On the Mathematical Theory of Fluid Dynamic Limits to Conservation Laws

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In memory of John A. Nohel

Abstract. These lectures discuss topics in the theory of hyperbolic systems of conservation laws focusing on the mathematical theory of fluid-dynamic limits. First, we discuss the emergence of the compressible Euler equations for an ideal gas in the fluid-dynamic limit of the Boltzmann equation or of the BGK model. Then we survey the current state of the mathematical theory of fluid-dynamic limits for BGK systems and for discrete velocity models of relaxation type. This is done for the case that the limit is a scalar conservation law or a system of two equations.

1 Introduction

These lecture notes deal with the subject of fluid dynamic limits from kinetic equations to conservation laws. The subject is motivated by the formal derivation of the compressible Euler equations for a mono-atomic gas as the zero mean-free-path limit of the Boltzmann equation. While the rigorous justification of the fluid-dynamic limit for the Boltzmann equation is a challenging open problem, it has prompted recent work on the derivation of hyperbolic systems of conservation laws from kinetic models, in simpler situations where the limits are scalar equations or systems of two conservation laws. The objective of the present article is to describe some of these recent results.

We begin in Section 2 with a discussion of the structure of the fluid dynamic limit for the Boltzmann equation. On the one hand the formalism of the fluid limit for Boltzmann circumscribes the framework that the mathematical analysis is challenged to elucidate. On the other hand this formalism introduces some fundamental notions like the collision invariants and associated evolutions of moments, the H-theorem, and the closure of the conservation laws in the fluid dynamic limit. These notions have been instrumental in the development of an analytical theory where such a theory is currently present.

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In Section 3 we discuss certain kinetic models that are equipped with one conservation law. In certain cases those models generate L^1-contractions and the Kruzhkov theory can be adapted to develop a rigorous derivation of the fluid limit to the entropy solution of a scalar multi-dimensional conservation law. This theory is currently well understood and applies to a variety of kinetic or discrete kinetic (of relaxation type) models. We also take the opportunity to discuss the zero mean free path limit of a special kinetic model that motivates the so-called kinetic formulation of scalar conservation laws. The kinetic formulation, proposed by Lions, Perthame and Tadmor, provides a notion of solution for hyperbolic equations (and systems) that is equivalent to the notion of entropy solution and connects the theory of conservation laws with the theory of transport equations.

In the final Section 4 we discuss some examples of kinetic models that converge to systems of two conservation laws in one space dimension. We emphasize the fact that some kinetic models are equipped with stronger dissipative structures than the H-theorem, what allows the use of compensated compactness to effect the derivation of the fluid limit.

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2 The Boltzmann Equation and its Fluid-Dynamical Limit

In the kinetic theory of gases the statistical description of a gas is given by its distribution function f. For a monoatomic gas f is a function of the positions $x \in \mathbb{R}^d$ and the momenta $\xi \in \mathbb{R}^d$ in the phase space. The product $f(t,x,\xi)\,dx\,d\xi$ describes the mean number of molecules in an element of the phase space centered at (x,ξ) of range dx and $d\xi$.

The Boltzmann equation describes the evolution of the distribution function f for a dilute gas and has the form

$$\partial_t f + \xi \cdot \nabla_x f = Q(f,f) \ . \tag{1}$$

The collision operator $Q(f,f)$ describes the gain and loss of the distribution function due to collisions and is given by

$$\begin{aligned} Q(f,f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} & b(\xi - \xi_*, \omega) \times \\ & \times \Big[f(t,x,\xi') f(t,x,\xi'_*) - f(t,x,\xi) f(t,x,\xi_*) \Big]\, d\xi_*\, d\omega \ . \end{aligned} \tag{2}$$

One arrives at the form of Q as follows: The incoming velocities ξ and ξ_* and outgoing velocities ξ' and ξ'_* entering a collision satisfy microscopic balances of mass, momentum and energy

$$\begin{aligned} \xi + \xi_* &= \xi' + \xi'_* \\ |\xi|^2 + |\xi_*|^2 &= |\xi'|^2 + |\xi'_*|^2 \ . \end{aligned} \tag{3}$$

These microscopic balances can be expressed with regard to the center of mass of the collision, noticing that (3) are equivalent to

$$\xi + \xi_* = \xi' + \xi'_* \,, \quad |\xi - \xi_*| = |\xi' - \xi'_*| \ .$$

A pair of incoming velocities ξ, ξ_* defines a sphere of center $\frac{\xi+\xi_*}{2}$ and radius $\frac{|\xi-\xi_*|}{2}$. Each outgoing pair of velocities ξ', ξ'_* compatible with (3) lies on this sphere and can be obtained by reflection with respect to a plane passing through the collision center. The planes are parametrized by their normal $\omega \in S^{d-1}$. The outgoing velocities are determined through the formulas

$$\begin{aligned} \xi' &= \xi \,+\, \omega \cdot (\xi_* - \xi)\omega \\ \xi'_* &= \xi_* - \omega \cdot (\xi_* - \xi)\omega \ . \end{aligned}$$

The process defines a map $T_\omega : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ taking the incoming into the outgoing velocity pairs

$$(\xi, \xi_*) \mapsto (\xi', \xi'_*) = T_\omega(\xi, \xi_*) \ .$$

T_ω has the properties: For $\omega \in S^{d-1}$,

(i) $T_\omega \circ T_\omega = Id$ (microreversibility)
(ii) $|\det T_\omega| = 1$
(iii) $T_\omega(\xi_*, \xi) = (\xi'_*, \xi')$ (symmetry).

The term $Q(f,f)$ describes the gains and losses in the distribution function due to collisions. Gains occur from collisions at x of the type $(\xi', \xi'_*) \mapsto (\xi, \xi_*)$, and losses from collisions at x of the type $(\xi, \xi_*) \mapsto (\xi', \xi'_*)$. In the form of (2), it is factored the assumption that only binary collisions are admitted and the hypothesis of detailed balancing, stating that the number of collisions $(\xi, \xi_*) \mapsto (\xi', \xi'_*)$ is equal, in equilibrium, to the number of collisions $(\xi', \xi'_*) \mapsto (\xi, \xi_*)$ (see [21]). The factor b is called Boltzmann collision kernel and models the microscopic physics of the collisional process. For mechanical reasons it has to satisfy $b \geq 0$, b is locally integrable and the symmetries

$$b(\xi - \xi_*, \omega) = b\left(|\xi - \xi_*|, |(\xi - \xi_*) \cdot \omega|\right) \ . \tag{5}$$

As an example, for collisions associated to a hard sphere potential $b(g, \omega) = |g \cdot \omega|$.

We outline without proof certain properties of the collision operator that are indicative of the structure of the equations. A function $\varphi = \varphi(\xi)$ is called a collision invariant if, along the collisions described by T_ω, it satisfies

$$\varphi(\xi) + \varphi(\xi_*) = \varphi(\xi') + \varphi(\xi'_*) \ . \tag{6}$$

This is written in the shorthand notation $\varphi + \varphi_* = \varphi' + \varphi'_*$.

Proposition 1. *The collision invariants are given by*

$$\varphi(\xi) = a + b \cdot \xi + c|\xi|^2 \,, \qquad a, c \in \mathbb{R}\,,\ b \in \mathbb{R}^d \ .$$

Proposition 2. *The collision operator satisfies:*

(i)

$$\begin{aligned}
&\int_{\mathbb{R}^d} Q(f,f)(\xi)\varphi(\xi)\, d\xi \\
&\quad = \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times S^{d-1}} (f' f'_* - f f_*)(\varphi + \varphi_* - \varphi' - \varphi'_*) b \, d\omega \, d\xi_* \, d\xi \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times S^{d-1}} f f_* (\varphi' + \varphi'_* - \varphi - \varphi_*) b \, d\omega \, d\xi_* \, d\xi
\end{aligned}$$

(ii)

$$\int_{\mathbb{R}^d} Q(f,f)(\xi) \ln f(\xi)\, d\xi \le 0$$

(iii) *If $b > 0$ then the equilibria f_{eq} of the collision operator, $Q(f_{eq}, f_{eq}) = 0$, are*

$$f_{eq}(\xi) = \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} \exp\left\{ -\frac{|\xi - u|^2}{2\theta} \right\} \tag{7}$$

for some $\rho, \theta > 0$ and $u \in \mathbb{R}^d$.

Macroscopic Balance Laws – H-Theorem

Let ρ, u and E stand for the macroscopic density, velocity and energy respectively, defined through the moments of f

$$\begin{aligned}
\rho(t,x) &= \int f(t,x,\xi)\, d\xi \\
\rho u(t,x) &= \int \xi f(t,x,\xi)\, d\xi \\
E(t,x) &= \int \frac{1}{2}|\xi|^2 f(t,x,\xi)\, d\xi \ .
\end{aligned} \tag{8}$$

Since 1, ξ and $|\xi|^2$ are collision invariants, ρ, u and E evolve according to the moment equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \\
\frac{\partial}{\partial t}\rho u + \operatorname{div}(\rho u \otimes u + P) &= 0 \\
\frac{\partial E}{\partial t} + \operatorname{div}(E u + P \cdot u + Q) &= 0
\end{aligned} \tag{9}$$

where $(u \otimes u)_{ij} = u_i u_j$ and the pressure (tensor) P, heat flux Q, total energy E and internal energy e are determined through the formulas

$$
\begin{aligned}
P_{ij} &= \int (\xi_i - u_i)(\xi_j - u_j) f \, d\xi \\
Q_i &= \frac{1}{2} \int (\xi_i - u_i) |\xi - u|^2 f \, d\xi \\
E &= \int \frac{1}{2} |\xi - u|^2 f \, d\xi + \frac{1}{2} \rho |u|^2 =: e + \frac{1}{2} \rho |u|^2 \ .
\end{aligned}
\tag{10}
$$

The equations (9) describe the evolution of the moments ρ, u and E and are not a closed system of equations. Because of the close connection with the balance equations of continuum physics they are called macroscopic balance laws.

