Kinetic decomposition of approximate solutions to conservation laws : application to relaxation and diffusion-dispersion approximations *

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1 Introduction

For scalar multi-dimensional conservation laws

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = 0\,, \qquad x \in \mathbb{R}^d,\, t \in \mathbb{R}^+. \tag{1.1}$$

there are available two equivalent notions for weak solutions: the Kruzhkov entropy solution [11], stating that u satisfies the entropy inequalities

$$\partial_t \eta(u) + \mathrm{div} q(u) \le 0 \quad \text{in } \mathcal{D}'\,, \tag{1.2}$$

for any entropy pair $\eta - q$ with η convex, and the kinetic formulation of Lions-Perthame-Tadmor [12]. Both concepts lead to uniqueness, stability theorems and error estimates for approximate entropy solutions [11, 15].

Starting with Tartar [21], entropy pairs are used to determine compactness for approximate solutions to (1.1). The compactness of a given family $\{u^\varepsilon\}$ of approximate solutions bounded in some L^p-norm ($p > 1$) appears to be determined by compactness of the entropy dissipation measure in the sense

$$\partial_t \eta(u^\varepsilon) + \mathrm{div} q(u^\varepsilon) \quad \text{is precompact in } H^{-1}_{loc,x,t}\,. \tag{1.3}$$

This has been proved in one-space dimension in both the L^∞ and L^p stability settings by Tartar [21] and Schonbek [19] (see [18] for a simplified proof using singular

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entropies and [22] for an analysis of the compensated compactness bracket in multi-d). Tartar's framework is quite versatile and applies even to approximations that do not yield entropy solutions. The goal of this article is to show how the kinetic formulation compactness framework of Lions-Perthame-Tadmor [12] can be easily adapted to analyze the structure (1.3) in multi-d.

For multi-dimensional conservation laws, convergence of approximate solutions is usually deduced by using a framework of DiPerna [4] and Szepessy [20]. The argument hinges on showing that a Young-measure solution (with certain regularity in time) that satisfies (1.2) for all convex η and is a Dirac mass at $t = 0$ is in fact a regular weak solution. It yields compactness for bounded families of approximate entropy solutions, *i.e.* approximate solutions $\{u^\varepsilon\}$ that satisfy the dissipation structure

$$\partial_t \eta(u^\varepsilon) + \mathrm{div} q(u^\varepsilon) \leq \mathcal{P}^\varepsilon(u^\varepsilon) \tag{1.4}$$

with $\mathcal{P}^\varepsilon(u^\varepsilon) \to 0$ in $\mathcal{D}'$ as $\varepsilon \to 0$. The framework (1.4) is naturally associated with monotone approximations that produce entropy solutions. By contrast, the framework (1.3) is quite general and even encompasses approximations that may produce non-entropic solutions, *i.e.* solutions that do not decrease all convex entropies, like the non-classical shocks in [6].

An alternative compactness framework is proposed in Lions-Perthame-Tadmor [12] by means of the kinetic formulation and averaging lemmas (*e.g.* [16, 17]). The framework in [12] is still developed for approximations that satisfy (1.4). Nevertheless, as we will see, it can be easily adapted to apply to the structure (1.3). This is achieved by first transforming the entropy dissipation structure of the problem at hand to an approximate transport equation. The process uses duality and we call it kinetic decomposition. It results to an approximate transport equation where the right hand side consists of "lower order terms"; then the averaging lemma of Perthame-Souganidis [17] yields compactness of moments in L^p spaces.

We pursue this approach in two examples: First, the relaxation approximation of scalar conservation laws

$$\begin{cases} \partial_t u + \sum_{j=1}^d \partial_{x_j} v_j &= 0 \\ \partial_t v_i + A_i^2 \partial_{x_i} u &= -\frac{1}{\varepsilon}(v_i - F_i(u)) \qquad i = 1, \cdots, d, \end{cases} \tag{1.5}$$

proposed in Jin-Xin [8]. Here $(u^\varepsilon, v^\varepsilon) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^d$ is the solution of (1.5) and the constants A_i satisfy the subcharacteristic condition

$$A_i - |F_i'(u)| > 0 \qquad i = 1, \cdots, d, \quad u \in \mathbb{R}. \tag{1.6}$$

We refer to [9, 14, 1] for convergence results of other (diagonalizable) L^1-contractive relaxation systems. In 1-d the system (1.5) is diagonalizable and L^1-contractive [13], in multi-d it is no longer diagonalizable [14, Remark 2.4]. We work in an L^2-stability framework, and use certain strong dissipation estimates from [23] valid under (1.6). The entropy production is turned into a kinetic form using duality (see section 2) and results to an approximate transport equation,

$$\partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon = \sum_{j=1}^{d} \partial_{x_j} \left(\bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon \right) + \partial_t \left(\bar{g}_0^\varepsilon + \partial_\xi g_0^\varepsilon \right) + \partial_\xi k^\varepsilon\,, \tag{1.7}$$

for the function $\chi^\varepsilon = \mathbb{1}(u^\varepsilon(x,t), \xi)$, where

$$\mathbb{1}(u,\xi) = \begin{cases} \mathbb{1}_{0<\xi<u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -\mathbb{1}_{u<\xi<0} & \text{if } u < 0 \end{cases} . \tag{1.8}$$

is the usual Maxwellian associated with the kinetic formulation of scalar conservation laws, $\bar{g}_i^\varepsilon$, $g_i^\varepsilon \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and k^ε is uniformly bounded in measures. Convergence is then obtained via the averaging lemma in [17]. In the limit $\varepsilon \to 0$, $\chi^\varepsilon \to \mathbb{1}(u, \xi) =: \chi$ which satisfies

$$\partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_\xi k \qquad \text{in} \quad D'_{x,t,\xi}, \tag{1.9}$$

with k a bounded measure. It is not clear whether the limit measure k is positive in multi-d (see Remark 4 for a discussion). In the one-d case, an alternative kinetic decomposition may be effected using extensions of entropies, which leads in the $\varepsilon \to 0$ limit to the kinetic formulation with k a positive measure (see section 2.2). In the multi-d case, entropies are in general non-extensible and this process fails.

