,\alpha) := -G_\alpha(\theta,\alpha)$ satisfies

$$\Big(u - j(\theta,\alpha)\Big)\Big(\alpha - h(u,\theta)\Big) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.29}$$

Lemma 2.1. *Given $h(u,\theta)$, (2.29) is satisfied if and only if $h(u,\theta)$ is strictly decreasing in u, $j(\theta,\alpha)$ is strictly decreasing in α, and $j = h^{-1}$ is the inverse function of h for θ fixed,*

$$j(\theta, h(u,\theta)) = u\,, \quad h(j(\theta,\alpha),\theta) = \alpha\,. \tag{2.30}$$

Proof. Suppose (2.29) is satisfied and let θ be fixed. If the graphs of the functions $\alpha = h(u,\theta)$ and $u = j(\theta,\alpha)$ do not coincide in the $\alpha - u$ plane then (2.29) is clearly violated. Since each function describes the same graph viewed from a different axis, either h is strictly decreasing in u or h is strictly increasing in u. Checking directly we see that if h is strictly increasing in u, then (2.29) is violated. We conclude that h is strictly decreasing in u, and j is the inverse function of h and is strictly decreasing in α.

Conversely, if h and j are as in the statement of the lemma, then

$$(u - j(\theta,\alpha))(\alpha - h(u,\theta)) = (j(\theta, h(u,\theta)) - j(\theta,\alpha))(\alpha - h(u,\theta)) \geq 0\,,$$

since j is decreasing in α. □

In the sequel, we assume the slightly stronger condition $h_u(u,\theta) < 0$ and note that the associated G is given by the formula

$$G(\theta,\alpha) = -\int_0^\alpha j(\theta,\zeta)d\zeta - \int_1^\theta s(z)\,dz\,,$$

where s is an arbitrary function of θ. The constitutive functions of the internal variable theory read

$$\begin{aligned} \rho_0\psi &= \rho_0\Psi(u,\theta,\alpha) = -\int_0^\alpha j(\theta,\zeta)d\zeta - \int_1^\theta s(z)\,dz + \alpha u + \int_0^u f(\xi,\theta)d\xi \\ \tau &= S(u,\theta,\alpha) = f(u,\theta) + \alpha \\ \rho_0\eta &= \rho_0 H(u,\theta,\alpha) = \int_0^\alpha j_\theta(\theta,\zeta)d\zeta + s(\theta) - \int_0^u f_\theta(\xi,\theta)d\xi \end{aligned} \tag{2.31}$$

In turn, the internal energy is determined by

$$\begin{aligned} \rho_0 e &= \rho_0 E(u,\theta,\alpha) = \rho_0(\Psi + \theta H)(u,\theta,\alpha) \\ &= \int_0^\alpha \big(\theta j_\theta - j\big)(\theta,\zeta)d\zeta + \Big(\theta s(\theta) - \int_1^\theta s(z)\,dz\Big) + \alpha u + \int_0^u \big(f - \theta f_\theta\big)(\xi,\theta)d\xi\,. \end{aligned} \tag{2.32}$$

As an application, we consider a model for a viscoelastic material where the total stress τ is decomposed into a viscoelastic part, evolving according to stress relaxation, and a viscous part with Newtonian viscosity,

$$\begin{aligned} \tau &= \sigma + \mu v_x\,, \quad \mu \geq 0 \\ \partial_t(\sigma - f(u,\theta)) &= -\lambda(\sigma - g(u,\theta))\,. \end{aligned} \tag{2.33}$$

The viscoelastic part of the stress may be put into the integral form,

$$\sigma(\cdot,t) = f(u,\theta)(\cdot,t) + \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\big(g(u,\theta) - f(u,\theta)\big)(\cdot,s)\,ds\,, \tag{2.34}$$

of a Maxwell type viscoelastic fluid with memory. The function $f(u,\theta)$ describes the instantaneous elastic stress-strain response, while $g(u,\theta)$ describes the equilibrium stress-strain response.

The inviscid version of (2.33) is formulated in the context of internal variables by setting

$$\begin{aligned} \sigma &= f(u,\theta) + \alpha \\ \partial_t\alpha &= -\lambda(\alpha - h(u,\theta)) \qquad \text{with } h(u,\theta) := g(u,\theta) - f(u,\theta), \end{aligned} \tag{2.35}$$

which fits under the previously discussed framework. The model is achieved from a theory consistent with the second law of thermodynamics if and only if the functions f and g satisfy $(g-f)(u,\theta)$ is strictly decreasing in u. Henceforth, we focus on functions satisfying

$$g_u(u,\theta) < f_u(u,\theta) \tag{2.36}$$

while the free energy ψ and entropy η are determined by (2.31) for $\alpha = \sigma - f(u,\theta)$.

The thermomechanical process $\big(y(x,t), \theta(x,t), \sigma(x,t)\big)$, associated to the material model (2.33), is described by the system of equations

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \rho_0 \partial_t v - \partial_x \sigma &= (\mu v_x)_x + \rho_0 f \\ \partial_t \Big(\frac{1}{2} \rho_0 v^2 + \rho_0 e \Big) - \partial_x (\sigma v) &= (\mu v_x v)_x + (k\theta_x)_x + \rho_0 f v + \rho_0 r \\ \partial_t \big(\sigma - f(u,\theta) \big) &= -\lambda (\sigma - g(u,\theta)) \end{aligned} \tag{2.37}$$

where the internal energy is determined by (2.32),

$$\rho_0 e = \rho_0 E(u,\theta,\sigma - f(u,\theta)) = \rho_0 \big(\Psi + \theta H \big)(u,\theta,\sigma - f(u,\theta)) \,. \tag{2.38}$$

A direct computation using (2.37), in conjunction with (2.17), (2.28) and (2.31), shows that the thermomechanical process is equipped with the dissipation estimate

$$\begin{aligned} \rho_0 \partial_t \Big(H(u,\theta,\sigma - f(u,\theta)) \Big) - \Big(\frac{k\theta_x}{\theta} \Big)_x =& \lambda \frac{1}{\theta} (u - h^{-1}(\theta,\alpha))(\alpha - h(u,\theta)) \Big|_{\alpha = \sigma - f(u,\theta)} \\ & + \frac{k\theta_x^2}{\theta^2} + \frac{\mu v_x^2}{\theta} + \frac{\rho_0 r}{\theta} \,, \end{aligned} \tag{2.39}$$

which, in view of (2.36) and (2.29), implies that smooth processes satisfy the Clausius-Duhem inequality, for all values of $\lambda > 0$ and μ, $k \geq 0$, and yields an estimate for the amount of dissipation.