The Boltzmann equation is equipped with an imbalance law, considered to capture the entropy dissipation in a rarefied gas. It is obtained by multiplying (1) by $(1 + \ln f)$ and using Proposition 2. The resulting identity yields the celebrated Boltzmann H-Theorem:

$$
\begin{aligned}
&\partial_t \int_{\mathbf{R}^d} f \ln f \, d\xi + \operatorname{div}_x \int_{\mathbf{R}^d} \xi f \ln f \, d\xi = \int_{\mathbf{R}^d} Q(f,f) \ln f \, d\xi =: S \\
&= -\frac{1}{4} \iiint_{\mathbf{R}^d \times \mathbf{R}^d \times S^{d-1}} (f' f'_* - f f_*)(\ln f' f'_* - \ln f f_*) \, b \, d\omega \, d\xi_* \, d\xi \leq 0 \, .
\end{aligned}
\tag{11}
$$

The term S is called entropy dissipation (rate) and is negative. Moreover $S = 0$ iff $f' f'_* = f f_*$ or equivalently if f is a Maxwellian.

Maxwellians

The solutions of $Q(f_{eq}, f_{eq}) = 0$ are called Maxwellians. They are determined by solving $f' f'_* = f f_*$ and thus $\ln f_{eq}$ is a collision invariant

$$
\ln f_{eq} = a + b \cdot \xi + c |\xi|^2
$$

and f_{eq} is expressed in the form (7). The constants ρ, u and θ entering in (7) are determined from the moments of $\mathcal{M}$ through the relations:

$$
\begin{aligned}
&\int \mathcal{M} = \rho \\
&\int (\xi - u) \mathcal{M} = 0 \quad \text{implies} \quad \int \xi \mathcal{M} = \rho u \\
&\int \frac{1}{2} |\xi - u|^2 \mathcal{M} = \frac{d}{2} \rho \theta \quad \text{implies} \quad E = \int \frac{1}{2} |\xi|^2 \mathcal{M} = \frac{1}{2} \rho |u|^2 + \frac{d}{2} \rho \theta \ .
\end{aligned}
\tag{12}
$$

Maxwellians are completely determined by their moments and are denoted $\mathcal{M} = \mathcal{M}(\rho, u, \theta, \xi)$. The pressure and heat flux can also be computed along

Maxwellians by

$$
\begin{aligned}
P_{ij} &= \int (\xi_i - u_i)(\xi_j - u_j)\mathcal{M}_{(\rho,u,\theta)} = \rho\theta\delta_{ij} \\
Q_i &= \frac{1}{2}\int (\xi_i - u_i)|\xi - u|^2\mathcal{M}_{(\rho,u,\theta)} = 0 \ .
\end{aligned}
\tag{13}
$$

Another characterization of Maxwellians is that they arise as minima of the H-functional for the constrained minimization problem:

$$
\begin{aligned}
&\min \int f \ln f \, d\xi \qquad \text{over all } f \geq 0 \text{ such that} \\
&\qquad\qquad \int f = \rho \, , \quad \int \xi f = \rho u \, , \quad \int \frac{|\xi|^2}{2} f = E \ .
\end{aligned}
\tag{14}
$$

The minimum of this minimization problem is attained at $\mathcal{M}(\rho, u, \theta, \xi)$.

It is instructive to outline a proof of this statement. The minimization problem leads to computing the critical points of the functional

$$
J(f) = \int f \ln f \, d\xi + a\big(\rho - \int f\big) + b \cdot \big(m - \int \xi f\big) + c\big(E - \int \frac{|\xi|^2}{2} f\big) \tag{15}
$$

where a, b, c are Lagrange multipliers. The critical points satisfy the equation

$$
0 = J'(f)v = \int_{\mathbf{R}^d} \Big(\ln f - (a-1) - b \cdot \xi - c\frac{|\xi|^2}{2} \Big) v
$$

for all test functions v, which implies

$$
\ln f = (a-1) + b \cdot \xi + c\frac{|\xi|^2}{2}
$$

or that f is a Maxwellian $\mathcal{M}(\rho, u, \theta)$ with ρ, u, θ computed from the moments of f.

Another proof is obtained via the inequality: $z \ln z - z \ln y + y - z \geq 0$ for $y, z > 0$. For $f \geq 0$ with moments as in (14) and for $\mathcal{M}$ a Maxwellian with the same moments, we have

$$
\begin{aligned}
\int f \ln f &\geq \int f \ln \mathcal{M} + \int f - \mathcal{M} \\
&\geq \int (f - \mathcal{M}) \ln \mathcal{M} + \int \mathcal{M} \ln \mathcal{M} = \int \mathcal{M} \ln \mathcal{M} \ .
\end{aligned}
$$

The Euler Limit

Next, we describe the formal fluid limit of the Boltzmann equation. We present the Euler limit which is the limit as $\varepsilon \to 0$ of the scaled Boltzmann equation

$$
\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon) \ . \tag{16}
$$

In this scaling ε stands for the Knudsen number the ratio of ''the mean free path" (a measure of the average distance between successive collisions) over a macroscopic length scale.

The moments ρ^ε, u^ε, E^ε, P^ε and Q^ε satisfy the macroscopic balance laws (9). In addition the H-theorem reads

$$\begin{aligned}&\partial_t \int_{\mathbf{R}^d} f^\varepsilon \ln f^\varepsilon \, d\xi + \operatorname{div}_x \int_{\mathbf{R}^d} \xi f^\varepsilon \ln f^\varepsilon \, d\xi \\ &+ \frac{1}{4\varepsilon} \iiint_{\mathbf{R}^d \times \mathbf{R}^d \times S^{d-1}} ({f^\varepsilon}' {f^\varepsilon}'_* - f^\varepsilon {f^\varepsilon}_*)\Big(\ln \frac{{f^\varepsilon}' {f^\varepsilon}'_*}{f^\varepsilon {f^\varepsilon}_*}\Big) b \, d\omega \, d\xi_* \, d\xi = 0 \ . \end{aligned} \tag{17}$$

The last term is positive and vanishes only along Maxwellians. It is then conceivable that in the fluid limit $\varepsilon \to 0$ the kinetic function approaches a Maxwellian $\mathcal{M}(\rho, u, \theta, \xi)$.

Using the relations (12)–(13), we see that the formal fluid limit becomes

$$\begin{aligned} \partial_t \rho + \operatorname{div}_x(\rho u) &= 0 \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u + \rho \theta \delta_{ij}) &= 0 \\ \partial_t\big(\frac{1}{2}\rho |u|^2 + \frac{d}{2}\rho\theta\big) + \operatorname{div}_x\big((\frac{1}{2}\rho|u|^2 + \frac{d}{2}\rho\theta)u + \rho\theta u\big) &= 0 \end{aligned} \tag{18}$$

which are the compressible Euler equations for an ideal monoatomic gas. In the fluid limit the H-theorem (17) gives the macroscopic entropy inequality

$$\partial_t\big(\rho \ln \frac{\rho}{\theta^{\frac{d}{2}}}\big) + \operatorname{div}_x\big(\rho u \ln \frac{\rho}{\theta^{\frac{d}{2}}}\big) \le 0 \ . \tag{19}$$

The justification of the fluid-limit for weak solutions is a challenging open problem, due to the presence of shocks and the poor understanding of the theory of weak solutions for the compressible Euler equations. In the forthcoming sections we discuss mathematical results for fluid-limits in simpler situations.

BGK Models

A class of collision models sharing some of the properties of the Boltzmann collision operator are the so-called BGK models (after Bhatganar, Gross and Krook). In the BGK model the collision operator is replaced by relaxation to a Maxwellian

$$\partial_t f + \xi \cdot \nabla_x f = -\frac{1}{\varepsilon}\Big(f - \mathcal{M}_{(\rho,u,\theta)}\Big) \tag{20}$$

where $\mathcal{M}_{(\rho,u,\theta)}(\xi)$ is a local Maxwellian with moments

$$\rho = \int f \,, \quad \rho u = \int \xi f \,, \quad \rho|u|^2 + d\rho\theta = \int |\xi|^2 f \ .$$

The BGK-collision operator $Q_{BGK}(f) = \mathcal{M}_{(\rho,u,\theta)} - f$ has the properties

$$\int Q_{BGK}(f) \begin{pmatrix} 1 \\ \xi \\ |\xi|^2 \end{pmatrix} = \int (\mathcal{M}_{(\rho,u,\theta)} - f) \begin{pmatrix} 1 \\ \xi \\ |\xi|^2 \end{pmatrix} = 0$$

$$\int Q_{BGK}(f)\ln f = \int (\mathcal{M}_{(\rho,u,\theta)} - f)(\ln f - \ln \mathcal{M}_{(\rho,u,\theta)}) \leq 0$$

with equality iff $f = \mathcal{M}_{(\rho,u,\theta)}$. As a result, (20) is equipped with the same macroscopic balance laws as the Boltzmann equation and also with an analog of the H-theorem:

$$\partial_t \int_{\mathbb{R}^d} f \ln f + \mathrm{div}_x \int_{\mathbb{R}^d} \xi f \ln f + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} (f - \mathcal{M}_{(\rho,u,\theta)}) \Big(\ln \frac{f}{\mathcal{M}_{(\rho,u,\theta)}} \Big) = 0\,.$$

The fluid limit for this model is again the compressible Euler equations (18)–(19) for an ideal mono-atomic gas.

Bibliographic remarks. The books of Lifshitz and Pitaevskii [21] and Cercignani, Illner and Pulvirenti [6] contain detailed accounts on the derivation of the Boltzmann equation. The surveys of Perthame [32] and Golse [14] contain proofs of several of the listed properties of the Boltzmann equation and detailed discussions of the formalism of the fluid limits.

3 Kinetic Models with One Conservation Law

In this section we consider a kinetic model that is equipped with one conservation law and develop the mathematical theory of its fluid dynamic limit. The presented results follow ideas developed in [30,18,27,4] for a series of kinetic and discrete-kinetic (of relaxation type) models. The equation reads

$$\begin{aligned} \partial_t f + a(\xi)\cdot\nabla_x f &= \frac{1}{\varepsilon} C(f(t,x,\cdot),\xi) \\ f(0,x,\xi) &= f_0(x,\xi) \end{aligned} \tag{21}$$

where $x \in \mathbb{R}^d$ and $C(f,\xi)$ is a functional on $f(t,x,\cdot)$ (depending on ξ) that encodes the detailed properties of a collision process. The variable ξ may be continuous ($\xi \in \mathbb{R}$) or it may take discrete values; in the latter case (21) becomes a discrete velocity kinetic model. Both cases are treated simultaneously and we retain a common notation. A theory is developed for the fluid limit based on structural assumptions of C without recourse to its detailed properties.