The second example is the diffusion-dispersion approximation

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = \varepsilon \Delta u + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u\,, \tag{1.10}$$

of scalar conservation laws, with F_j globally Lipschitz. Convergence of (1.10) has been established for $\delta = O(\varepsilon^2)$ in the 1-d case by Schonbek [19], and for $\delta = o(\varepsilon^2)$ in the multi-d case by Kondo-LeFloch [10] (see also [3]). The analysis of [10] uses the framework of [4, 20] and is based on the dissipative structure (1.4), valid only on the range $\delta = o(\varepsilon^2)$. We consider here a family of smooth solutions $\{u^\varepsilon\}$ to (1.10) in the range $\delta = O(\varepsilon^2)$ and show in section 3 that $\chi^\varepsilon = \mathbb{1}(u^\varepsilon, \xi)$ satisfies the transport equation

$$\partial_t \chi^\varepsilon + a(\xi)\cdot\nabla\chi^\varepsilon = \sum_{j=1}^{d} \partial_{x_j}\left(\bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon\right) + \partial_\xi k^\varepsilon \quad \text{in } D'_{x,t,\xi}, \tag{1.11}$$

with $\bar{g}_j^\varepsilon$, g_j^ε, k^ε as before. The convergence $u_\varepsilon \to u$ almost everywhere then follows from the averaging lemma.

As $\varepsilon \to 0$, $\chi^\varepsilon \to \chi = \mathbb{1}(u,\xi)$ a.e. and χ satisfies the transport equation (1.9). It turns out that if $\delta = o(\varepsilon^2)$ the bounded measure k is positive and u an entropy solution. By contrast, on the range $\delta \sim \varepsilon^2$ the measure k might in general be nonpositive. We note that for 1-d scalar conservation laws, when the flux is not genuinely nonlinear in the sense of Lax, diffusive-dispersive approximations can produce in the range $\delta \sim \varepsilon^2$ nonclassical shocks that dissipate the energy but do not dissipate all the convex entropies (see Hayes-Lefloch [6] and [5] for a survey of this subject). The form (1.9) with k non-positive retains in the limit information on the approximation process, and may prove useful in analyzing limit processes that yield non-classical shocks.

2 Relaxation Approximation

Consider the relaxation approximation of the scalar conservation law

$$\left\{\begin{array}{lcl} \partial_t u + \displaystyle\sum_{j=1}^{d} \partial_{x_j} v_j & = & 0 \\ \\ \partial_t v_i + A_i^2 \partial_{x_i} u & = & -\frac{1}{\varepsilon}(v_i - F_i(u)) \qquad i = 1,\cdots,d. \end{array}\right. \tag{2.1}$$

where $u, v_i : \mathbb{R}_x^d \times \mathbb{R}_t^+ \longrightarrow \mathbb{R}$, and the flux $F = (F_1(u),\ldots,F_d(u))$ is a smooth function. Let $a_i = F_i'$ and assume that the constants A_i satisfy the subcharacteristic condition

$$A_i > |a_i(u)| \qquad i = 1,\cdots,d\,, \quad u \in \mathbb{R}. \tag{2.2}$$

In particular, this assumption implies F_i are globally Lipschitz.

Existence of solutions for (2.1) follows from general considerations. The system (2.1) can be put into a symmetric hyperbolic form

$$B_0\partial_t U + \sum_{j=1}^{d} B_j \partial_{x_j} U = Q(U)$$

where B_0 is positive definite symmetric and B_j, $j = 1, \ldots, d$, are symmetric. The local existence and uniqueness is well known for linear or semi-linear symmetric hyperbolic systems. Since in our case $Q(U)$ is globally Lipschitz continuous, it is easy to obtain global existence (for example by a direct contraction mapping argument). In the sequel we need solutions having derivatives in L^2; this is achieved by assuming the same smoothness for the initial data.

2.1 Convergence in the multi-d case

The objective is to show that solutions $(u^\varepsilon, v^\varepsilon)$ of the relaxation system (2.1) converge as $\varepsilon \to 0$ toward a weak solution (u, v), $v = F(u)$, of the scalar conservation law

$$\partial_t u + \mathrm{div} F(u) = 0\,, \tag{2.3}$$

In preparation, recall that η-q with $q = (q_j(u))_{j=1,\ldots d}$ is an entropy-entropy flux pair if $q_j' = a_j \eta'$. Such pairs describe the nonlinear structure of (2.3) and are represented in terms of the kernel $\mathbb{1}(u, \xi)$ by the formulas

$$\begin{aligned} \eta(u) - \eta(0) &= \int_\xi \mathbb{1}(u,\xi)\eta'(\xi)d\xi\,, \\ q_j(u) - q_j(0) &= \int_\xi \mathbb{1}(u,\xi) a_j(\xi)\eta'(\xi)d\xi\,. \end{aligned} \tag{2.4}$$

where

$$\mathbb{1}_u(\xi) = \mathbb{1}(u,\xi) = \begin{cases} \mathbb{1}_{0<\xi<u} & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -\mathbb{1}_{u<\xi<0} & \text{if } u < 0 \end{cases}\,. \tag{2.5}$$

The system (2.1) can be written in the form of regularization by a wave operator

$$\begin{aligned} \partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) &= \sum_{j=1}^{d} \partial_{x_j}(F_j(u) - v_j) \\ &= \varepsilon\Big(\sum_{j=1}^{d} A_j^2 \partial_{x_j x_j} u - \partial_{tt} u\Big)\,. \end{aligned} \tag{2.6}$$

We start with a dissipation estimate for solutions of (2.1). Such estimates hold for general classes of systems regularized by wave operators (see [23]).