The stress relaxation model (2.37) gives rise to a hierarchy of thermomechanical theories as the parameters describing the viscosity μ and heat-conductivity k tend to zero, and to a second hierarchy of theories as the relaxation parameter λ tends to infinity. In the limit $\lambda \to \infty$, one formally obtains the theory of thermoviscoelasticity. As both $\lambda \to \infty$ and μ and/or k tend to zero one can obtain a variety of thermomechanical theories. Along all these limiting processes (2.39) holds, and any limiting non-smooth processes inherit the limiting form of the entropy dissipation estimate (2.39).

We close by considering the case of isothermal motions, that is processes along which $\theta = \theta_0$ is kept constant and $Q = 0$. The process is described now by equations,

$$
\begin{aligned}
&\partial_t u - \partial_x v = 0 \\
&\rho_0 \partial_t v - \partial_x \sigma = (\mu v_x)_x + \rho_0 f \qquad (2.40)\\
&\partial_t\big(\sigma - f(u,\theta_0)\big) = -\lambda(\sigma - g(u,\theta_0))\,,
\end{aligned}
$$

that are pertinent to a purely mechanical process. From a mechanical viewpoint, isothermal processes are attained by externally controlling the radiation heat transfer r so that $\theta = \theta_0$ and $Q = 0$. The balance of energy and the entropy production equations imply

$$
\begin{aligned}
&\partial_t\Big(\frac{1}{2}\rho_0 v^2 + \rho_0 \Psi(u,\theta_0,\sigma - f(u,\theta_0))\Big) - \partial_x(\sigma v) \\
&\quad + \mu v_x^2 + \lambda(u - h^{-1}(\theta_0,\alpha))(\alpha - h(u,\theta_0))\Big|_{\alpha = \sigma - f(u,\theta_0)} = (\mu v_x v)_x + \rho_0 f v
\end{aligned}
\qquad (2.41)
$$

The theory emerging in the zero-viscosity $\mu \to 0$ is described by

$$
\begin{aligned}
&\partial_t u - \partial_x v = 0 \\
&\rho_0 \partial_t v - \partial_x \sigma = \rho_0 f \qquad (2.42)\\
&\partial_t\big(\sigma - f(u,\theta_0)\big) = -\lambda(\sigma - g(u,\theta_0))\,.
\end{aligned}
$$

It is known that stress relaxation equation exerts a subtle dissipative effect on smooth processes, and as a result the system admits smooth solutions for initial data close to equilibrium. By contrast, for data away from equilibrium shock waves can develop in finite time, *e.g* [Da$_1$]. The inviscid theory inherits the dissipative structure

$$
\begin{aligned}
&\partial_t\Big(\frac{1}{2}\rho_0 v^2 + \rho_0 \Psi(u,\theta_0,\sigma - f(u,\theta_0))\Big) - \partial_x(\sigma v) \\
&\quad + \lambda(u - h^{-1}(\theta_0,\alpha))(\alpha - h(u,\theta_0))\Big|_{\alpha = \sigma - f(u,\theta_0)} \le \rho_0 f v
\end{aligned}
\qquad (2.43)
$$

with equality for the case of smooth isothermal processes.

In the limit $\lambda \to \infty$, the internal variable theory (2.42) yields the equations of one-dimensional isothermal elasticity,

$$
\begin{aligned}
&\partial_t u - \partial_x v = 0 \\
&\rho_0 \partial_t v - \partial_x g(u,\theta_0) = \rho_0 f\,,
\end{aligned}
\qquad (2.44)
$$

a strictly hyperbolic system when $g_u > 0$. If $f_u > g_u$ the internal variable theory is consistent with the Clausius-Duhem inequality. (It is remarkable that this is precisely the subcharacteristic condition for the associated relaxation process, *i.e* consistency with the second law of thermodynamics implies, in this context, the subcharacteristic condition). The function

$$
\rho_0 \Psi(u,\theta_0,\alpha) = -\int_0^\alpha h^{-1}(\theta_0,\zeta)\,d\zeta + \alpha u + \int_0^u f(\xi,\theta_0)\,d\xi \qquad (2.45)
$$

provides an "entropy" function for the associated relaxation process, which is convex in (u,α) if $-\partial_\alpha h^{-1}\partial_u f \ge 1$ for all u and α.

2.e A model with relaxation of internal energy.

Next, we consider a model of relaxation of internal energy. The total internal energy is assumed to be decomposed into two parts

$$\begin{aligned} E(u,\theta,\alpha) &= \mathcal{E}(u,\theta) + \alpha \\ \partial_t \alpha &= -\lambda(\alpha - h(u,\theta))\,. \end{aligned} \tag{2.46}$$

The part $\mathcal{E}(u,\theta)$ is called active-mode energy while α is called vibrational energy. In this model the variable α is an internal variable and thus $\alpha > 0$. It evolves according to the differential law (2.46), which is of exponentially dissipative type with relaxation time $\frac{1}{\lambda}$ and equilibria $\alpha_{eq} = h(u,\theta)$. (See Coquel-Perthame [CP] for a model with analogous features).

From part (ii) of Section 2.c, we see that (2.46) can be achieved by an internal variable theory compatible with the Clausius-Duhem inequality if and only if there is a function $F(u,\alpha)$ such that $1 - F_\alpha(u,\alpha) =: K(u,\alpha)$ satisfies

$$\Big(\frac{1}{\theta} - K(u,\alpha)\Big)\Big(\alpha - h(u,\theta)\Big) \geq 0 \quad \text{for all } u,\theta,\alpha. \tag{2.47}$$

We set $\beta = \frac{1}{\theta}$, $H(u,\frac{1}{\theta}) = h(u,\theta)$ and recast (2.47) to the form

$$\Big(\beta - K(u,\alpha)\Big)\Big(\alpha - H(u,\beta)\Big) \geq 0 \quad \text{for all } u,\alpha,\beta. \tag{2.48}$$

Lemma 2.1 implies that (2.48) is solvable iff H is strictly decreasing as a function of β, or equivalently h is strictly increasing as a function of θ.