It is assumed that C satisfies: $C(0(\cdot),\xi) = 0$, and

$$\int_{\mathbb{R}} C(f,\xi)\,d\xi = 0 \tag{hyp1}$$

so that (21) is equipped with a macroscopic balance law

$$\partial_t u + \mathrm{div}_x \int a(\xi) f\,d\xi = 0 \tag{22}$$

for the "mass" u defined by

$$u := \int f \, d\xi \ . \tag{23}$$

Second, the equilibria of (21) are parametrized in terms of exactly one scalar parameter w, which may be associated to the mass u. More precisely, the solutions of $C(f,\xi) = 0$ are

$$f_{eq} = \mathcal{M}(w,\xi) \qquad \text{where} \quad \int_\xi \mathcal{M}(w,\xi)\, d\xi = u = b(w) \tag{hyp2}$$

where b is a strictly increasing function, so that the map $w \to u$ is invertible.

3.1 Kinetic Models that Generate L^1-Contractions

It is assumed that the collision operator satisfies

$$\int_{\mathbf{R}} \Big(C(f(\cdot),\xi) - C(\bar f(\cdot),\xi) \Big) \operatorname{sgn}(f - \bar f)(\xi)\, d\xi \le 0 \ , \tag{hyp3}$$

for all $f(\cdot)$, $\bar f(\cdot)$. This hypothesis guarantees that the space-homogeneous variant of (hyp3) is a contraction in L^1_ξ.

The property is also preserved in the space-nonhomogeneous case and, as a consequence, the kinetic model (21) is endowed with a class of "kinetic entropies".

Theorem 3. *Under hypotheses* (hyp1)–(hyp3)*:*

(i) *The kinetic model is a contraction in* $L^1(\mathbb{R}^d_x \times \mathbb{R}_\xi)$.
(ii) *For all* $\kappa \in \mathbb{R}$, *we have*

$$\partial_t \int_\xi |f - \mathcal{M}(\kappa,\xi)| + \operatorname{div}_x \int_\xi a(\xi)|f - \mathcal{M}(\kappa,\xi)| \le 0 \quad \text{in } \mathcal{D}'. \tag{24}$$

(iii) *If for some* a, b *it is* $\mathcal{M}(a,\xi) \le \mathcal{M}(b,\xi)$ *for all* ξ, *then the domain* $\prod_\xi [\mathcal{M}(a,\xi), \mathcal{M}(b,\xi)]$ *is positively invariant.*

Proof. Let f and $\bar f$ be two solutions. By subtracting the corresponding equations, multiplying by $\operatorname{sgn}(f - \bar f)$, and using (hyp3), we obtain

$$\partial_t \int |f - \bar f|\, d\xi + \operatorname{div} \int a(\xi)|f - \bar f| = \frac{1}{\varepsilon} \int \big(C(f(\cdot),\xi) - C(\bar f(\cdot),\xi) \big) \operatorname{sgn}(f - \bar f) \le 0\, . \tag{25}$$

This shows that any two solutions f and $\bar f$ satisfy the L^1-contraction property:

$$\int_x \int_\xi |f - \bar f|(t,x,\xi)\, dx\, d\xi \quad \text{is decreasing in } t.$$

Since $\int_x \int_\xi (f - \bar{f})\, dx\, d\xi$ is a conserved quantity, we have

$$\int_x \int_\xi (f - \bar{f})^+(t,x,\xi)\, dx\, d\xi \quad \text{is decreasing in } t,$$

and as a result

$$\text{if } f_0 \leq \bar{f}_0 \text{ then } f \leq \bar{f} \ .$$

A special class of solutions of (21) are the global Maxwellians $\mathcal{M}(\kappa, \xi)$. These may be used as comparison functions. For instance

$$\text{if } f_0(x,\xi) \leq \mathcal{M}(a,\xi), \text{ for some } a \in \mathbb{R}, \text{ then } \quad f(t,x,\xi) \leq \mathcal{M}(a,\xi) \ .$$

From this property part (iii) follows. Finally, if $\bar{f} = \mathcal{M}(\kappa, \xi)$ in (25) then

$$\int_\xi (\partial_t + a(\xi) \cdot \nabla_x) |f - \mathcal{M}(\kappa,\xi)|\, d\xi = \frac{1}{\varepsilon} \int C(f(\cdot), \xi) \mathrm{sgn}\,(f - \mathcal{M}(\kappa,\xi)) \leq 0$$

which shows (24). ◻

We present next two specific models that satisfy hypotheses (hyp1)–(hyp3).

I. A Discrete Velocity Model. Consider the system

$$\begin{aligned} \partial_t f_0 + a_0 \cdot \nabla_x f_0 &= \frac{1}{\varepsilon} \sum_{i=1}^{d} \big(f_i - h_i(f_0)\big) \\ \partial_t f_i + a_i \cdot \nabla_x f_i &= -\frac{1}{\varepsilon} \big(f_i - h_i(f_0)\big) \qquad i = 1, \dots, d. \end{aligned} \tag{26}$$

for the evolution of $f = (f_0, f_1, \dots, f_d)$ where $a_0, a_1, \dots, a_d \in \mathbb{R}^d$. This discrete velocity model of relaxation type is developed in [18] as a relaxation approximation for the scalar multi-dimensional conservation law.

We assume that

$$h_i(w) \ \text{ are strictly increasing, } i = 1, \dots, d, \tag{A}$$

and let $u = f_0 + \sum_k f_k$. The Maxwellian functions are

$$\begin{aligned} f_{eq} = \mathcal{M}(w, j)_{j=0,1,\dots,d} &= (w, h_1(w), \dots, h_d(w))\,, \\ \text{where } \ u = w + \sum_i h_i(w) &=: b(w) \ . \end{aligned}$$

Clearly (hyp1) and (hyp2) are satisfied. To see Hypothesis (hyp3), note that

$$
\begin{aligned}
I &= \sum_{j=0}^{d} \Big(C(f,j) - C(\bar f, j) \Big) \operatorname{sgn}(f_j - \bar f_j) \\
&= \Big[\sum_{i=1}^{d} \big(f_i - \bar f_i - (h_i(f_0) - h_i(\bar f_0))\big) \Big] \operatorname{sgn}(f_0 - \bar f_0) \\
&\quad - \sum_{i=1}^{d} \big(f_i - \bar f_i - (h_i(f_0) - h_i(\bar f_0)\big) \operatorname{sgn}(f_i - \bar f_i) \\
&= \sum_{i=1}^{d} \Big(f_i - \bar f_i - (h_i(f_0) - h_i(\bar f_0)) \Big) \big(\operatorname{sgn}(f_0 - \bar f_0) - \operatorname{sgn}(f_i - \bar f_i) \big) \le 0
\end{aligned}
$$

where the last inequality follows from (A).

Under (A) the model (26) is also equipped with a globally defined entropy function

$$
\partial_t \Big(\frac{1}{2} f_0^2 + \sum_{i=1}^{d} \Psi_i(f_i) \Big) + \operatorname{div} \Big(a_0 \frac{1}{2} f_0^2 + \sum_{i=1}^{n} a_i \Psi_i(f_i) \Big) + \frac{1}{\varepsilon} \sum_{i=1}^{d} \phi_i(f_0, f_i) = 0 \tag{27}
$$

where

$$
\Psi_i(f_i) = \int_0^{f_i} h_i^{-1}(\tau)\, d\tau \,,
$$

is positive and strictly convex, while

$$
\phi_i(f_0, f_i) = (f_0 - h_i^{-1}(f_i))(h_i(f_0) - f_i)
$$

satisfies $\phi_i \ge 0$ and $\phi_i = 0$ if and only if f is a Maxwellian: $f_j = \mathcal{M}(w, j)$ for some w. The identity provides control of the distance of solutions from equilibria: If $\frac{dh_i}{dw} \ge c$ then $\phi_i \ge c(h_i(f_0) - f_i)^2$ and (27) leads to

$$
\int_0^\infty \int_{\mathbb{R}^d} \sum_{i=1}^{d} (h_i(f_0) - f_i)^2 \, dx\, dt \le O(\varepsilon) \quad .
$$

II. A BGK Model. Consider next the kinetic model of BGK type

$$
\begin{gathered}
\partial_t f + a(\xi) \cdot \nabla_x f = -\frac{1}{\varepsilon} \big(f - \mathcal{M}(u, \xi) \big) \\
\text{with } \ u = \int_\xi f
\end{gathered}
\tag{28}
$$

where $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}$. The model is introduced in [30] for the special choice of Maxwellian function $\mathcal{M}(u, \xi) = \mathbb{1}(u, \xi)$. The general case is developed in [27,4].

It is here assumed that $\mathcal{M}(u,\xi)$ is smooth and satisfies

$$\mathcal{M}(\cdot,\xi) \text{ is strictly increasing,} \qquad u = \int \mathcal{M}(u,\xi) \ . \tag{B}$$

Then (hyp1) and (hyp2) are clearly fulfilled. The monotonicity of $\mathcal{M}$ states $\mathcal{M}(u,\xi) > \mathcal{M}(\bar{u},\xi)$ iff $u > \bar{u}$, and, hence,

$$\begin{aligned}\int_\xi |\mathcal{M}(u,\xi) - \mathcal{M}(\bar{u},\xi)| &= \operatorname{sgn}(u-\bar{u})\Big(\int_\xi \mathcal{M}(u,\xi) - \mathcal{M}(\bar{u},\xi)\Big) \\ &= |u-\bar{u}| = \Big|\int_\xi f - \bar{f}\Big| \le \int_\xi |f-\bar{f}| \ .\end{aligned}$$

In turn that implies (hyp3):

$$\begin{aligned} I &= \int_\xi \Big[C(f(\cdot),\xi) - C(\bar{f}(\cdot),\xi)\Big] \operatorname{sgn}(f-\bar{f}) \\ &= -\int_\xi |f-\bar{f}| + \int_\xi (\mathcal{M}(u,\xi) - \mathcal{M}(\bar{u},\xi)) \operatorname{sgn}(f-\bar{f}) \le 0 \ .\end{aligned}$$

The model also possesses an analog of the H-theorem. If we multiple the equation (28) by $(\mathcal{M}^{-1}(f,\xi) - u)$, integrate over $\xi \in \mathbb{R}$ and denote by

$$\mu(f,\xi) = \int_0^f \mathcal{M}^{-1}(g,\xi)\,dg$$

then $\mu(\cdot,\xi)$ is convex and we have

$$\begin{aligned}\partial_t \int \mu(f) + div_x \int a(\xi)\mu(f) - u\Big(\partial_t u + \operatorname{div}_x \int a(\xi) f\Big) \\ + \frac{1}{\varepsilon}\int (\mathcal{M}^{-1}(f,\xi) - u)(f - \mathcal{M}(u,\xi)) = 0 \ .\end{aligned} \tag{29}$$

The third term vanishes due to the conservation law, the last term is positive due to the monotonicity assumption. If we further assume that $\partial_u \mathcal{M} \ge c$ then the last equation yields the bound

$$\int_0^T \int_x \int_\xi c|f - \mathcal{M}(u,\xi)|^2 \le \int_0^T \int_x \int_\xi (\mathcal{M}^{-1}(f,\xi) - u)(f - \mathcal{M}(u,\xi)) \le O(\varepsilon) \tag{30}$$

stating that the Maxwellians are enforced in the fluid limit $\varepsilon \to 0$.