Proposition 1 *If* (2.2) *holds, then solutions* (u, v) *of (2.1) satisfy the dissipation identity*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2}(u+\varepsilon\partial_t u)^2 + \frac{1}{2}\varepsilon^2(\partial_t u)^2 + \varepsilon^2 \sum_{j=1}^d A_j^2(\partial_{x_j}u)^2 dx \\
(2.7) \qquad &+ \int_0^t \int_{\mathbb{R}^d} \varepsilon^3 |\partial_{tt}u - \sum_{j=1}^d A_j^2 \partial_{x_j x_j} u|^2 + \varepsilon \sum_{j=1}^d \big(A_j^2 - F_j'(u)^2\big)(\partial_{x_j}u)^2 dx ds \\
&\le \int_{\mathbb{R}^d} \frac{1}{2}(u(0)+\varepsilon\partial_t u(0))^2 + \frac{1}{2}\varepsilon^2(\partial_t u(0))^2 + \varepsilon^2 \sum_{j=1}^d A_j^2 (\partial_{x_j} u(0))^2 dx.
\end{aligned}$$

Proof. Let $u = u^\varepsilon(x,t)$ be the solution of (2.1). Multiplying (2.6) by u, we obtain

$$\begin{aligned}
&\partial_t(\frac{1}{2}u^2 + \varepsilon u u_t) + \sum_{j=1}^d \partial_{x_j} Q_j(u) \\
(2.8) \qquad &\qquad + \varepsilon(\sum_{j=1}^d A_j^2 (\partial_{x_j}u)^2 - |u_t|^2) = \varepsilon \sum_{j=1}^d \partial_{x_j}(A_j^2 u u_{x_j}),
\end{aligned}$$

where $Q_j'(u) = uF_j'(u)$. Second, we multiply (2.6) by u_t, and obtain

$$\begin{aligned}
&\partial_t(\frac{1}{2}\varepsilon {u_t}^2 + \frac{1}{2}\varepsilon \sum_{j=1}^d A_j^2 (\partial_{x_j}u)^2) \\
(2.9) \qquad &\qquad + u_t(u_t + \mathrm{div} F(u)) = \varepsilon \sum_{j=1}^d \partial_{x_j}(A_j^2 u_t u_{x_j}).
\end{aligned}$$

Combining (2.8) and (2.9), we deduce

$$\begin{aligned}
&\partial_t(\frac{1}{2}(u+\varepsilon u_t)^2 + \frac{1}{2}\varepsilon^2 (u_t)^2 + \varepsilon^2 \sum_{j=1}^d A_j^2 (\partial_{x_j}u)^2) + \sum_{j=1}^d \partial_{x_j} Q_j(u) \\
(2.10) \qquad &+ \quad \varepsilon |u_t + \mathrm{div} F(u)|^2 + \varepsilon \sum_{j=1}^d (A_j^2 - F_j'(u)^2)(\partial_{x_j}u)^2 \\
&= \quad \sum_{j=1}^d \partial_{x_j}(\varepsilon A_j^2 u u_{x_j} + 2\varepsilon^2 A_j^2 u_t u_{x_j}).
\end{aligned}$$

The subcharacteristic condition (2.2) yields the identity (2.7). □

In the sequel, we use the notation $u^\varepsilon \in_b X$ to denote sequences that are uniformly bounded in the norm of the Banach space X.

Corollary 2 *If* (2.2) *holds and the initial data satisfy the uniform bound*

$$\|u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon\|\partial_t u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon \sum_{j=1}^d \|\partial_{x_j} u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} = O(1)\,, \tag{2.11}$$

then the solutions of (2.1) *satisfy the uniform estimates*

$$u^\varepsilon(x,t) \quad \in_b \quad L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d)) \tag{2.12}$$

$$\varepsilon \sum_{j=1}^d (\partial_{x_j} u^\varepsilon(x,t))^2 \quad \in_b \quad L^1(\mathbb{R}^d \times \mathbb{R}^+) \tag{2.13}$$

$$\varepsilon(\partial_t u^\varepsilon(x,t))^2 \quad \in_b \quad L^1(\mathbb{R}^d \times \mathbb{R}^+) \tag{2.14}$$

Proof. Equation (2.7) directly gives the bounds for u and $\varepsilon^{\frac{1}{2}} u_{x_j}$. The bound of $\varepsilon^{\frac{1}{2}} u_t$ follows from (2.7) and

$$\varepsilon |u_t|^2 = \varepsilon\Big(-\operatorname{div} F(u) + \varepsilon\big(\sum_{j=1}^d A_j^2 \partial_{x_j x_j} u - \partial_{tt} u\big)\Big)^2 \in_b L^1(\mathbb{R}^d \times \mathbb{R}^+)$$

□

Remark 1 For the 1-d variant of (2.1) there are available L^∞ estimates (see [13]). These estimates are derived from diagonalized forms of the 1-d version of (2.1), and remain true for other multi-d relaxation systems of diagonal form (see [9], [14]). The multi-d version of (2.1) is not diagonalizable and L^∞ estimates are not available. Hence, we work on the natural L^2 stability-framework suggested by (2.7).

Remark 2 Let $\mathbb{1}(u,\xi)$ be the entropy kernel. Since $u^\varepsilon \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ we have for K compact subset of $\mathbb{R}^d \times \mathbb{R}^+$

$$\int_K \Big(\int_\xi |\mathbb{1}(u^\varepsilon, \xi)| d\xi \Big)^2 dx dt = \int_K |u^\varepsilon|^2 dx dt \le C$$

and thus $\mathbb{1}(u^\varepsilon, \xi) \in L^2_{loc}(\mathbb{R} \times \mathbb{R}^+; L^1(\mathbb{R}))$.