In the sequel, we assume the condition $h_\theta(u,\theta) > 0$ and define the function $K(u,\alpha)$ as the inverse function of H for u fixed, or equivalently by

$$K(u,h(u,\theta)) = \frac{1}{\theta} \tag{2.49}$$

Then (2.47) is satisfied and F is given by

$$F(u,\alpha) = \alpha - \int_1^\alpha K(u,z)dz - \phi(u) \tag{2.50}$$

where $\phi(u)$ is an arbitrary function.

The constitutive functions of the internal variable theory are

$$\begin{aligned} \psi = \Psi(u,\theta,\alpha) &= -\theta\int_1^\alpha K(u,z)dz - \theta\phi(u) - \theta\int_1^\theta \frac{1}{\zeta^2}\mathcal{E}(u,\zeta)d\zeta + \alpha \\ \frac{1}{\rho_0}\tau = \frac{1}{\rho_0}S(u,\theta,\alpha) &= -\theta\int_1^\alpha K_u(u,z)dz - \theta\phi_u(u) - \theta\int_1^\theta \frac{1}{\zeta^2}\mathcal{E}_u(u,\zeta)d\zeta \\ \eta = H(u,\theta,\alpha) &= \int_1^\alpha K(u,z)dz + \phi(u) + \int_1^\theta \frac{1}{\zeta^2}\mathcal{E}(u,\zeta)d\zeta + \frac{1}{\theta}\mathcal{E}(u,\theta) \\ e = E(u,\theta,\alpha) &= \mathcal{E}(u,\theta) + \alpha \end{aligned} \tag{2.51}$$

while the equations describing the relaxation process are

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\rho_0 \partial_t v - \partial_x S(u,\theta,\alpha) &= \rho_0 f \\
\partial_t \Big(\frac{1}{2}\rho_0 v^2 + \rho_0 (\mathcal{E}(u,\theta) + \alpha) \Big) - \partial_x (S(u,\theta,\alpha) v) &= (k\theta_x)_x + \rho_0 f v + \rho_0 r \\
\partial_t \alpha &= -\lambda(\alpha - h(u,\theta))
\end{aligned}
\tag{2.52}
$$

When $h_\theta(u,\theta) > 0$ this model is consistent with the Clausius-Duhem inequality and, by (2.16), it satisfies the dissipation estimate

$$
\rho_0 \partial_t H(u,\theta,\alpha) - \Big(\frac{k\theta_x}{\theta}\Big)_x = \rho_0 \lambda \Big(\frac{1}{\theta} - K(u,\alpha)\Big)\Big(\alpha - h(u,\theta)\Big) + \frac{k\theta_x^2}{\theta^2} + \frac{\rho_0 r}{\theta}\,. \tag{2.53}
$$

3. Relaxation of a viscoelastic model to the equations of isothermal elastodynamics

In this section, we address the problem of constructing weak solutions of the equations of isothermal elasticity with $g_u > 0$,

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\partial_t v - \partial_x g(u) &= 0\,,
\end{aligned}
\tag{3.1}
$$

as $\varepsilon \to 0$ limits of the relaxation system

$$
\begin{aligned}
\partial_t u - \partial_x v &= 0 \\
\partial_t v - \partial_x \sigma &= 0 \\
\partial_t (\sigma - Eu) &= -\frac{1}{\varepsilon}(\sigma - g(u))\,.
\end{aligned}
\tag{3.2}
$$

The model (3.2) is suggested as an approximating model for the equations of isothermal elastodynamics in [FM] and is a semilinear version of the isothermal viscoelasticity model (2.42).

We work under the standing hypotheses $g(0) = 0$ and $0 < g_u < E$, in which case (3.2) admits globally defined smooth solutions, if the initial data are smooth. The hypothesis $g_u < E$ can be motivated in two ways: First, it guarantees that the internal variable theory described by (3.2) is consistent with the Clausius-Duhem inequality (see Sec. 2). Second, it is motivated by the analog of the Champan-Enskog expansion for the relaxation process.

In the Chapman-Enskog expansion one seeks to identify the effective response of the relaxation process as it approaches the surface of local equilibria. It is postulated that the relaxing variable σ^ε can be described in an asymptotic expansion that involves *only* the local macroscopic values u^ε, v^ε and their derivatives, *i.e.*

$$\sigma^\varepsilon = g(u^\varepsilon) + \varepsilon S(u^\varepsilon, v^\varepsilon, u^\varepsilon_x, v^\varepsilon_x, \dots) + O(\varepsilon^2) \tag{3.3}$$

To calculate the form of S, we use (3.2),

$$\begin{aligned} \partial_t u^\varepsilon - \partial_x v^\varepsilon &= 0 \\ \partial_t v^\varepsilon - \partial_x g(u^\varepsilon) &= \varepsilon S_x + O(\varepsilon^2) \\ \partial_t (g(u^\varepsilon) - E u^\varepsilon) + O(\varepsilon) &= -S + O(\varepsilon)\,, \end{aligned} \tag{3.4}$$

whence we obtain

$$S = [E - g_u(u^\varepsilon)] v^\varepsilon_x + O(\varepsilon)\,, \tag{3.5}$$

and we conclude that the effective equations describing the process are

$$\begin{aligned} \partial_t u^\varepsilon - \partial_x v^\varepsilon &= 0 \\ \partial_t v^\varepsilon - \partial_x g(u^\varepsilon) &= \varepsilon \partial_x \big([E - g_u(u^\varepsilon)] v^\varepsilon_x \big) + O(\varepsilon^2)\,. \end{aligned} \tag{3.6}$$

This is a stable parabolic system provided the condition $g_u < E$ is satisfied.

According to Section 2.d, when $g_u < E$ the system (3.2) describes a theory with internal variables that is consistent with the second law of thermodynamics. Smooth solutions (u, v, σ) satisfy the energy dissipation identity

$$\partial_t \Big(\frac{1}{2} v^2 + \Psi(u, \sigma - Eu) \Big) - \partial_x(\sigma v) + \frac{1}{\varepsilon} (u - h^{-1}(\alpha))(\alpha - h(u)) \Big|_{\alpha = \sigma - Eu} = 0 \tag{3.7}$$

where

$$\Psi(u, \alpha) = - \int_0^\alpha h^{-1}(\zeta) d\zeta + \alpha u + \int_0^u E \xi d\xi\,, \tag{3.8}$$

$h(u) = g(u) - Eu$, $h(0) = 0$, and h^{-1} is the inverse function of h. The function Ψ provides an "entropy" function for the associated relaxation process, which is convex in (u, α) if $-\partial_\alpha h^{-1} \partial_u f \geq 1$ for all u and α.