We next consider a family of solutions f^ε of (21) and study their limiting behavior $\varepsilon \to 0$. Let $u^\varepsilon = \int f^\varepsilon$ and set

$$w^\varepsilon = b^{-1}\Big(\int f^\varepsilon\Big)$$

From (22) we obtain

$$\partial_t\, b(w^\varepsilon) + \mathrm{div}_x \int_\xi a(\xi)\mathcal{M}(w^\varepsilon,\xi) = \mathrm{div}_x\Big(\int_\xi a(\xi)\big(\mathcal{M}(w^\varepsilon,\xi) - f^\varepsilon\big)\Big) \ .$$

We will see in the next section that the L^1-contraction property and the conservation laws allow to conclude that $\{u^\varepsilon\}$ is precompact in $L^1_{loc,x,t}$ and (along a subsequence) $u^\varepsilon \to u$ and, since b is strictly increasing, $w^\varepsilon \to w = b^{-1}(u)$ a.e.

To conclude a hypothesis is needed dictating that Maxwellian distributions are enforced in the limit $\varepsilon \to 0$:

$$\int_\xi a(\xi)(f^\varepsilon - \mathcal{M}(w^\varepsilon,\xi)) \to 0 \quad \text{in } \mathcal{D}' , \text{ where } \quad w^\varepsilon = b^{-1}\Big(\int_\xi f^\varepsilon\Big). \tag{hyp4}$$

Both (26) and (28) verify such a hypothesis due to (27) and (30) (see Sec 3.2 for the model (28)). Under this framework it is a technical issue to show that the limiting $u = b(w)$ satisfies the scalar conservation law

$$\partial_t b(w) + \mathrm{div}_x \int_\xi a(\xi)\mathcal{M}(w,\xi) = 0 \ . \tag{31}$$

3.2 The Fluid-Dynamic Limit for a Kinetic BGK-Model

We provide the main technical details for the fluid dynamic limit of the BGK-model

$$\begin{aligned} \partial_t f^\varepsilon + a(\xi)\cdot\nabla_x f^\varepsilon &= -\frac{1}{\varepsilon}\big(f^\varepsilon - \mathcal{M}(u^\varepsilon,\xi)\big) \\ f^\varepsilon(0,x,\xi) &= f_0^\varepsilon(x,\xi) \end{aligned} \tag{32}$$

where $u^\varepsilon = \int f^\varepsilon$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}$. It is assumed that $a(\xi)$ is uniformly bounded and that the Maxwellians are smooth functions that satisfy $\mathcal{M}(0,\xi) = 0$,

$$\begin{gathered} \mathcal{M}(u,\cdot) \in L^1_\xi \,, \quad \mathcal{M}(\cdot,\xi) \ \text{ is strictly increasing} \\ u = \int_{\mathbb{R}} \mathcal{M}(u,\xi)\, d\xi \ . \end{gathered} \tag{a}$$

Then (hyp1) and (hyp2) and (hyp3) are fulfilled. Let $\omega(\tau)$ be a positive, increasing function denoting a modulus of continuity, $\limsup_{\tau\to 0+}\omega(\tau) = 0$.

Theorem 4. *Let $|a(\xi)| \le M$ and assume the initial data satisfy*

$$\begin{gathered} \mathcal{M}(a,\xi) \le f_0^\varepsilon(x,\xi) \le \mathcal{M}(b,\xi) \qquad \textit{for some } a < b \\ \int_x\int_\xi |f_0^\varepsilon(x,\xi)|\, dx\, d\xi \le C \\ \int_x\int_\xi |f_0^\varepsilon(x+h,\xi) - f_0^\varepsilon(x,\xi)|\, dx\, d\xi \le \omega(|h|) \quad \textit{for } h \in \mathbb{R}^d \ . \end{gathered} \tag{33}$$

Then

$$u^\varepsilon = \int_\xi f^\varepsilon \to u \quad \text{a.e. and in } L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^d) \text{ for } 1 \le p < \infty \ . \tag{34}$$

The limiting $u \in C\big([0,T]; L^1(\mathbb{R}^d)\big) \cap L^\infty((0,T) \times \mathbb{R}^d)$ *is an entropy solution:*

$$\partial_t |u - k| + \mathrm{div}_x (F(u) - F(k)) \,\mathrm{sgn}\,(u-k) \le 0 \quad \text{in } \mathcal{D}', \text{ for } k \in \mathbb{R} \ , \tag{35}$$

where $F(u) = \int_\xi a(\xi)\mathcal{M}(u,\xi)\,d\xi$.

Proof. The proof proceeds in three steps. From the L^1 contraction property, the invariance under translations, and the use of Maxwellians as comparison functions we have

$$\begin{gathered} \mathcal{M}(a,\xi) \le f^\varepsilon(t,x,\xi) \le \mathcal{M}(b,\xi) \qquad \text{for } a < b \\ \int_x |u^\varepsilon(t,x)| \le \int_x \int_\xi |f^\varepsilon| \le \int_x \int_\xi |f_0^\varepsilon| \le C \end{gathered} \tag{36}$$

and

$$\begin{aligned} \int_x |u^\varepsilon(t,x+h) - u^\varepsilon(t,x)| &\le \int_x \int_\xi |f^\varepsilon(t,x+h,\xi) - f^\varepsilon(t,x,\xi)| \\ &\le \int_x \int_\xi |f_0^\varepsilon(x+h,\xi) - f_0^\varepsilon(x,\xi)| \le \omega(|h|) \ . \end{aligned} \tag{37}$$

Using the lemma in the appendix and the bound $|a(\xi)| \le M$ we conclude that for $k > 0$

$$\int_x |u^\varepsilon(t+k,x) - u^\varepsilon(t,x)|\,dx \le C\omega(k) \ . \tag{38}$$

From (36), (37) and (38) we obtain that u^ε is precompact in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ and, along a subsequence, $u^\varepsilon \to u$ a.e. From (38) and Fatou's lemma $u \in C\big([0,T]; L^1(\mathbb{R}^d)\big)$.

Note that $a \le u^\varepsilon \le b$ is uniformly bounded and that (30) implies $|f^\varepsilon - \mathcal{M}(u^\varepsilon,\xi)| \to 0$ a.e (t,x,ξ). Using (36) we conclude

$$\int_\xi |f^\varepsilon - \mathcal{M}(u^\varepsilon,\xi)| \to 0 \quad \text{a.e. } (t,x) \ .$$

Along a further subsequence, $f^\varepsilon \to \mathcal{M}(u,\xi)$ a.e., and passing to the limit in the kinetic entropies (24) we see that

$$\partial_t \int_\xi |\mathcal{M}(u,\xi) - \mathcal{M}(k,\xi)| + \mathrm{div}_x \int_\xi a(\xi)|\mathcal{M}(u,\xi) - \mathcal{M}(k,\xi|\,d\xi \le 0$$

in $\mathcal{D}'$. The latter inequality is recast in the form (35) by using (a) and the property $\mathrm{sgn}\,(\mathcal{M}(u,\xi) - \mathcal{M}(k,\xi)) = \mathrm{sgn}\,(u-k)$. Since u is an entropy solution, it is unique and the whole family $u^\varepsilon \to u$ in L^p_{loc}, $1 \le p < \infty$. $\square$

3.3 The Connection with the Kinetic Formulation

Consider the scalar conservation law

$$\begin{cases} \partial_t u + \mathrm{div} F(u) = 0 \\ \qquad u(x,0) = u_0(x) \end{cases} \tag{39}$$

with data $u_0 \in L^1 \cap L^\infty$. There are two equivalent notions of solution for this problem: The notion of Kruzhkov entropy solution stating that u is an entropy solution of the initial value problem (39) if

$$\text{ess}\ \lim_{t\to 0} \int |u(x,t) - u_0(x)|\, dx = 0 \tag{40}$$

and u satisfies the entropy conditions

$$\partial_t \eta(u) + \mathrm{div}\, q(u) \le 0 \tag{41}$$

in $\mathcal{D}'$ for all entropy pairs η, q with η convex. (Recall that entropy pairs η, q are required to satisfy $q_i'(u) = \eta'(u) F_i'(u)$ with $i = 1, \dots, d$.)

The second notion is the kinetic formulation of Lions-Perthame-Tadmor [22] and is based on the Maxwellian

$$\mathbb{1}(u,\xi) = \begin{cases} \mathbb{1}_{0<\xi<u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -\mathbb{1}_{u<\xi<0} & \text{if } u < 0 \end{cases} \quad = \quad \frac{1}{2}\big[\mathrm{sgn}\,(u-\xi) + \mathrm{sgn}\,\xi\big]$$

It is equivalent to the notion of Kruzhkov entropy solution and states that u is a solution of (39) if it takes the initial data as in (40) and there exist a positive bounded measure $m = m(t,x,\xi)$ on $\mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{R}_\xi$ so that

$$\partial_t \mathbb{1}(u,\xi) + a(\xi)\cdot \nabla_x \mathbb{1}(u,\xi) = \partial_\xi m \ . \tag{42}$$

Moreover, the measure m is supported on the shocks.