In the sequel we use the limiting case of the averaging lemma proved in Perthame-Souganidis [17], see also [16] :

Theorem 3 *Let* $\{f_n\}, \{g_{i,n}\}$ *be two sequences of solutions to the transport equation*

$$\partial_t f_n + a(\xi)\cdot \nabla_x f_n = \partial_t \partial_\xi^k g_{0,n} + \sum_{i=1}^d \partial_{x_i} \partial_\xi^k g_{i,n} \tag{2.15}$$

where $k \in N$. *Assume that* $a(\xi) \in C^\infty(\mathbb{R})$ *satisfies the non-degeneracy condition: for* $R > 0$

$$\omega(\beta) = \sup_{\alpha\in\mathbb{R},\omega\in S^{d-1}} \int_{\{|\xi|\le R\}} \Big(|\alpha + \frac{a(\xi)\cdot\omega}{\beta}|^2 + 1\Big)^{-1} d\xi \to 0\,, \qquad \text{as } \beta \to 0. \tag{2.16}$$

If $\{f_n\}$ *is bounded in* $L^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, *for some* $1 < q < \infty$, *and* $\{g_{i,n}\}$ *is precompact in* $L^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, *then the average*

$$\int_{\mathbb{R}} \psi(\xi) f_n(t,x,\xi)d\xi \qquad \text{is precompact in } L^q(\mathbb{R}^d \times \mathbb{R}^+)\,,$$

for any $\psi \in C_c^\infty(\mathbb{R})$.

Remark 3
1. The non-degeneracy condition (2.16) is equivalent to for all $R > 0$

$$\text{meas}\{\xi \in B_R \mid \alpha + a(\xi)\cdot\omega = 0\} = 0, \quad \forall \alpha \in \mathbb{R},\ \omega \in S^{d-1}, \tag{2.17}$$

where $B_R = \{|\xi| \le R\}$. The condition (2.17) can be interpreted geometrically, and means that the curve $\xi \mapsto a(\xi)\cdot\omega + \alpha$ is not locally contained in any hyperplane.
2. An assumption on the behavior of $a(\xi)$ is necessary; there would no improvement of regularity in the case $a(\xi) = a = const$.
3. By using cut-off functions, it is easy to show a variant of theorem 3 stating that under the same hypotheses if $\{f_n\}$ is bounded in $L^q_{loc}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and $\{g_{i,n}\}$ are precompact in $L^q_{loc}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ then the averages are precompact in $L^q_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$ for any $\psi \in C_c^\infty(\mathbb{R})$.

Now we state the main theorem of this section.

Theorem 4 *Assume that* F_i *are globally Lipschitz functions that satisfy the genuine-nonlinearity condition (2.16)(or (2.17)).*
Let A_i *be selected so that (2.2) holds, and let* u^ε *be a family of solutions to (2.1) generated by data subject to the uniform bounds* (2.11).

Then, along a subsequence if necessary, u^ε *converges to* u *in* $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$, $1 < p < 2$, *and* u *is a weak solution of (2.3).*

Proof. Let $(u^\varepsilon, v^\varepsilon)$ be a family of solutions to (2.1). The proof proceeds in three steps.

Step 1. Let $\eta - q$ be an entropy-entropy flux pair. Multiplying (2.6) by $\eta'(u^\varepsilon)$ and rearranging the terms we compute the entropy dissipation

$$\begin{aligned}\partial_t\eta(u^\varepsilon) + \mathrm{div} q(u^\varepsilon) &= \varepsilon(\sum_{j=1}^{d} A_j^2 \partial_{x_j x_j}\eta(u^\varepsilon)) - \varepsilon\partial_{tt}\eta(u^\varepsilon)\\ &- \varepsilon\eta''(u^\varepsilon)(\sum_{j=1}^{d} A_j^2(\partial_{x_j}u^\varepsilon)^2) + \varepsilon\eta''(u^\varepsilon)(\partial_t u^\varepsilon)^2.\end{aligned} \tag{2.18}$$

Fix a test function $\varphi(x,t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and regard $\eta'(\xi) \in C_c^\infty(\mathbb{R})$ as a test function in velocity space. By (2.4), we may write

$$\begin{aligned}- \int_{x,t,\xi} & \big[\mathbb{1}(u^\varepsilon,\xi)\eta'(\xi)\big]\varphi_t(x,t) + \Big[\sum_{j=1}^{d} a_j(\xi)\mathbb{1}(u^\varepsilon,\xi)\eta'(\xi)\Big]\varphi_{x_j}(x,t)\, d\xi dx dt\\ = & \int_{x,t,\xi} \varepsilon\mathbb{1}(u^\varepsilon,\xi)\eta'(\xi)\Big[\sum_{j=1}^{d} A_j^2\varphi_{x_j x_j}(x,t) - \varphi_{tt}(x,t))\Big]\, d\xi dx dt\\ & - \int_{x,t}\Big(\varepsilon\sum_{j=1}^{d} A_j^2(\partial_{x_j}u^\varepsilon)^2 - \varepsilon(\partial_t u^\varepsilon)^2\Big)\,\eta''(u^\varepsilon(x,t))\,\varphi(x,t)\, dx dt\,,\end{aligned} \tag{2.19}$$

which is viewed as describing the action on tensor products $\varphi \otimes \eta'$.