Henceforth, we assume the initial data (u_0, v_0, σ_0) are smooth (of compact support or decaying fast at infinity) and the function $g(u) \in C^3$ satisfies

$$0 < \gamma \leq g_u(u) \leq \Gamma < E\,, \tag{h}$$

for some positive constants γ and Γ. It is easy to check that (3.2) admits global smooth solutions, and we proceed to study the $\varepsilon \to 0$ relaxation process. Equation (3.7) provides uniform stability in L^2 for the relaxation process.

Lemma 3.1. *Under hypothesis* (h),

$$\int_{\mathbb{R}}(u^2+v^2+\sigma^2)dx+\frac{1}{\varepsilon C}\int_0^t\int_{\mathbb{R}}\big(\sigma-g(u)\big)^2dxdt\le C\int_{\mathbb{R}}(u_0^2+v_0^2+\sigma_0^2)dx \tag{3.9}$$

for some C *independent of* ε *and* t.

Proof. From (3.8) we have

$$\begin{aligned}\Psi(u,\sigma-Eu)&=-\int_0^{\sigma-Eu}h^{-1}(\zeta)d\zeta+\frac{\sigma^2}{2E}-\frac{1}{2E}(\sigma-Eu)^2\\&=\int_0^{\sigma-Eu}\kappa(\zeta)d\zeta+\frac{\sigma^2}{2E}\end{aligned} \tag{3.10}$$

where $\kappa(\alpha)=-\frac{1}{E}\alpha-h^{-1}(\alpha)$. Hypothesis (h) implies

$$\frac{\gamma}{E(E-\gamma)}\le\frac{d\kappa}{d\alpha}=\frac{g_u}{E(E-g_u)}\le\frac{\Gamma}{E(E-\Gamma)}$$

and thus there is a constant C, depending only on γ, Γ and E, so that

$$\frac{1}{C}\big((\sigma-Eu)^2+\sigma^2\big)\le\Psi(u,\sigma-Eu)\le C\big((\sigma-Eu)^2+\sigma^2\big) \tag{3.11}$$

Furthermore, since $-\frac{d}{d\alpha}h^{-1}(\alpha)=\frac{1}{E-g_u}\ge\frac{1}{E}$, we have

$$(u-h^{-1}(\alpha))(\alpha-h(u))\ge\frac{1}{E}(\alpha-h(u))^2 \tag{3.12}$$

The result now follows from (3.7), upon using (3.11) and (3.12). □

The following lemma indicates that, under (h), the dissipative strength of the present relaxation process is comparable to that of viscosity approximations. In preparation, note that solutions of (3.12) satisfy

$$\begin{aligned}&\partial_t u-\partial_x v=0\\&\partial_t v-\partial_x g(u)=\partial_x(\sigma-g(u))=\varepsilon(Ev_{xx}-v_{tt})\end{aligned} \tag{3.13}$$

i.e. (3.2) is equivalent to approximation of (3.1) via a wave equation.

Lemma 3.2. *Suppose the initial data satisfy*

$$\begin{aligned}\int_{\mathbb{R}}v_0^2+u_0^2+\sigma_0^2dx&\le O(1)\,,\\ \varepsilon^2\int_{\mathbb{R}}u_{0x}^2+v_{0x}^2+\sigma_{0x}^2dx&\le O(1)\,.\end{aligned} \tag{a}$$

Under hypothesis (h), *solutions* (u,v,σ) *of (3.2) satisfy the* ε *independent estimates*

$$\varepsilon\int_0^t\int_{\mathbb{R}}u_x^2+v_x^2+\sigma_x^2dxdt\le O(1)\,. \tag{3.14}$$

Proof. We multiply $(3.13)_1$ by $g(u)$ and $(3.13)_2$ by v. Adding and rearranging the terms we obtain the energy identity

$$\partial_t\big(\frac{1}{2}v^2 + W(u) + \varepsilon v v_t\big) - \partial_x\big(vg(u)\big) + \varepsilon(Ev_x^2 - v_t^2) = \varepsilon\partial_x(Evv_x)\,, \tag{3.15}$$

where the stored energy function $W(u)$ is given by

$$W(u) = \int_0^u g(\xi)d\xi\,. \tag{3.16}$$

The problem is that the term $Ev_x^2 - v_t^2$ is not positive definite. To compensate for that, we first multiply $(3.13)_2$ by v_t to obtain

$$v_t^2 - g_u u_x v_t = \varepsilon\Big[(Ev_tv_x)_x - \partial_t\big(\frac{1}{2}Ev_x^2 + \frac{1}{2}v_t^2\big)\Big]$$

and, in turn

$$\varepsilon^2\partial_t\big(Ev_x^2 + v_t^2\big) + \varepsilon(2v_t^2 - 2g_uu_xv_t) = 2\varepsilon^2\partial_x(Ev_tv_x)\,. \tag{3.17}$$

Using once again $(3.13)_2$ and the identity $a_xb_t - a_tb_x = \partial_t(a_xb) - \partial_x(a_tb)$, we have

$$\begin{aligned} g_uu_x^2 &= u_x\partial_t(v+\varepsilon v_t) - \varepsilon Eu_xv_{xx} \\ &= \Big[u_t\partial_x(v+\varepsilon v_t) + \partial_t\big(u_x(v+\varepsilon v_t)\big) - \partial_x\big(u_t(v+\varepsilon v_t)\big)\Big] - \varepsilon\partial_t\big(\frac{1}{2}Eu_x^2\big)\,, \end{aligned}$$

which in turn yields

$$\varepsilon^2\partial_t\big(\frac{1}{2}E^2u_x^2 - \frac{1}{2}Ev_x^2\big) - \varepsilon\partial_t\big(Eu_x(v+\varepsilon v_t)\big) + \varepsilon(Eg_uu_x^2 - Ev_x^2) = -\varepsilon\partial_x\big(Eu_t(v+\varepsilon v_t)\big)\,. \tag{3.18}$$