This notion arises naturally as the $\varepsilon \to 0$ limit of the BGK model

$$\begin{aligned} \partial_t f^\varepsilon + a(\xi)\cdot \nabla_x f^\varepsilon &= -\frac{1}{\varepsilon}\big(f^\varepsilon - \mathbb{1}(u^\varepsilon,\xi)\big) \\ f^\varepsilon(0,x,\xi) &= \mathbb{1}(u_0(x),\xi)\,. \end{aligned} \tag{43}$$

The variable $\xi \in \mathbb{R}$ and the model (43) is a special case of (21). The special form of the Maxwellian allows to calculate the kinetic equation that the limiting f satisfies. Following [30,22] we show:

Theorem 5. *As $\varepsilon \to 0$,*

$$u^\varepsilon \to u \quad a.e, \qquad f^\varepsilon \to F = \mathbb{1}(u,\xi) \quad a.e.$$

and F satisfies (42) *for some positive bounded measure m.*

Proof. As before the BGK-model (43) defines an L^1-contraction and one shows $u^\varepsilon \to u$ a.e. Comparisons with the Maxwellians $1\!\!1(V,\xi)$ and $1\!\!1(-V,\xi)$, $V = \sup |u_0(x)|$, give

$$
\begin{gathered}
-1 \le f^\varepsilon \le 1\,, \qquad \mathrm{supp}_\xi f^\varepsilon \subset [-V,V] \\
f^\varepsilon \ge 0 \ \text{ for } \xi > 0\,, \quad f^\varepsilon \le 0 \ \text{ for } \xi < 0 \ .
\end{gathered}
$$

Next introduce m^ε by

$$
\partial_\xi m^\varepsilon = \frac{1}{\varepsilon}\big(1\!\!1(u^\varepsilon,\xi) - f^\varepsilon\big) = \begin{cases} > 0 & \text{for } \xi < u^\varepsilon \\ < 0 & \text{for } \xi > u^\varepsilon \end{cases}
$$

The function

$$
m^\varepsilon = \int_{-\infty}^{\xi} \frac{1}{\varepsilon}\big(1\!\!1(u^\varepsilon,\xi) - f^\varepsilon(\xi)\big)\, d\xi
$$

satisfies $m^\varepsilon(-\infty) = 0$, $m^\varepsilon(+\infty) = 0$ by conservation and $m^\varepsilon > 0$ for $\xi \in \mathbb{R}$.

We write the BGK-model in the form

$$
\partial_t f^\varepsilon + a(\xi)\cdot\nabla_x f^\varepsilon = \partial_\xi m_\varepsilon \ .
$$

We multiply by ξ and integrate over $[0,T]\times\mathbb{R}^d\times\mathbb{R}$. Taking account of the compact support in ξ we obtain

$$
\begin{aligned}
\int_0^T \int_x \int_\xi m^\varepsilon &= -\int_x\int_\xi \xi f^\varepsilon(t,x,\xi)d\xi\, dx + \int_x\int_\xi \xi f_0(x,\xi)d\xi\, dx \\
&\le V\int_x\int_\xi |f^\varepsilon| + V\int_x\int_\xi |f_0| \le C \ .
\end{aligned}
$$

Using the relations

$$
u^\varepsilon \to u \ \text{ a.e.}, \quad f^\varepsilon - 1\!\!1(u^\varepsilon,\xi) \to 0 \ \text{ in } \mathcal{D}'\,, \quad 1\!\!1(u^\varepsilon,\xi) \to 1\!\!1(u,\xi) \ \text{ a.e.}
$$

and the property (along subsequences)

$$
m^\varepsilon \rightharpoonup m \quad \text{weak-}\star \text{ in measures}
$$

we pass to the limit $\varepsilon \to 0$ in $\mathcal{D}'$ to obtain (42). $\square$

Bibliographic remarks. The development of the model (43) is due to Perthame-Tadmor [30], the model (26) is produced in Katsoulakis-Tzavaras [18] as a discrete velocity approximation of the scalar multi-d conservation law. The discrete kinetic version of the BGK-model (28) is proposed in Natalini [26,27] while the continuous variant is developed in Bouchut [4]; see also Bouchut-Guarguaglini-Natalini [5] for a discussion of the diffusive limit. The convergence in the hyperbolic limit is based on the Kruzhkov theory [20]. Early kinetic theory motivated schemes for scalar conservation laws appear in [13,2].

Related issues appear in relaxation limits to scalar one-dimensional conservation laws [26,40]. Applications of fluid-limits can be found in derivations of hydrodynamic limits for stochastic interacting particle systems [31,19].

The kinetic formulation of Lions-Perthame-Tadmor [22] provides a notion of solution for scalar conservation laws equivalent to the Kruzhkov entropy solution [20]. It provides a description of the regularizing effect [22] through the use of the averaging lemma, proofs of uniqueness and error estimates [33], and a perspective to issues of propagation and cancellation of oscillations [34]. It is a rapidly developing subject both in the context of scalar equations, *e.g.* [3,29], but also for systems of two conservation laws, *e.g.* [23,24,16,34].

4 Fluid Limits to Systems of Two Conservation Laws

In this section we discuss certain discrete kinetic models whose fluid limits are systems of two conservation laws in one space dimension. These results are motivated from the theory of relaxation approximations for hyperbolic systems. We begin with a system for the evolution of the kinetic vector variable $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$

$$\begin{aligned} \partial_t f_1 - c\partial_x f_1 &= -\frac{1}{\varepsilon}\big(f_1 - \mathcal{M}_1(u,v)\big) \\ \partial_t f_2 + c\partial_x f_2 &= -\frac{1}{\varepsilon}\big(f_2 - \mathcal{M}_2(u,v)\big) \\ \partial_t f_3 &= -\frac{1}{\varepsilon}\big(f_3 - \mathcal{M}_3(u,v)\big) \end{aligned} \tag{44}$$

where

$$u = f_1 + f_2 + f_3\,, \quad v = cf_1 - cf_2\,, \quad \sigma = c^2 f_1 + c^2 f_2$$

are the first three moments. The system is a discrete kinetic model of BGK type. Under the hypotheses

$$\begin{aligned} \mathcal{M}_1(u,v) + \mathcal{M}_2(u,v) + \mathcal{M}_3(u,v) &= u \\ c\mathcal{M}_1(u,v) - c\mathcal{M}_2(u,v) &= v \end{aligned} \tag{45}$$

it is equipped with two conservation laws

$$u_t - v_x = 0\,, \quad v_t - \sigma_x = 0\ .$$

The Maxwellians are now selected as

$$\mathcal{M}_1 = \frac{g(u)}{2c^2} + \frac{v}{2c}\,, \quad \mathcal{M}_2 = \frac{g(u)}{2c^2} - \frac{v}{2c}\,, \quad \mathcal{M}_3 = u - \frac{g(u)}{c^2}\ . \tag{46}$$

where $g(u)$ is a strictly increasing function with $g(0) = 0$. This choice is consistent with (45) and $g(u) = \mathcal{M}_1(u,v)c^2 + \mathcal{M}_2(u,v)c^2$. The system governing the evolution of the moments is closed and reads

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x \sigma &= 0 \\ \partial_t(\sigma - c^2 u) &= -\frac{1}{\varepsilon}\big(\sigma - g(u)\big) \ . \end{aligned} \tag{47}$$

The kinetic model (44) can be viewed as the system for the evolution of the Riemann invariants

$$f_1 = \frac{\sigma}{2c^2} + \frac{v}{2c} \,, \quad f_2 = \frac{\sigma}{2c^2} - \frac{v}{2c} \,, \quad f_3 = u - \frac{\sigma}{c^2} \,, \tag{48}$$

associated with the hyperbolic operator in (47).

The formal limit of a solution $(u^\varepsilon, v^\varepsilon, \sigma^\varepsilon)$ of the equation (47) is the system of isothermal elasticity

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x g(u) &= 0 \ . \end{aligned} \tag{49}$$

Insight on the nature of the approximation process is obtained via an adaptation of the Chapman-Enskog expansion familiar from the theory of the Boltzmann equation. One seeks to identify the effective response of the relaxation process as it approaches the surface of local equilibria. It is postulated that the relaxing variable σ^ε can be described in an asymptotic expansion that involves *only* the local macroscopic values u^ε, v^ε and their derivatives, *i.e.*

$$\sigma^\varepsilon = g(u^\varepsilon) + \varepsilon S(u^\varepsilon, v^\varepsilon, u^\varepsilon_x, v^\varepsilon_x, \dots) + O(\varepsilon^2) \ .$$

To calculate the form of S, we substitute the expansion in (47),

$$\begin{aligned} \partial_t u^\varepsilon - \partial_x v^\varepsilon &= 0 \\ \partial_t v^\varepsilon - \partial_x g(u^\varepsilon) &= \varepsilon S_x + O(\varepsilon^2) \\ \partial_t(g(u^\varepsilon) - c^2 u^\varepsilon) + O(\varepsilon) &= -S + O(\varepsilon) \,, \end{aligned}$$

whence we obtain

$$S = [c^2 - g_u(u^\varepsilon)] v^\varepsilon_x + O(\varepsilon) \,,$$

and conclude that the effective equations describing the process are

$$\begin{aligned} \partial_t u^\varepsilon - \partial_x v^\varepsilon &= 0 \\ \partial_t v^\varepsilon - \partial_x g(u^\varepsilon) &= \varepsilon \partial_x\big([c^2 - g_u(u^\varepsilon)] v^\varepsilon_x\big) + O(\varepsilon^2) \ . \end{aligned} \tag{50}$$

This is a stable parabolic system when $g_u < c^2$ is satisfied. The formal expansion suggests that the limit of (47) will be the equations (49) provided $0 < g_u < c^2$, a condition stating that the characteristic speeds $\pm\sqrt{g_u}$ of the hyperbolic system (49) lie between (and not in resonance to) the characteristic speed $\pm c, 0$ of the hyperbolic system (47).

We work under the standing hypotheses

$$g(0)=0\,,\quad 0<\gamma\le g_u\le \Gamma<c^2, \tag{h}$$

for some constants γ, Γ. The system (47) may be viewed as a model in viscoelasticity. Motivated from deliberations of consistency of constitutive theories of materials with internal variables with the second law of thermodynamics [12,42], one can check that smooth solutions (u, v, σ) of this viscoelasticity model satisfy the energy dissipation identity

$$\partial_t\Big(\frac{1}{2}v^2+\Psi(u,\sigma-c^2u)\Big)-\partial_x(\sigma v)+\frac{1}{\varepsilon}(u-h^{-1}(\alpha))(\alpha-h(u))\Big|_{\alpha=\sigma-c^2u}=0$$

where

$$\Psi(u,\alpha)=-\int_0^\alpha h^{-1}(\zeta)\,d\zeta+\alpha u+\int_0^u E\xi\,d\xi$$

$\alpha=\sigma-c^2u$, $h(u)=g(u)-c^2u$ and h^{-1} is the inverse function of h. We remark that the associated constitutive theory is compatible with the second law of thermodynamics iff $g_u<c^2$, and that the function $\frac{1}{2}v^2+\Psi$ provides an "entropy" function for the associated relaxation process, which is convex in (v, u, α) iff $g_u>0$.