We proceed to interpret (2.19) as an equation in $\mathcal{D}'_{x,t,\xi}$. Let

$$\begin{aligned}\chi^\varepsilon(x,t,\xi) &= \mathbb{1}(u^\varepsilon,\xi)\\ G^\varepsilon(x,t) &= \varepsilon\Big(\sum_{j=1}^{d} A_j^2(\partial_{x_j}u^\varepsilon)^2 - (\partial_t u^\varepsilon)^2\Big)\,.\end{aligned} \tag{2.20}$$

$G^\varepsilon \in_b L^1(\mathbb{R}^d \times \mathbb{R}^+)$ from corollary 2. We wish to define $\delta(u^\varepsilon - \xi)G^\varepsilon$ as a distribution in $\mathcal{D}'_{x,t,\xi}$ by its action on tensor products

$$< \delta(u^\varepsilon - \xi)G^\varepsilon, \varphi \otimes \eta' >= \int_{x,t} G^\varepsilon(x,t)\varphi(x,t)\eta'(u^\varepsilon(x,t))dxdt\,. \tag{2.21}$$

This follows from the Schwartz kernel theorem (*e.g.* [7, Sec 5.2]) as follows: Define the linear map

$$\mathcal{K} : C_c^\infty(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^+) \quad \text{by} \quad \mathcal{K}\psi = G^\varepsilon(x,t)\psi(u^\varepsilon(x,t))$$

If $\psi_j \to 0$ in $C_c^\infty(\mathbb{R})$ then $\mathcal{K}\psi_j \to 0$ in $\mathcal{D}'_{x,t}$. The kernel theorem implies that $\delta(u^\varepsilon - \xi)G^\varepsilon$ is well defined as a distribution in $\mathcal{D}'_{x,t,\xi}$ and acts on tensor products via (2.21). Moreover,

$$< \partial_\xi\delta(u^\varepsilon - \xi)G^\varepsilon, \varphi \otimes \eta' >= -\int_{x,t} G^\varepsilon(x,t)\varphi(x,t)\eta''(u^\varepsilon(x,t))dxdt\,. \tag{2.22}$$

Thus (2.19) is written as

$$
\begin{aligned}
& < \partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon \, , \, \eta'(\xi)\varphi(x,t) > \\
= \; & < \sum_{j=1}^{d} \partial_{x_j} \Big(\varepsilon A_j^2 \partial_{x_j} \chi^\varepsilon \Big) - \partial_t \big(\varepsilon \partial_t \chi^\varepsilon \big) \, , \, \eta'(\xi)\varphi(x,t) > \\
& + < \partial_\xi (\delta(u^\varepsilon - \xi) G^\varepsilon \, , \, \eta'(\xi)\varphi(x,t) > .
\end{aligned}
\tag{2.23}
$$

Since the subspace generated by the direct sum test functions $\varphi \otimes \eta'$ is dense in $C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, the bracket (2.23) is extended to test functions $\theta(x,t,\xi) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. So, we have

$$
\begin{aligned}
\partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon &= \sum_{j=1}^{d} \partial_{x_j} \big(\varepsilon A_j^2 \partial_{x_j} \chi^\varepsilon \big) - \partial_t \big(\varepsilon \partial_t \chi^\varepsilon \big) + \partial_\xi \delta(u^\varepsilon - \xi) G^\varepsilon \\
&=: \sum_{j=1}^{d} \partial_{x_j} \pi_j^\varepsilon + \partial_t \pi_0^\varepsilon + \partial_\xi k^\varepsilon \qquad \text{in} \quad D'_{x,t,\xi}.
\end{aligned}
\tag{2.24}
$$

where

$$
\pi_j^\varepsilon = \varepsilon A_j^2 \partial_{x_j} \chi^\varepsilon \, , \quad \pi_0^\varepsilon = -\varepsilon \partial_t \chi^\varepsilon \, , \quad k^\varepsilon = \delta(u^\varepsilon - \xi) G^\varepsilon \, .
\tag{2.25}
$$

Step 2. The next objective is to estimate the terms k^ε and π_j^ε. For the term k^ε, let $\theta(x,t,\xi) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and note that

$$
\begin{aligned}
| < k^\varepsilon, \theta > | &= | < \delta(u^\varepsilon - \xi) G^\varepsilon \, , \, \theta > | \\
&= | \int_{x,t} G^\varepsilon(x,t) \theta(x,t,u^\varepsilon(x,t)) dx dt | \\
&\leq \sup_{x,t,\xi} |\theta(x,t,\xi)| \cdot \|G^\varepsilon\|_{L^1(\mathbb{R}^d \times \mathbb{R}^+)}
\end{aligned}
$$

By corollary 2,

$$
| < k^\varepsilon, \theta > | \leq C \|\theta\|_{C^0}
$$

and k^ε lies in a bounded set of the space of bounded measures $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ (the dual of $C_0(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, the continuous functions that vanish at infinity). The Sobolev embedding theorem implies $\mathcal{M}$ is embedded in $W^{-1,p}$, $1 \leq p < \frac{d+2}{d+1}$ and thus k^ε is precompact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for $1 \leq p < \frac{d+2}{d+1}$.

Consider now the terms $\pi_j^\varepsilon(x,t,\xi)$, $j=0,1,\ldots,d$. By (2.21), we have

$$\begin{aligned}
<\partial_{x_j}\mathbb{1}(u^\varepsilon(x,t),\xi),\eta'(\xi)\varphi(x,t)> &= -<\mathbb{1}(u^\varepsilon(x,t),\xi),\eta'(\xi)\partial_{x_j}\varphi(x,t)> \\
&= -\int_{x,t}\partial_{x_j}\varphi(x,t)\eta(u^\varepsilon(x,t))dxdt \\
&= \int_{x,t}\varphi(x,t)\eta'(u^\varepsilon(x,t))\partial_{x_j}u^\varepsilon(x,t)dxdt \\
&= <\delta(u^\varepsilon(x,t)-\xi)\,\partial_{x_j}u^\varepsilon(x,t),\varphi(x,t)\eta'(\xi)>
\end{aligned}$$

and thus

$$\varepsilon\partial_{x_j}\mathbb{1}(u^\varepsilon,\xi)=\varepsilon\delta(u^\varepsilon-\xi)\partial_{x_j}u^\varepsilon \quad \text{in} \quad D'_{x,t,\xi}\,. \tag{2.26}$$