Adding (3.15), (3.17) and (3.18), we arrive at

$$\begin{aligned} &\partial_t\Big(\frac{1}{2}(v+\varepsilon v_t - \varepsilon Eu_x)^2 + \frac{1}{2}\varepsilon^2(v_t^2 + Ev_x^2) + W(u)\Big) - \partial_x(vg(u)) \\ &\qquad + \varepsilon\big[v_t^2 - 2g_uu_xv_t + Eg_uu_x^2\big] = \varepsilon^2(Ev_tv_x)_x \end{aligned} \tag{3.19}$$

Under (h), the third term in (3.19) is positive definite

$$\varepsilon\big[v_t^2 - 2g_uu_xv_t + Eg_uu_x^2\big] \geq \varepsilon g_u(E-g_u)u_x^2 \geq 0\,. \tag{3.20}$$

Therefore, we conclude

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2}(v+\varepsilon v_t - \varepsilon Eu_x)^2 + \frac{1}{2}\varepsilon^2(v_t^2 + Ev_x^2) + W(u)dx \\ &\qquad + \varepsilon\int_0^t\int_{\mathbb{R}} g_u(E-g_u)u_x^2dxdt \\ &\qquad \leq \int_{\mathbb{R}} \frac{1}{2}(v_0 + \varepsilon\sigma_{0x} - \varepsilon Eu_{0x})^2 + \frac{1}{2}\varepsilon^2(\sigma_{0x}^2 + Ev_{0x}^2) + W(u_0)dx \leq O(1) \end{aligned} \tag{3.21}$$

and, due to (h) and (a),

$$\varepsilon \int_0^t \int_{\mathbb{R}} g_u(E - g_u) u_x^2 \, dx dt \leq O(1)$$

In turn, (3.17) and (3.15) imply

$$\varepsilon \int_0^t \int_{\mathbb{R}} \sigma_x^2 \, dx dt \leq O(1)$$
$$\varepsilon \int_0^t \int_{\mathbb{R}} v_x^2 \, dx dt \leq O(1)$$

and (3.14) follows. □

We come next to the convergence Theorem.

Theorem 3.3. *Let $g \in C^3$ satisfy the subcharacteristic condition (h) and*

$$g''(u_0) = 0 \text{ and } g''(u) \neq 0 \text{ for } u \neq u_0, \tag{h$_1$}$$

$$g'', g''' \in L^2 \cap L^\infty . \tag{h$_2$}$$

Let $(u^\varepsilon, v^\varepsilon, \sigma^\varepsilon)$ be a family of smooth solutions of (3.2) on $I\!R \times [0,T]$ emanating from smooth initial data subject to the bounds (a). Then, along a subsequence if necessary,

$$u^\varepsilon \to u \,, \quad v^\varepsilon \to v \,, \quad \text{a.e. } (x,t) \text{ and in } L^p_{loc}(I\!R \times (0,T)), \text{ for } p < 2 \,, \tag{3.22}$$

and (u,v) is a weak solution of (3.1).

Proof. Let $\eta(u,v)$, $q(u,v)$ be an entropy pair for the equations of isothermal elasticity. Using (3.13) we obtain

$$\begin{aligned} \partial_t \eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon) &= \eta_v \partial_x(\sigma - g(u)) \\ &= \partial_x(\eta_v(\sigma - g(u))) - (\eta_{vu}\varepsilon^{\frac12} u_x + \eta_{vv}\varepsilon^{\frac12} v_x)\frac{\sigma - g(u)}{\varepsilon^{\frac12}} \\ &= I_1 + I_2 \end{aligned} \tag{3.23}$$

If the approximating solutions satisfy the uniform L^∞ bound,

$$|u^\varepsilon| + |v^\varepsilon| \leq C \,, \tag{H}$$

then, by (3.9) and (3.14), the term I_1 lies in a compact of H^{-1}, the term I_2 is uniformly bounded in L^1, and the sum $I_1 + I_2$ is uniformly bounded in $W^{-1,\infty}$. One concludes using the framework of compensated compactness, Tartar [Ta], and the lemma of Murat [M]

$$\partial_t \eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon) \quad \text{lies in a compact of } H^{-1}_{loc}. \tag{3.24}$$

Then, from DiPerna [Dp], we obtain, along a subsequence, $u^\varepsilon \to u$ and $v^\varepsilon \to v$ a.e. (x,t).

In the present case estimate (H) is not available and the natural stability framework is in L^2,

$$\int_{\mathbf{R}} \frac{1}{2}v^2 + W(u)dx + \varepsilon \int_0^t \int_{\mathbf{R}} g_u u_x^2 + v_x^2 dx d\tau \le O(1)\,. \tag{3.25}$$

Convergence of viscosity approximations to the equations of elastodynamics in the natural energy framework are carried out in Lin [Ln], Shearer [Sh] (for full viscosity matrices) and Serre-Shearer [SSh] (for the physical viscosity matrix). In [Sh, SSh] two classes of entropies with controlled growth at infinity are constructed and used to show that the support of the (generalized) Young measure is a point mass. We conclude the convergence proof using the results of these works. Under (3.9), (3.25) and (h) one easily proves (3.24) for entropy pairs η, q satisfying the growth restrictions

$$\eta,\;\; q,\;\; \eta_v,\;\; \eta_{vv},\;\; \eta_{vu}/g_u^{\frac{1}{2}} \in L^\infty$$

It follows from [Sh, Lemmas 2, 3] and [SSh, Lemma 3, Sec 5] that the above class contains sufficient entropies to allow the reduction of the generalized Young measures to point masses and show strong convergence in L^p_{loc} for $p < 2$. The hypotheses $(h_1 - h_2)$ reflect the assumptions of the main Theorem in [SSh]. $\square$

Remark. It is of interest to consider whether the solution (u,v) of (3.1) in Theorem 3.3 satisfies the energy dissipation inequality

$$\partial_t\big(\frac{1}{2}v^2 + W(u)\big) - \partial_x\big(g(u)v\big) \le 0\,, \qquad \text{in } \mathcal{D}'. \tag{3.26}$$