It follows that under (h) the viscoelasticity system (47) admits global smooth solutions (for smooth data) which satisfy the ε-independent bounds

$$\int_{\mathbb{R}}(u^2+v^2+\sigma^2)\,dx+\frac{1}{\varepsilon C}\int_0^t\int_{\mathbb{R}}\big(\sigma-g(u)\big)^2\,dx\,dt\le C\int_{\mathbb{R}}(u_0^2+v_0^2+\sigma_0^2)\,dx \tag{51}$$

for some C independent of ε and t. This estimate indicates that the natural stability estimate of the problem is in L^2. It is also clear that while this estimate is in a sense the analog of the H-theorem in this simplified context, and while it provides a control of the distance from equilibrium as $\varepsilon\to 0$, it is not sufficient to guarantee strong convergence.

However, as it turns out (47) is endowed with a stronger dissipative structure, analogous to the one associated with viscosity approximations of the equations (49). To this end, note that (47) can be put in the form,

$$\begin{aligned}&\partial_t u-\partial_x v=0\\&\partial_t v-\partial_x g(u)=\partial_x(\sigma-g(u))=\varepsilon(c^2v_{xx}-v_{tt})\,,\end{aligned} \tag{52}$$

of an approximation of (49) via the wave equation.

Lemma 6. *For initial data satisfying*

$$\int_{\mathbb{R}}v_0^2+u_0^2+\sigma_0^2\,dx+\varepsilon^2\int_{\mathbb{R}}u_{0x}^2+v_{0x}^2+\sigma_{0x}^2\,dx\le O(1) \tag{d}$$

and under hypothesis (h) *solutions of* (47) *satisfy the ε independent estimates*

$$\varepsilon\int_0^t\int_{\mathbb{R}}u_x^2+v_x^2+\sigma_x^2\,dx\,dt\le O(1)\ . \tag{53}$$

Proof. From (52) we obtain the natural energy identity

$$\partial_t\Big(\frac{1}{2}v^2 + W(u) + \varepsilon v v_t\Big) - \partial_x\big(vg(u)\big) + \varepsilon(c^2 v_x^2 - v_t^2) = \varepsilon\partial_x(c^2 v v_x)$$

where the stored energy function $W(u) = \int_0^u g(\xi)\,d\xi$. The term $c^2v_x^2 - v_t^2$ is not positive definite. To compensate, we multiply the second equation in (52) by the natural multiplier of the wave equation v_t to obtain

$$\varepsilon^2\partial_t\big(c^2v_x^2 + v_t^2\big) + \varepsilon(2v_t^2 - 2g_u u_x v_t) = 2\varepsilon^2\partial_x(c^2 v_t v_x)\ .$$

Using the identity $a_x b_t - a_t b_x = \partial_t(a_x b) - \partial_x(a_t b)$, we have

$$\begin{aligned} g_u u_x^2 &= u_x\partial_t(v + \varepsilon v_t) - \varepsilon c^2 u_x v_{xx} \\ &= \Big[u_t\partial_x(v + \varepsilon v_t) + \partial_t\big(u_x(v + \varepsilon v_t)\big) - \partial_x\big(u_t(v + \varepsilon v_t)\big)\Big] - \varepsilon\partial_t\big(\frac{1}{2}c^2u_x^2\big)\ , \end{aligned}$$

and, in turn,

$$\begin{aligned} \varepsilon^2\partial_t\Big(\frac{1}{2}c^4u_x^2 - \frac{1}{2}c^2v_x^2\Big) - \varepsilon\partial_t\big(c^2u_x(v + \varepsilon v_t)\big) + \varepsilon(c^2g_u u_x^2 - c^2v_x^2) \\ = -\varepsilon\partial_x\big(c^2u_t(v + \varepsilon v_t)\big)\ . \end{aligned}$$

Adding the identities, we arrive at the strengthened dissipation estimate

$$\begin{aligned} \partial_t\Big(\frac{1}{2}(v + \varepsilon v_t - \varepsilon c^2u_x)^2 + \frac{1}{2}\varepsilon^2(v_t^2 + c^2v_x^2) + W(u)\Big) - \partial_x(vg(u)) \\ + \varepsilon\big[v_t^2 - 2g_u u_x v_t + c^2 g_u u_x^2\big] = \varepsilon^2(c^2v_tv_x)_x\ . \end{aligned}$$

Because of (h) the second term is positive definite:

$$\varepsilon\big[v_t^2 - 2g_u u_x v_t + c^2g_u u_x^2\big] \geq \varepsilon g_u(c^2 - g_u)u_x^2 \geq 0\ .$$

We conclude

$$\begin{aligned} &\int_{\mathbf{R}} \frac{1}{2}(v + \varepsilon v_t - \varepsilon c^2u_x)^2 + \frac{1}{2}\varepsilon^2(v_t^2 + c^2v_x^2) + W(u)\,dx \\ &\qquad + \varepsilon\int_0^t\int_{\mathbf{R}} g_u(c^2 - g_u)u_x^2\,dx\,dt \\ &\qquad \leq \int_{\mathbf{R}} \frac{1}{2}(v_0 + \varepsilon\sigma_{0x} - \varepsilon c^2u_{0x})^2 + \frac{1}{2}\varepsilon^2(\sigma_{0x}^2 + c^2v_{0x}^2) + W(u_0)\,dx \\ &\qquad \leq O(1) \end{aligned}$$

and, due to (h) and (d),

$$\varepsilon\int_0^t\int_{\mathbf{R}} g_u(c^2 - g_u)u_x^2\,dx\,dt \leq O(1)$$

and

$$\varepsilon \int_0^t \int_{\mathbb{R}} \sigma_x^2 \, dx \, dt \leq O(1)$$
$$\varepsilon \int_0^t \int_{\mathbb{R}} v_x^2 \, dx \, dt \leq O(1)$$

which completes the proof of (53). □

We come next to the convergence Theorem.

Theorem 7. *Let $g \in C^3$ satisfy the subcharacteristic condition* (h) *and*

$$g''(u_0) = 0 \text{ and } g''(u) \neq 0 \text{ for } u \neq u_0 \ ,$$
$$g'' \ , \ g''' \in L^2 \cap L^\infty \ .$$

Let $(u^\varepsilon, v^\varepsilon, \sigma^\varepsilon)$ be a family of smooth solutions of (47) *on $\mathbb{R} \times [0,T]$ emanating from smooth initial data subject to the bounds* (d). *Then, along a subsequence if necessary,*

$$u^\varepsilon \to u \ , \quad v^\varepsilon \to v \ , \quad \text{a.e. } (x,t) \text{ and in } L^p_{\mathrm{loc}}(\mathbb{R} \times (0,T)), \text{ for } p < 2 \ ,$$

and (u,v) is a weak solution of (49).

Sketch of Proof. Let $\eta(u,v)$, $q(u,v)$ be an entropy pair for the equations of isothermal elasticity. Using (52) we obtain

$$\begin{aligned}
\partial_t \eta(u^\varepsilon, v^\varepsilon) &+ \partial_x q(u^\varepsilon, v^\varepsilon) \\
&= \eta_v \partial_x (\sigma^\varepsilon - g(u^\varepsilon)) \\
&= \partial_x (\eta_v (\sigma^\varepsilon - g(u^\varepsilon))) - (\eta_{vu} \varepsilon^{\frac{1}{2}} u_x^\varepsilon + \eta_{vv} \varepsilon^{\frac{1}{2}} v_x^\varepsilon) \, \frac{\sigma^\varepsilon - g(u^\varepsilon)}{\varepsilon^{\frac{1}{2}}} \\
&= I_1^\varepsilon + I_2^\varepsilon \ .
\end{aligned}$$

The dissipation measure $\partial_t \eta^\varepsilon + \partial_x q^\varepsilon$ is tested for entropy pairs $\eta - q$ that are uniformly bounded up to second order derivatives. Due to (51) and (53) the term I_1^ε lies in a compact of H^{-1}, the term I_2^ε is uniformly bounded in L^1, and the sum $I_1^\varepsilon + I_2^\varepsilon$ is uniformly bounded in $W^{-1,\infty}$. One concludes from Murat's lemma

$$\partial_t \eta(u^\varepsilon, v^\varepsilon) + \partial_x q(u^\varepsilon, v^\varepsilon) \quad \text{lies in a compact of } H^{-1}_{\mathrm{loc}} \ .$$

Using the L^p theory of compensated compactness developed in [38,36] for the equations of elasticity, one obtains strong convergence $u^\varepsilon \to u$ and $v^\varepsilon \to v$ a.e. (x,t) along a subsequence (see [42] for the details). □

Next we present a second discrete kinetic model that may be treated with techniques of similar flavor. For κ, $\lambda > 0$ consider

$$\begin{aligned} \partial_t f_1 - \kappa \partial_x f_1 &= -\frac{1}{\varepsilon}\big(f_1 - \mathcal{M}_1\big) \\ \partial_t f_2 + \kappa \partial_x f_2 &= -\frac{1}{\varepsilon}\big(f_2 - \mathcal{M}_2\big) \\ \partial_t f_3 - \lambda \partial_x f_3 &= -\frac{1}{\varepsilon}\big(f_3 - \mathcal{M}_3\big) \\ \partial_t f_4 + \lambda \partial_x f_4 &= -\frac{1}{\varepsilon}\big(f_4 - \mathcal{M}_4\big) \end{aligned} \tag{54}$$

where $\mathcal{M}_i = \mathcal{M}_i(u,v)$, $i = 1,\dots,4$, are Maxwellians depending on the "moments"

$$u = \kappa f_1 - \kappa f_2\,, \quad v = \lambda f_3 - \lambda f_4 \ , \tag{55}$$

that are assumed to satisfy

$$u = \kappa\mathcal{M}_1(u,v) - \kappa\mathcal{M}_2(u,v)\,, \quad v = \kappa\mathcal{M}_3(u,v) - \kappa\mathcal{M}_4(u,v) \ . \tag{56}$$