(This step requires that $\partial_{x_j}u^\varepsilon\in L^1_{loc}$.) For $\theta\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$,

$$\begin{aligned}
<\varepsilon\partial_{x_j}\mathbb{1}(u^\varepsilon,\xi),\theta> &= <\varepsilon\delta(u^\varepsilon-\xi)\partial_{x_j}u^\varepsilon(x,t),\theta> \\
&= \int_{x,t}\varepsilon\partial_{x_j}u^\varepsilon(x,t)\theta(x,t,u^\varepsilon)dxdt \\
&\le \sqrt{\varepsilon}(\varepsilon\int_{x,t}(\partial_{x_j}u^\varepsilon)^2dxdt)^{\frac12}(\int_{x,t}\theta^2(x,t,u^\varepsilon(x,t))dxdt)^{\frac12}
\end{aligned}$$

But since

$$\begin{aligned}
\int_{x,t}\theta^2(x,t,u^\varepsilon)dxdt &= \int_{x,t}\int_{-\infty}^{u^\varepsilon(x,t)}2\theta\theta_\xi d\xi dxdt \\
&\le 2\int_{x,t}(\int_{-\infty}^{u^\varepsilon}\theta^2d\xi)^{\frac12}(\int_{-\infty}^{u^\varepsilon}(\partial_\xi\theta)^2d\xi)^{\frac12}dxdt\le\|\theta\|^2_{L^2_{x,t}(H^1_\xi)}\,, \qquad (2.27)
\end{aligned}$$

it follows that

$$|<\varepsilon\partial_{x_j}\mathbb{1}(u^\varepsilon,\xi),\theta>|\le\sqrt{\varepsilon}C\|\theta\|_{L^2_{x,t}(H^1_\xi)}\,.$$

We conclude

$$\begin{aligned}
\varepsilon\partial_{x_j}\mathbb{1}(u^\varepsilon,\xi)\to 0 \quad &\text{in } L^2_{x,t}(H_\xi^{-1}) \\
\varepsilon\partial_t\mathbb{1}(u^\varepsilon,\xi)\to 0 \quad &\text{in } L^2_{x,t}(H_\xi^{-1})
\end{aligned}$$

the latter coming from a similar argument.

Step 3. In summary, the function $\chi^\varepsilon=\mathbb{1}(u^\varepsilon,\xi)$ satisfies the (approximate) transport equation

(2.28)

$$\partial_t\chi^\varepsilon+a(\xi)\cdot\nabla\chi^\varepsilon=\sum_{j=1}^d\partial_{x_j}\left(\bar{g}_j^\varepsilon+\partial_\xi g_j^\varepsilon\right)+\partial_t\left(\bar{g}_0^\varepsilon+\partial_\xi g_0^\varepsilon\right)+\partial_\xi k^\varepsilon \qquad \text{in} \quad D'_{x,t,\xi},$$

where $\bar{g}_i^\varepsilon$, $g_i^\varepsilon \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ while k^ε is bounded in measures (not necessarily positive) and precompact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for $1 \le p < \frac{d+2}{d+1}$. By the averaging lemma (Theorem 3),

$$\int_\xi \mathbb{1}(u^\varepsilon, \xi)\psi(\xi)d\xi \quad \text{is precompact in } L^p_{loc},\ 1 < p < \frac{d+2}{d+1}$$

for $\psi(\xi) \in C_c^\infty(\mathbb{R})$.

Let R be a large positive number and consider $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $(-R, R)$ and $0 \le \psi \le 1$. Then

$$\begin{aligned}\Big|u^\varepsilon - \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\psi(\xi)d\xi\Big| &= \Big|\int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)(1 - \psi(\xi))d\xi\Big| \\ &\le \int_R^\infty |\mathbb{1}(u^\varepsilon, \xi)|d\xi + \int_{-\infty}^{-R} |\mathbb{1}(u^\varepsilon, \xi)|d\xi = (u^\varepsilon - R)^+ + (u^\varepsilon + R)^-\end{aligned}$$

Moreover,

$$\begin{aligned}\int (u^\varepsilon - R)^+ + (u^\varepsilon + R)^- dxdt &\le \int_{|u^\varepsilon|>R} |u^\varepsilon|\, dxdt \\ &\le \frac{1}{R}\int_0^T \int |u^\varepsilon|^2 dxdt \le \frac{C}{R}\end{aligned}$$

We conclude that $\{u^\varepsilon\}$ is Cauchy in $L^1_{loc,x,t}$.

Since $u^\varepsilon \in_b L^\infty(L^2)$, it follows that (along subsequences) $u^\varepsilon \to u$ in L^p_{loc}, $p < 2$, and almost everywhere and that $u \in L^\infty(L^2)$. Integrating (2.28) with respect to ξ yields

$$\partial_t \int \chi^\varepsilon d\xi + \operatorname{div} \int a(\xi)\chi^\varepsilon d\xi = \sum_{j=1}^d \varepsilon A_j^2 \partial_{x_j}^2 \int \chi^\varepsilon d\xi - \varepsilon \partial_t^2 \int \chi^\varepsilon d\xi\,;$$

and with limit $\varepsilon \to 0$, u satisfies the scalar conservation law.