To this end note that, if

$$u^\varepsilon \to u\,, \quad v^\varepsilon \to v \quad \text{in } L^2_{loc}(\mathbb{R}\times(0,T))\,, \tag{3.27}$$

and the data satisfy

$$\text{(b)} \qquad u_0^\varepsilon \to u_0\,,\; v_0^\varepsilon \to v_0 \;\text{ in } L^2\,, \quad \varepsilon^2(u_{0x}^\varepsilon)^2 + \varepsilon^2(v_{0x}^\varepsilon)^2 + \varepsilon^2(\sigma_{0x}^\varepsilon)^2 \to 0 \;\text{ in } L^1\,, \;\text{ as } \varepsilon \to 0,$$

then (3.19) yields, for φ a positive test function with compact support in $[0,T)\times\mathbb{R}$,

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}} &\varphi_t\big[\frac{1}{2}(v^\varepsilon + \varepsilon v_t^\varepsilon - \varepsilon E u_x^\varepsilon)^2 + \frac{1}{2}\varepsilon^2({v_t^\varepsilon}^2 + E{v_x^\varepsilon}^2) + W(u^\varepsilon)\big] - \varphi_x(v^\varepsilon g(u^\varepsilon))\,dxdt \\
&+ \varepsilon \int_0^T \int_{\mathbb{R}} \varphi\big[{v_t^\varepsilon}^2 - 2g_u u_x^\varepsilon v_t^\varepsilon + E g_u {u_x^\varepsilon}^2\big]\,dxdt \\
&- \int_{\mathbb{R}} \varphi(x,0)\big[\frac{1}{2}(v_0^\varepsilon + \varepsilon\sigma_{0x}^\varepsilon - \varepsilon E u_{0x}^\varepsilon)^2 + \frac{1}{2}\varepsilon^2({\sigma_{0x}^\varepsilon}^2 + E{v_{0x}^\varepsilon}^2) + W(u_0^\varepsilon)\big]\,dx \\
&= -\varepsilon^2 \int_0^T \int_{\mathbb{R}} \varphi_x(E v_t^\varepsilon v_x^\varepsilon)\,dxdt\,.
\end{aligned}$$

In turn, (3.22), (3.27), (a) and (b) imply

$$-\int_0^T\int_{\mathbb{R}} \varphi_t\big[\frac{1}{2}v^2 + W(u)\big] - \varphi_x v g(u)\,dxdt \le \int_{\mathbb{R}} \varphi(x,0)\big[\frac{1}{2}v_0^2 + W(u_0)\big]\,dx\,,$$

that is (3.26) is derived. However, alluding to (3.22), we see that (3.27) is not validated within the framework of Theorem 3.3; that is concentrations in the L^2 norm cannot be excluded by the L^2 theory of approximate solutions for (3.1). That obstructs the derivation of (3.26).

4. Relaxation to symmetric hyperbolic systems

Consider the symmetric hyperbolic system

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = 0\,, \quad x \in \mathbb{R}^d,\ t > 0, \tag{4.1}$$

where $u(x,t)$ takes values in $\mathbb{R}^n$ and $F_j'(u) = \nabla F_j(u)$ are symmetric $n\times n$ matrices for $j = 1,\dots,d$. Symmetric hyperbolic systems admit the special entropy entropy-flux pairs, Lax [La],

$$\mathcal{H}(u) = \frac{1}{2}\,|u|^2 \qquad \mathcal{Q}_j(u) = u\cdot F_j(u) - g_j(u)\,, \tag{4.2}$$

where g_j is a potential for F_j satisfying $F_j(u) = g_j'(u)$.

i. Relaxation in one space dimension

We take up first the one-dimensional case,

$$\partial_t u + \partial_x F(u) = 0\,, \tag{4.3}$$

and consider the semilinear relaxation approximation for (4.1), suggested in Jin-Xin [JX] for constructing relaxing schemes for conservation laws,

$$\begin{aligned} &\partial_t u + \partial_x v = 0 \\ &\partial_t v + A\partial_x u = -\frac{1}{\varepsilon}(v - F(u)) \end{aligned} \tag{4.4}$$

where A is a positive definite symmetric $n\times n$ matrix. This approximation can be written in the form of regularization by wave equations

$$\partial_t u + \partial_x F(u) = \varepsilon(Au_{xx} - u_{tt}) \tag{4.5}$$

We prove that, under the subcharacteristic condition

$$A - F'(u)^2 \ge \nu I \quad \text{for some } \nu > 0, \tag{S}$$

the relaxation system (4.4) (or (4.5)) satisfies a strong dissipation estimate, similar to the one induced by viscosity approximations. This follows from two estimates: First, taking the inner product of (4.5) with u we obtain

$$\partial_t\Big(\frac{1}{2}|u|^2 + \varepsilon u \cdot u_t\Big) + \partial_x \mathcal{Q}(u) + \varepsilon\Big(u_x \cdot A u_x - |u_t|^2\Big) = \varepsilon \partial_x (u \cdot A u_x)\,, \tag{4.6}$$

where $\mathcal{Q}(u) = u \cdot F(u) - g(u)$ with $F(u) = g'(u)$. Second, taking the inner product with u_t we obtain, after rearranging the terms,

$$\partial_t\Big(\frac{1}{2}\varepsilon|u_t|^2 + \frac{1}{2}\varepsilon u_x \cdot A u_x\Big) + \Big(|u_t|^2 + u_t \cdot F'(u) u_x\Big) = \partial_x(\varepsilon u_t \cdot A u_x) \tag{4.7}$$

Combining (4.6) with (4.7) and using the symmetry of $F'(u)$ and A, we deduce

$$\begin{aligned} &\partial_t\Big(\frac{1}{2}|u+\varepsilon u_t|^2 + \frac{1}{2}\varepsilon^2|u_t|^2 + \varepsilon^2 u_x \cdot A u_x\Big) + \partial_x \mathcal{Q}(u) + \varepsilon|u_t + F'(u)u_x|^2 \\ &\qquad + \varepsilon u_x \cdot \big(A - F'(u)^2\big) u_x = \partial_x\big(\varepsilon u \cdot A u_x + 2\varepsilon^2 u_t \cdot A u_x\big) \end{aligned} \tag{4.8}$$

In summary:

Proposition 4.1. *Under Hypothesis (S), smooth solutions of (4.4) (or (4.5)), that decay fast at infinity, satisfy*

$$\begin{aligned} &\int_{\mathbf{R}} \frac{1}{2}|u+\varepsilon u_t|^2 + \frac{1}{2}\varepsilon^2|u_t|^2 + \varepsilon^2 u_x \cdot A u_x \, dx + \int_0^t \int_{\mathbf{R}} \varepsilon|u_t + F(u)_x|^2 + \varepsilon\nu|u_x|^2 dx d\tau \\ &\qquad \le \int_{\mathbf{R}} \frac{1}{2}|u_0 - \varepsilon v_{0x}|^2 + \frac{1}{2}\varepsilon^2|v_{0x}|^2 + \varepsilon^2 u_{0x} \cdot A u_{0x} \, dx \end{aligned} \tag{4.9}$$

Remarks.

a/. Condition (S) and a condition that is equivalent to (S_d), below, are derived in [JX] by calculating an effective equation for the relaxation process using the formalism of the Chapman-Enskog expansion.

b/. By placing assumptions on the data, (4.9) yields control of $\varepsilon^{1/2}u_x$,

$$\int_0^t \int_{\mathbf{R}} \varepsilon|u_x|^2 dx d\tau \le O(1).$$

In turn, use of (4.7), together with condition (S) and the bound $A \le NI$ for the matrix A, leads to control of $\varepsilon^{1/2}u_t$,

$$\int_0^t \int_{\mathbf{R}} \varepsilon|u_t|^2 dx d\tau \le \int_0^t \int_{\mathbf{R}} \varepsilon|F'(u)u_x|^2 dx d\tau + O(1) \le O(1)\,.$$

c/. If $u^\varepsilon \to u$ boundedly a.e. and the initial data satisfy

$$u_0^\varepsilon \to u_0 \text{ in } L^2\,, \quad \varepsilon^2 (u_{0x}^\varepsilon)^2 + \varepsilon^2 (v_{0x}^\varepsilon)^2 \to 0 \text{ in } L^1\,,$$

then (4.8) implies that u satisfies the entropy inequality

$$\partial_t \frac{1}{2}|u|^2 + \partial_x \mathcal{Q}(u) \le 0\,, \quad \text{in } \mathcal{D}'. \tag{4.10}$$

d/. Proposition 4.1 provides a simple proof for the convergence of the relaxation system

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + a^2 u_x &= -\frac{1}{\varepsilon}(f(u) - v) \end{aligned} \tag{4.11}$$

where $u, v \in \mathbb{R}$, to the scalar conservation law

$$u_t + f(u)_x = 0\,, \tag{4.12}$$

under the hypotheses for the data (suppressing the ε-dependence):

$$\begin{aligned} |u_0| + |v_0| &\le O(1)\,, \\ \int_{\mathbb{R}} u_0^2 + v_0^2 + \varepsilon^2 u_{0x}^2 + \varepsilon^2 v_{0x}^2 &\le O(1)\,. \end{aligned} \tag{4.13}$$

Under the subcharacteristic condition $|f'| < a$, solutions (u, v) of (4.11) satisfy the uniform L^∞-bound $|u^\varepsilon| \le C$, [CLL, N_1], and, from (4.9) and (4.7),

$$\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 + u_t^2 \, dx dt \le O(1)\,. \tag{4.14}$$

Let η, q be an entropy pair for the scalar conservation law. From (4.5) we obtain

$$\partial_t \eta(u) + \partial_x q(u) = \varepsilon \partial_x \eta'(u) u_x - \varepsilon \partial_t \eta'(u) u_t - \varepsilon \eta''(u) u_x^2 + \varepsilon \eta''(u) u_t^2 \tag{4.15}$$

The control of the dissipation measure and convergence to (4.12) follows from the argument of Tartar [Ta].

ii. The multi-dimensional case

Consider the relaxation approximations for (4.1) in the multi-dimensional case, [JX],

$$\begin{aligned} \partial_t u + \sum_{j=1}^d \partial_{x_j} v_j &= 0 \\ \partial_t v_i + A_i \partial_{x_i} u &= -\frac{1}{\varepsilon}(v_i - F_i(u))\,, \quad i = 1, \dots, d, \end{aligned} \tag{4.16}$$

where $u, v_1, \dots, v_d$ take values in $\mathbb{R}^n$ and A_i, $i = 1, \dots, d$ are positive definite and symmetric $n \times n$ matrices. Again this may be written as approximation by wave equations

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = \varepsilon\Big(\sum_{j=1}^{d} A_j u_{x_j x_j} - u_{tt}\Big)\,. \tag{4.17}$$

The same steps as in the one-dimensional case yield the identity

$$\begin{aligned} &\partial_t\Big(\frac{1}{2}|u+\varepsilon u_t|^2 + \frac{1}{2}\varepsilon^2|u_t|^2 + \varepsilon^2\sum_{j=1}^{d} u_{x_j}\cdot A_j u_{x_j}\Big) + \sum_{j=1}^{d}\partial_{x_j}\mathcal{Q}_j(u) \\ &\quad + \varepsilon|u_t + \sum_{j=1}^{d}\partial_{x_j}F_j(u)|^2 + \varepsilon\Big[\sum_{i=1}^{d} u_{x_i}\cdot A_i u_{x_i} - \big|\sum_{i=1}^{d} F_i'(u)u_{x_i}\big|^2\Big] \\ &\quad = \sum_{j=1}^{d}\partial_{x_j}\big(\varepsilon u\cdot A_j u_{x_j} + 2\varepsilon^2 u_t\cdot A_j u_{x_j}\big)\,. \end{aligned} \tag{4.18}$$

The natural dissipation condition is, for some $\nu_i > 0$, $i = 1, \dots, d$,

$$\sum_{i=1}^{d}\xi_i\cdot A_i\xi_i - \big|\sum_{i=1}^{d}F_i'(u)\xi_i\big|^2 \ge \sum_{i=1}^{d}\nu_i|\xi_i|^2\,, \quad \text{for } \xi_1,\dots,\xi_d \in \mathbb{R}^n. \tag{S$_d$}$$