Then (54) is equipped with two conservation laws for u, v. In fact, if we introduce also the moments

$$\kappa^2(f_1 + f_2) = a\,, \quad \lambda^2(f_3 + f_4) = b\,, \tag{57}$$

the moments (u,v,a,b) evolve according to the closed system

$$\begin{aligned} u_t - a_x &= 0 \\ a_t - \kappa^2 u_x &= -\frac{1}{\varepsilon}\big(a - \kappa^2(\mathcal{M}_1 + \mathcal{M}_2)\big) \\ v_t - b_x &= 0 \\ b_t - \lambda^2 v_x &= -\frac{1}{\varepsilon}\big(b - \lambda^2(\mathcal{M}_3 + \mathcal{M}_4)\big)\,. \end{aligned} \tag{58}$$

If the Maxwellians are selected

$$\begin{aligned} \mathcal{M}_1 &= \frac{u}{2\kappa} + \frac{v}{2\kappa^2} \qquad & \mathcal{M}_3 &= \frac{v}{2\lambda} + \frac{g(u)}{2\lambda^2} \\ \mathcal{M}_2 &= -\frac{u}{2\kappa} + \frac{v}{2\kappa^2} \qquad & \mathcal{M}_4 &= -\frac{v}{2\lambda} + \frac{g(u)}{2\lambda^2} \end{aligned}$$

a choice consistent with (56), the system (58) takes the form

$$\begin{aligned} u_t - a_x &= 0 \\ v_t - b_x &= 0 \\ a_t - \kappa^2 u_x &= -\frac{1}{\varepsilon}(a - v) \\ b_t - \lambda^2 v_x &= -\frac{1}{\varepsilon}(b - g(u)) \end{aligned} \tag{59}$$

of a relaxation approximation for the equation of elasticity (49) of the type proposed in Jin-Xin [17]. The convergence of (59) to (49) when $\kappa = \lambda$ is carried out in Serre [37]. The proof utilizes the L^∞ compensated compactness framework of DiPerna [11], but differs in the methodology for controlling the dissipation measure from the one presented above. It is based on the idea of extending entropy pairs of (49) by viewing them as "equilibrium" entropy pairs for the hyperbolic operator in (59), an idea first developed for scalar conservation laws by Chen, Levermore and Liu [7,8]. It is proved in [37] that, remarkably, the system (59) is endowed with invariant regions and its solutions converge to the entropy solutions of (49).

For the model (59) one can also give a convergence proof based on strengthened dissipation estimates (see Gosse-Tzavaras [15]). One proceeds by writing (59) in the form

$$\begin{aligned} u_t - v_x &= \varepsilon\big(\kappa^2 u_{xx} - u_{tt}\big) \\ v_t - g(u)_x &= \varepsilon\big(\lambda^2 v_{xx} - v_{tt}\big) \end{aligned} \tag{60}$$

of an approximation of (49) by two wave equations. It turns out that under the subcharacteristic condition $\min\{\kappa^2, \lambda^2\} > g_u$ the system (60) is endowed with a stronger dissipative structure, similar to the one characteristic of viscosity approximations. This is the key observation allowing to carry out the $\varepsilon \to 0$ limit to the equations of elasticity (see [15]).

Such strong dissipation estimates are known in more general contexts of approximations by wave operators [42]. Consider the hyperbolic system

$$\partial_t u + \partial_x F(u) = 0\,, \quad x \in \mathbb{R},\, t > 0 \tag{61}$$

where $u(x,t)$ takes values in $\mathbb{R}^n$ and assume that it is endowed with a strictly convex entropy $\eta(u)$. Consider the approximation of (61) by a wave operator,

$$\partial_t u + \partial_x F(u) = \varepsilon(A u_{xx} - u_{tt}) \tag{62}$$

where A is a positive definite symmetric $n \times n$ matrix. This can be written in the form of the relaxation approximation

$$\begin{aligned} &\partial_t u - \partial_x v = 0 \\ &\partial_t v - A\partial_x u = -\frac{1}{\varepsilon}\big(v - F(u)\big)\ . \end{aligned} \tag{63}$$

Using the notations $\eta_u := \nabla\eta$, η_{uu} for the Hessian of η, and I for the $n \times n$ identity matrix, we prove:

Proposition 8. *Assume that* (61) *is equipped with a strictly convex entropy* $\eta(u)$ *that satisfies, for some* $\alpha > 0$,

$$\eta_{uu}(u) \le \alpha\, I\,,$$

and suppose that the positive definite, symmetric matrix A satisfies

$$\frac{1}{2}\big(A^T\eta_{uu}(u)+\eta_{uu}(u)A\big)-\alpha {F'}^T(u)F'(u)\geq \nu I \ .$$

Then smooth solutions of (62)*, that decay fast at infinity satisfy the dissipation estimate*

$$\begin{aligned}\int_{I\!R}\eta(u+\varepsilon u_t)+\frac{1}{2}\varepsilon^2\alpha|u_t|^2+\varepsilon^2\alpha u_x\cdot Au_x\,dx&\\ +\int_0^t\int_{I\!R}\varepsilon^3\alpha|u_{tt}-Au_{xx}|^2+\varepsilon\nu|u_x|^2\,dx\,d\tau&\\ \leq\int_{I\!R}\eta(u_0+\varepsilon u_t(0))+c\varepsilon^2|u_t(0)|^2+\varepsilon^2u_{0x}\cdot Au_{0x}\,dx&\end{aligned}$$

where c is a constant independent of ε.

Proof. Taking the inner product of (62) with u_t, we obtain

$$\partial_t\Big(\frac{1}{2}\varepsilon|u_t|^2+\frac{1}{2}\varepsilon u_x\cdot Au_x\Big)+\Big(|u_t|^2+u_t\cdot F'(u)u_x\Big)=\partial_x(\varepsilon u_t\cdot Au_x)$$

Next, taking the inner product with η_u, we arrive at

$$\partial_t\Big(\eta(u)+\varepsilon\eta_u\cdot u_t\Big)+\partial_x q(u)+\varepsilon\Big(\eta_{uu}u_x\cdot Au_x-u_t\cdot\eta_{uu}u_t\Big)=\varepsilon\partial_x(\eta_u\cdot Au_x) \ . \quad (64)$$

We multiply the first identity by $2\alpha\varepsilon$, add the second identity, and use that

$$\eta(u+\varepsilon u_t)=\eta(u)+\varepsilon\eta_u(u)\cdot u_t+\varepsilon u_t\cdot\Big(\int_0^1\int_0^s\eta_{uu}(u+\varepsilon\tau u_t)\,d\tau\,ds\Big)\varepsilon u_t$$

to obtain, after some rearrangements of terms,

$$\begin{aligned}\partial_t\Big(\eta(u+\varepsilon u_t)&+\varepsilon^2u_t\cdot\Big[\frac{1}{2}\alpha I-\int_0^1\int_0^s\eta_{uu}(u+\varepsilon\tau u_t)\,d\tau\,ds\Big]u_t\\ &+\frac{1}{2}\varepsilon^2\alpha|u_t|^2+\varepsilon^2\alpha u_x\cdot Au_x\Big)+\partial_xq(u)+\varepsilon u_t\cdot\big(\alpha I-\eta_{uu}\big)u_t\\ &+\varepsilon\alpha|u_t+F'(u)u_x|^2+\varepsilon u_x\cdot\big(\eta_{uu}A-\alpha {F'}^TF'\big)u_x\\ &=\partial_x\big(\varepsilon\eta_u\cdot Au_x+2\varepsilon^2\alpha u_t\cdot Au_x\big) \ .\end{aligned}$$

In view of the hypotheses

$$u_t\cdot\Big[\frac{1}{2}\alpha I-\int_0^1\int_0^s\eta_{uu}(u+\varepsilon\tau u_t)\,d\tau\,ds\Big]u_t\geq 0\,,$$

and (64) follows. □

Bibliographic remarks. Dissipation induced by damping appears in a variety of subjects from kinetic theory and continuum physics, prime examples being the theory of viscoelasticity and many models in the kinetic theory of gases. There is a prolific literature in the domain of viscoelastiicity (*c.f.* Renardy, Hrusa and Nohel [35]) and in particular related issues appear in studies of weak solutions for conservation laws with memory [10,28,9]. On the realm of relaxation, the importance of the subcharacteristic condition was recognized from the early studies of Liu [25] and Whitham [43]. A general framework for investigating relaxation to processes containing shocks is proposed in Chen-Levermore-Liu [7,8], and the mechanism is exploited in Jin-Xin [17] to introduce a class of nonoscillatory numerical schemes. We refer to Yong [44] for a discussion of stability conditions for general relaxation systems.

The connection between discrete kinetic models and relaxation approximations is exploited in Aregba-Driollet and Natalini [27,1] in order to develop numerical schemes. The problem of constructing entropies for relaxation and kinetic systems can be systematically addressed by considerations motivated by either continuum physics [12,41] as well as by kinetic theory [4]. The existence of strong dissipative estimates for certain relaxation models is noted in [42]. Convergence results to weak solutions of systems of two conservation laws can be found in Tzavaras [42], Serre [37] and Gosse-Tzavaras [15], and in Slemrod-Tzavaras [39] for self-similar limits of the Broadwell system.

5 Appendix: An Estimate of S.N. Kruzhkov

In this Appendix we consider the conservation law

$$\partial_t u + \mathrm{div} f = \mu \Delta g \tag{65}$$

and we prove that an L^1-modulus of continuity for the functions u, f and g in the variable x induces a modulus of continuity in t of the function u. This idea was introduced in one of the central lemmas of the celebrated work of Kruzhkov [20]. The presented version indicates that there is no loss of modulus of continuity in t.

In what follows we use the notation

$$\omega_w(h) = \sup_{|y|<h} \int_{\mathbf{R}^d} |w(x+y) - w(x)|\, dx \tag{66}$$

for the L^1-modulus of continuity of the function w. Also, for $k, h > 0$ and $t > 0$ we let M_f denote the quantity

$$\begin{aligned} M_f(k,h) &= \int_t^{t+k} \sup_{|y|<h} \int_{\mathbf{R}^d} |f(x+y,\tau) - f(x,\tau)|\, dx\, d\tau \\ &= \int_t^{t+k} \omega_f(h,\tau)\, d\tau \end{aligned} \tag{67}$$

associated to the function f. We prove the following lemma.