Next, we pass to the limit $\varepsilon \to 0$ in (2.28). Note that

$$\begin{aligned}&\chi^\varepsilon = \mathbb{1}(u^\varepsilon, \xi) \to \chi = \mathbb{1}(u, \xi) && \text{a.e. and in } L^p_{loc,x,t}(L^1_\xi),\ 1 \le p < 2 \\ &k^\varepsilon = G^\varepsilon \delta(u^\varepsilon - \xi) \rightharpoonup k && \text{weak-}\star \text{ in } \mathcal{M}_{x,t,\xi}\end{aligned} \tag{2.29}$$

and χ satisfies

$$\partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_\xi k \qquad \text{in} \quad D'_{x,t,\xi}. \tag{2.30}$$

□

Remark 4

1. The limit k is a bounded measure but we do not know whether k is a positive measure. Ultimately, the sign of k depends on the sign of

$$G^\varepsilon = \varepsilon\Big(\sum_{j=1}^{d} A_j^2(\partial_{x_j} u^\varepsilon)^2 - \big[\mathrm{div}F(u^\varepsilon) - \varepsilon\big(A^2\cdot\Delta u^\varepsilon - u^\varepsilon_{tt})\big]^2\Big)$$

One may formally argue that $u_t^\varepsilon \sim \mathrm{div}F(u^\varepsilon)$ as $\varepsilon \to 0$ and postulate that

$$G^\varepsilon \sim \varepsilon\Big(\sum_{j=1}^{d} A_j^2(\partial_{x_j} u^\varepsilon)^2 - (\mathrm{div}F(u^\varepsilon))^2\Big) \geq 0\,,$$

by some version of the subcharacteristic condition. This argument may well be misleading, since all the available information comes from (2.10), and it could in principle happen that the terms $\varepsilon\big(\mathrm{div}F(u^\varepsilon)\big)^2$ and $\varepsilon^3\left(A^2\cdot\Delta u^\varepsilon - u^\varepsilon_{tt}\right)^2$ are comparable near shocks. This situation should also be compared with the behavior of diffusion-dispersion approximations when $\delta = O(\varepsilon^2)$.

2. One may derive a (fairly weak) version of entropy inequality for the energy of the problem. Using corollary 2,

$$\varepsilon^2(u_t)^2 + \varepsilon^2\sum_{j=1}^{d} A_j^2(\partial_{x_j}u)^2 \to 0 \quad \text{in } \mathcal{D}'_{x,t}$$
$$\varepsilon u u_t \to 0, \quad \varepsilon u u_{x_j} \to 0, \quad \varepsilon^2 u_t u_{x_j} \to 0 \quad \text{in } \mathcal{D}'_{x,t}$$

Also, there exist measures $\bar{H}$ and $\bar{Q}_j$ so that

$$\frac{1}{2}u^{\varepsilon 2} \rightharpoonup \bar{H}\,, \quad Q_j(u^\varepsilon) \rightharpoonup \bar{Q}_j \qquad \text{weak-}\star \text{ in measures}$$

Passing to the limit $\varepsilon \to 0$ in (2.10) we obtain

$$\partial_t \bar{H} + \mathrm{div}\bar{Q} \leq 0 \quad \text{in } \mathcal{D}'_{x,t}. \tag{2.31}$$

2.2 A Kinetic Decomposition based on Extension of Entropies

The 1-d version of (2.1) yields when $\varepsilon \to 0$ entropy solutions [13]. In this section, we outline an alternative method for obtaining a kinetic decomposition, that is based on the idea of extending entropies (see [2]) and produces when $\varepsilon \to 0$ the kinetic formulation with k a bounded, positive measure. This process complements the one described in section 2.1. It has the advantage that it produces $k \geq 0$, but, since it is based on entropy extensions, its applicability is limited to the one dimensional case.

Consider the relaxation system

$$\left\{ \begin{array}{lcl} \partial_t u + \partial_x v & = & 0 \\ \\ \partial_t v + a^2 \partial_x u & = & -\frac{1}{\varepsilon}(v - f(u)) \end{array} \right. \tag{2.32}$$

where $u, v : \mathbb{R}_x \times \mathbb{R}_t^+ \longrightarrow \mathbb{R}$, and $f(u)$ is a smooth function with $f(0) = 0$ and satisfies the subcharacteristic condition

$$a > |f'(u)| \qquad u \in \mathbb{R}\,. \tag{2.33}$$

This relaxation system converges to the scalar equation

$$\partial_t u + \partial_x f(u) = 0\,. \tag{2.34}$$

The system (2.32) may be diagonalized, by setting $f_1^\varepsilon = \frac{1}{2}(u^\varepsilon - \frac{v^\varepsilon}{a})$, $\quad f_2^\varepsilon = \frac{1}{2}(u^\varepsilon + \frac{v^\varepsilon}{a})$. We then find

$$\left\{ \begin{array}{lcl} \partial_t f_1^\varepsilon - a \partial_x f_1^\varepsilon & = & -\frac{1}{\varepsilon}(f_1^\varepsilon - \mathcal{M}_1(u^\varepsilon)) \\ \\ \partial_t f_2^\varepsilon + a \partial_x f_2^\varepsilon & = & -\frac{1}{\varepsilon}(f_2^\varepsilon - \mathcal{M}_2(u^\varepsilon)) \end{array} \right. \tag{2.35}$$

where $\mathcal{M}_1(u^\varepsilon) = \frac{1}{2}(u^\varepsilon - \frac{f(u^\varepsilon)}{a})$, $\quad \mathcal{M}_2(u^\varepsilon) = \frac{1}{2}(u^\varepsilon + \frac{f(u^\varepsilon)}{a})$. Clearly we recover the original unknowns by the inverse transform $u^\varepsilon = f_1^\varepsilon + f_2^\varepsilon$ and $v^\varepsilon = a(f_2^\varepsilon - f_1^\varepsilon)$. If we set $\mu_i(f_i) = \int_0^{f_i} \mathcal{M}_i^{-1}(g)dg$, then $\mu_i(\cdot)$ is convex and we have the estimate

$$\begin{aligned} \partial_t \sum_{i=1}^{2} \mu_i(f_i) + \partial_x(-a\mu_1(f_1) + a\mu_1(f_1)) - u(\partial_t u + \partial_x v) \\ + \frac{1}{\varepsilon} \sum_{i=1}^{2} (\mathcal{M}_i^{-1}(f_i) - u)(f_i - \mathcal{M}_i(u)) = 0 \end{aligned} \tag{2.36}$$