Proposition 4.2. *Under Hypothesis (S_d), smooth solutions of (4.16), that decay fast at infinity, satisfy*

$$\begin{aligned} &\int_{\mathbf{R}^d}\frac{1}{2}|u+\varepsilon u_t|^2 + \frac{1}{2}\varepsilon^2|u_t|^2 + \varepsilon^2\sum_{j=1}^{d}u_{x_j}\cdot A_j u_{x_j}\,dx \\ &\quad + \int_0^t\int_{\mathbf{R}^d}\varepsilon^3|u_{tt} - \sum_{j=1}^{d}A_j u_{x_j x_j}|^2 + \varepsilon\sum_{i=1}^{d}\nu_i|u_{x_i}|^2\,dx\,d\tau \\ &\quad \le \int_{\mathbf{R}^d}\frac{1}{2}|u(0)+\varepsilon u_t(0)|^2 + \frac{1}{2}\varepsilon^2|u_t(0)|^2 + \varepsilon^2\sum_{j=1}^{d}u_{x_j}(0)\cdot A_j u_{x_j}(0)\,dx \end{aligned} \tag{4.19}$$

5. Systems equipped with a strictly convex entropy

It is well known that hyperbolic systems equipped with a strictly convex entropy are symmetrizable, Friedrichs-Lax [FL], and through a change of the dependent variable can be put into the form,

$$\partial_t G(v) + \partial_x F(v) = 0 \tag{5.1}$$

with $F'(u)$ symmetric, $G'(u)$ symmetric and positive definite, of the class of systems suggested by Godunov [Go]. Next, we pursue the analog of Proposition 4.1 for the case of symmetrizable systems. We only present the one-dimensional case; the multi-dimensional case is an easy generalization. Although the estimate is motivated by the results of Section 4 and the symmetrizability properties of systems equipped with a strictly convex entropy, a direct derivation is presented, as most systems in applications do not come naturally in their symmetric form.

We consider the hyperbolic system

$$\partial_t u + \partial_x F(u) = 0\,, \quad x \in \mathbb{R},\, t > 0, \tag{5.2}$$

where $u(x,t)$ takes values in $\mathbb{R}^n$. We assume that (5.2) is equipped with the entropy entropy-flux pair $\eta(u)$, $q(u)$, with $\eta(u)$ strictly convex. Consider the approximation of (5.2) by wave equation,

$$\partial_t u + \partial_x F(u) = \varepsilon(A u_{xx} - u_{tt})\,, \tag{5.3}$$

where A is a positive definite symmetric $n \times n$ matrix; this is equivalent to the relaxation system (4.4). In the sequel we use the notation $\eta_u := \nabla \eta$, η_{uu} for the Hessian of η, and I for the $n \times n$ identity matrix. We prove.

Proposition 5.1. *Assume that (5.2) is equipped with a strictly convex entropy $\eta(u)$ that satisfies, for some $\alpha > 0$,*

$$\eta_{uu}(u) \le \alpha\, I\,, \tag{H}$$

and that the positive definite, symmetric matrix A satisfies

$$\frac{1}{2}\big(A^T \eta_{uu}(u) + \eta_{uu}(u) A\big) - \alpha {F'}^T(u) F'(u) \ge \nu I \tag{S$'$}$$

Then smooth solutions of (5.3), that decay fast at infinity satisfy the dissipation estimate

$$\begin{aligned}\int_{\mathbf{R}} \eta(u + \varepsilon u_t) + \frac{1}{2}\varepsilon^2 \alpha |u_t|^2 + \varepsilon^2 \alpha u_x \cdot A u_x \, dx + \int_0^t \int_{\mathbf{R}} \varepsilon^3 \alpha |u_{tt} - A u_{xx}|^2 + \varepsilon \nu |u_x|^2 dx d\tau \\ \le \int_{\mathbf{R}} \eta(u_0 + \varepsilon u_t(0)) + c\varepsilon^2 |u_t(0)|^2 + \varepsilon^2 u_{0x} \cdot A u_{0x}\, dx\end{aligned} \tag{5.4}$$

where c is a constant independent of ε.

Proof. The system (5.3) has the following estimates: Taking the inner product with u_t, we obtain as in (4.7),

$$\partial_t \Big(\frac{1}{2}\varepsilon |u_t|^2 + \frac{1}{2}\varepsilon u_x \cdot A u_x\Big) + \Big(|u_t|^2 + u_t \cdot F'(u) u_x\Big) = \partial_x (\varepsilon u_t \cdot A u_x) \tag{5.5}$$

Next, taking the inner product with η_u, we arrive at

$$\partial_t\Big(\eta(u) + \varepsilon\eta_u \cdot u_t\Big) + \partial_x q(u) + \varepsilon\Big(\eta_{uu}u_x \cdot Au_x - u_t \cdot \eta_{uu}u_t\Big) = \varepsilon\partial_x(\eta_u \cdot Au_x)\,, \tag{5.6}$$

We multiply (5.5) by $2\alpha\varepsilon$, add (5.6), and use the identity

$$\eta(u+\varepsilon u_t) = \eta(u) + \varepsilon\eta_u(u)\cdot u_t + \varepsilon u_t \cdot \Big(\int_0^1\int_0^s \eta_{uu}(u+\varepsilon\tau u_t)d\tau ds\Big)\varepsilon u_t$$

to obtain, after some rearrangements of terms,

$$\begin{aligned}
&\partial_t\Big(\eta(u+\varepsilon u_t) + \varepsilon^2 u_t \cdot \big[\frac{1}{2}\alpha I - \int_0^1\int_0^s \eta_{uu}(u+\varepsilon\tau u_t)d\tau ds\big]u_t + \frac{1}{2}\varepsilon^2\alpha|u_t|^2 + \varepsilon^2\alpha u_x\cdot Au_x\Big)\\
&\quad + \partial_x q(u) + \varepsilon u_t\cdot\big(\alpha I - \eta_{uu}\big)u_t + \varepsilon\alpha|u_t + F'(u)u_x|^2 + \varepsilon u_x \cdot \big(\eta_{uu}A - \alpha {F'}^T F'\big)u_x\\
&\quad = \partial_x\big(\varepsilon\eta_u\cdot Au_x + 2\varepsilon^2\alpha u_t\cdot Au_x\big)\,.
\end{aligned} \tag{5.7}$$

In view of (H),

$$u_t \cdot \big[\frac{1}{2}\alpha I - \int_0^1\int_0^s \eta_{uu}(u+\varepsilon\tau u_t)d\tau ds\big]u_t \geq 0\,,$$

and (5.4) follows from (H) and (S'). $\square$

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