Lemma 9. *Let u, g and f^j, $j = 1, \dots, d$, in $L^1((0,T) \times I\!R^d)$ satisfy (65) in the sense of distributions. If*

$$\omega_u(\varepsilon) = \sup_{|z| \le 1} \int_{I\!R^d} |u(x + \varepsilon z, t) - u(x,t)| \, dx \tag{68}$$

is the L^1-modulus of continuity of the function u then for $t > 0$, $k > 0$ (with $t + k < T$) and any $\varepsilon > 0$ we have

$$\int_{I\!R^d} |u(x, t+k) - u(x,t)| \, dx \le C\Big(\omega_u(\varepsilon) + M_f(k, \varepsilon) \frac{1}{\varepsilon} + \mu M_g(k, \varepsilon) \frac{1}{\varepsilon^2} \Big) \tag{69}$$

and

$$\begin{aligned} \int_{I\!R^d} |u(x, t+k) - u(x,t)| \, dx \le C \min_{\varepsilon, k > 0} \Big(\omega_u(\varepsilon, t) + k \frac{1}{\varepsilon} \sup_{t \le \tau \le t+k} \omega_f(\varepsilon, \tau) \\ + \mu k \frac{1}{\varepsilon^2} \sup_{t \le \tau \le t+k} \omega_g(\varepsilon, \tau) \Big) \end{aligned} \tag{70}$$

Proof. From the weak formulation of (65)

$$- \int \int u \varphi_t + f \cdot \nabla \varphi + \mu g \Delta \varphi \, dx \, dt = 0 \, , \qquad \varphi \in C_c^\infty((0,T) \times \mathbb{R}^d) \, ,$$

we readily obtain using Lebesgue's theorem that

$$\begin{aligned} \int \big(u(x, t+k) - u(x,t) \big) \psi(x) \, dx = \int_t^{t+k} \int f(x, \tau) \cdot \nabla \psi(x) \\ + \mu g(x, \tau) \Delta \psi(x) \, dx \, d\tau \end{aligned} \tag{71}$$

for any $0 < k < T - t$ and for $\psi = \psi(x) \in C^2(\mathbb{R}^d)$.

Step 1. We proceed to estimate the right hand side of (71). In what follows we use extensively mollifiers. Let ρ_1 be a positive symmetric kernel supported on $(0,1)$ with $\int_{\mathbb{R}} \rho_1 = 1$ and consider the positive symmetric kernel on $[0,1]^d$ defined by $\rho(x) = \rho_1(x_1) \dots \rho_1(x_d)$. The kernel ρ generates a sequence of mollifiers $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho(\frac{x}{\varepsilon})$ which satisfy

$$\rho_\varepsilon \in C_c^\infty(\mathbb{R}^d) \, , \quad \mathrm{supp} \rho_\varepsilon \subset B_\varepsilon \, , \quad \int \rho_\varepsilon \, dx = 1 \, , \quad \rho_\varepsilon \ge 0 \ \ .$$

Let $\rho_\varepsilon * \psi$ denote the convolution of ρ_ε and ψ.

Consider the splitting of the integral

$$\begin{aligned} &\int_t^{t+k} \int f(x, \tau) \cdot \nabla \psi(x) \, dx \, d\tau \\ &\quad = \int_t^{t+k} \int f(x, \tau) \cdot \Big(\nabla \psi(x) - \nabla(\rho_\varepsilon * \psi)(x) \Big) \, dx \, d\tau \\ &\qquad + \int_t^{t+k} \int f(x, \tau) \cdot \nabla(\rho_\varepsilon * \psi)(x) \, dx \, d\tau \\ &\quad = I_1 + I_2 \ \ . \end{aligned}$$

The term I_1 is rewritten as

$$\begin{aligned}I_1 &= \int_t^{t+k} \int f(x,\tau)\cdot\Big(\nabla\psi(x) - \nabla\int_z \rho(z)\psi(x-\varepsilon z)\,dz\Big)\,dx\,d\tau \\ &= \int_t^{t+k}\int_x\int_z \Big(f(x,\tau)-f(x+\varepsilon z,\tau)\Big)\cdot\nabla\psi(x)\rho(z)\,dx\,dz\,d\tau\end{aligned}$$

and is estimated by

$$\begin{aligned}|I_1| &\le \Big(\int_t^{t+k}\sup_{|z|\le 1}\int_x \big|f(x,\tau)-f(x+\varepsilon z,\tau)\big|\,dx\,d\tau\Big)\sup_x|\nabla\psi(x)|\int_z\rho\,dz \\ &= M_f(k,\varepsilon)\,\sup_x|\nabla\psi(x)|\end{aligned} \tag{72}$$

Using the property $\int_z \nabla_z\rho(z)\,dz = 0$ the term I_2 is rewritten as

$$\begin{aligned}I_2 &= \int_t^{t+k}\int_x f(x,\tau)\cdot\frac{1}{\varepsilon}\int_z\nabla_z\rho(z)\psi(x-\varepsilon z)\,dz\,dx\,d\tau \\ &= \int_t^{t+k}\int_x f(x,\tau)\cdot\frac{1}{\varepsilon}\Big[\int_z\nabla_z\rho(z)\psi(x-\varepsilon z)\,dz - \int_z\nabla_z\rho(z)\psi(x)\,dz\Big]\,dx\,d\tau \\ &= \int_t^{t+k}\int_x\int_z\Big(f(x+\varepsilon z,\tau)-f(x,\tau)\Big)\cdot\frac{1}{\varepsilon}\nabla_z\rho(z)\psi(x)\,dz\,dx\,d\tau\end{aligned}$$

which in turn yields

$$\begin{aligned}|I_2| &\le \Big(\int_t^{t+k}\sup_{|z|\le 1}\int_x\big|f(x,\tau)-f(x+\varepsilon z,\tau)\big|\,dx\,d\tau\Big)\frac{1}{\varepsilon}\sup_x|\psi(x)|\Big(\int_z|\nabla\rho|\,dz\Big) \\ &\le CM_f(k,\varepsilon)\,\frac{1}{\varepsilon}\sup_x|\psi(x)|\quad .\end{aligned} \tag{73}$$

In a similar fashion the last integral in (71) is split in two terms

$$\begin{aligned}\int_t^{t+k}\int g(x,\tau)\Delta\psi(x)\,dx\,d\tau &= \int_t^{t+k}\int g(x,\tau)\Big(\Delta\psi(x)-\Delta(\rho_\varepsilon*\psi)(x)\Big)\,dx\,d\tau \\ &\quad + \int_t^{t+k}\int g(x,\tau)\Delta(\rho_\varepsilon*\psi)(x)\,dx\,d\tau \\ &= J_1 + J_2\quad .\end{aligned}$$

The terms J_1 and J_2 are estimated by similar arguments as the terms I_1 and I_2 using the property $\int_z \Delta\rho(z)\,dz = 0$. This leads to the bounds

$$|J_1| \le \Big(\int_t^{t+k}\sup_{|z|\le 1}\int_x\big|g(x,\tau)-g(x+\varepsilon z,\tau)\big|\,dx\,d\tau\Big)\sup_x|\Delta\psi(x)| \tag{74}$$

$$|J_2| \le C\Big(\int_t^{t+k}\sup_{|z|\le 1}\int_x\big|g(x,\tau)-g(x+\varepsilon z,\tau)\big|\,dx\,d\tau\Big)\frac{1}{\varepsilon^2}\sup_x|\psi(x)| \tag{75}$$

Finally, combining (71) with (72), (73), (74) and (75) we have

$$\Big| \int \big(u(x,t+k) - u(x,t)\big)\psi(x)\,dx \Big| \\ \le M_f(k,\varepsilon)\Big(\frac{C}{\varepsilon}\sup_x |\psi(x)| + \sup_x |\nabla\psi(x)|\Big) \\ + \mu M_g(k,\varepsilon)\Big(\frac{C}{\varepsilon^2}\sup_x |\psi(x)| + \sup_x |\Delta\psi(x)|\Big) \ . \tag{76}$$

Step 2. Set now

$$w(x) = u(x,t+k) - u(x,t)\,, \quad v(x) = \operatorname{sgn} w(x) = \operatorname{sgn}\big(u(x,t+k) - u(x,t)\big)\,,$$

and consider the choice $\psi = \rho_\varepsilon * v$. Since v is bounded we have

$$\sup_x |\psi(x)| \le 1\,, \quad \sup_x |\nabla\psi(x)| \le \frac{C}{\varepsilon}\,, \quad \sup_x |\Delta\psi(x)| \le \frac{C}{\varepsilon^2} \ .$$

For $\psi = \rho_\varepsilon * v$ the estimate (76) gives

$$\Big| \int w(x)\big(\rho_\varepsilon * \operatorname{sgn} w\big)(x)\,dx \Big| \le \frac{C}{\varepsilon}M_f(k,\varepsilon) + \mu\frac{C}{\varepsilon^2}M_g(k,\varepsilon) \ . \tag{77}$$

On the other hand the identity

$$\int |w(x)| - w(x)(\rho_\varepsilon * \operatorname{sgn} w)(x)\,dx \\ = \int_x w(x)\operatorname{sgn} w(x) - w(x)\Big(\int_z \rho(z)\operatorname{sgn} w(x-\varepsilon z)\,dz\Big)\,dx \\ = \int_x\int_z \big(w(x-\varepsilon z) - w(x)\big)\operatorname{sgn} w(x-\varepsilon z)\rho(z)\,dz\,dx$$

yields the estimate

$$\Big| \int |w(x)| - w(x)(\rho_\varepsilon * \operatorname{sgn} w)(x)\,dx \Big| \\ \le \sup_{|z|\le 1}\int_x |w(x-\varepsilon z) - w(x)|\,dx \le 2\omega_u(\varepsilon,t) \ . \tag{78}$$

Combining (77) and (78) we obtain the bound

$$\int_{\mathbb{R}^d} |w(x)|\,dx \le C\Big(\omega_u(\varepsilon) + M_f(k,\varepsilon)\frac{1}{\varepsilon} + \mu M_g(k,\varepsilon)\frac{1}{\varepsilon^2}\Big)$$

which is precisely (69). Then (70) follows from the estimation

$$M_f(k,\varepsilon;t) \le k \sup_{t\le\tau\le t+k} \omega_f(\varepsilon,\tau) \ .$$

□

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