For $f_i^\varepsilon(x, 0) \in L^2(\mathbb{R}), i = 1, 2$, by the subcharacteristic condition (2.33), we obtain the bound

$$\begin{aligned} \int_{x,t} |v^\varepsilon - f(u^\varepsilon)|^2 dxdt &\leq C \int_{x,t} \sum_{i=1}^{2} |f_i - \mathcal{M}_i(u)|^2 dxdt \\ &\leq C \int_{x,t} \sum_{i=1}^{2} (\mathcal{M}_i^{-1}(f_i) - u)(f_i - \mathcal{M}_i(u)) dxdt = O(\varepsilon) \end{aligned} \tag{2.37}$$

Let $H(u, v), Q(u, v) \in C^1(\mathbb{R} \times \mathbb{R})$ be an entropy pair for the relaxation system that extends the entropy pair $\eta - q$ of (2.34) :

$$\begin{cases} Q_v &= H_u \\ Q_u &= a^2 H_v \end{cases} \tag{2.38}$$

$$\begin{cases} H(u,f(u)) &= \eta(u) \\ Q(u,f(u)) &= q(u) \end{cases} \tag{2.39}$$

$H - Q$ have the property that (smooth) solutions of (2.32) satisfy

$$\partial_t H(u,v) + \partial_x Q(u,v) + \frac{1}{\varepsilon} H_v(u,v)(v - f(u)) = 0 \tag{2.40}$$

and when $\varepsilon \to 0$ they are expected to converge to the entropy dissipation of the limit conservation law.

The solution of (2.38)-(2.39) is computed easily: The general solution of (2.38) is

$$\begin{cases} H(u,v) &= F(au+v) + G(au-v) \\ Q(u,v) &= aF(au+v) - aG(au-v) \end{cases} \tag{2.41}$$

for some functions F, G. Since $\eta'(u)f'(u) = q'(u)$, we have

$$\begin{cases} F'(au+f(u)) &= \frac{1}{2a}\eta'(u) \\ G'(au-f(u)) &= \frac{1}{2a}\eta'(u) \end{cases} \tag{2.42}$$

Define the functions $\hat{w}(u) = au + f(u)$, $\bar{z}(u) = au - f(u)$. By the subcharacteristic condition, $\hat{w}$ and $\bar{z}$ are invertible with inverse functions $u = \hat{u}(w)$, $u = \bar{u}(z)$, that are strictly increasing and globally Lipschitz and satisfy $\hat{u}(0) = \bar{u}(0) = 0$. Hence,

$$\begin{cases} F'(w) &= \frac{1}{2a}\eta'(\hat{u}(w)) \\ G'(z) &= \frac{1}{2a}\eta'(\bar{u}(z)) \end{cases} \tag{2.43}$$

For the 1-d case we obtain the following approximate transport equation:

Theorem 5 *Let (2.33) hold and the initial data satisfy the bounds*

$$\|u_0^\varepsilon\|_{L^2(\mathbb{R})} + \|v_0^\varepsilon\|_{L^2(\mathbb{R})} \leq O(1)\,. \tag{2.44}$$

We then have

$$\begin{aligned} &\partial_t \frac{1}{2a}\Big(1\!\!1(au+v, as+f(s))(a+f'(s)) + 1\!\!1(au-v, as-f(s))(a-f'(s))\Big) \\ &\quad + \partial_x \frac{1}{2}\Big(1\!\!1(au+v, as+f(s))(a+f'(s)) - 1\!\!1(au-v, as-f(s))(a-f'(s))\Big) \\ &= \partial_s\Big(\frac{1}{2a\varepsilon}\big[1\!\!1(au+v, as+f(s)) - 1\!\!1(au-v, as-f(s))\big](v - f(u))\Big) \qquad (2.45) \\ &=: \partial_s m^\varepsilon(u,v,s) \end{aligned}$$

where m^ε is a bounded family of positive measures.

Proof. The proof uses representation formulas obtained via the extension of entropies. Let $\eta' \in C_c^\infty(\mathbb{R})$ be viewed as a test function. Then, from (2.4) and (2.43),

$$
\begin{aligned}
H(u,v) &= \int 1\!\!1(au+v,\xi)\frac{1}{2a}\eta'(\hat{u}(\xi))d\xi + \int 1\!\!1(au-v,\xi)\frac{1}{2a}\eta'(\bar{u}(\xi))d\xi \\
&= \int 1\!\!1(au+v, as+f(s))\frac{1}{2a}(a+f'(s))\eta'(s)ds \\
&\quad + \int 1\!\!1(au-v, as-f(s))\frac{1}{2a}(a-f'(s))\eta'(s)ds
\end{aligned}
$$

A similar formula is obtained from $Q(u,v)$. Finally,

$$
\begin{aligned}
H_v(u,v) &= \frac{1}{2a}\eta'\big(\hat{u}(au+v)\big) - \frac{1}{2a}\eta'\big(\bar{u}(au-v)\big) \\
&= \int \frac{1}{2a}\Big(1\!\!1(\hat{u}(au+v),\xi) - 1\!\!1(\bar{u}(au-v),\xi)\Big)\eta''(\xi)d\xi\,,
\end{aligned}
$$

which implies that, for $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$,

$$
\int_{x,t} \frac{1}{\varepsilon}H_v(u,v)(v-f(u))\varphi\,dxdt = -<\partial_\xi m^\varepsilon, \varphi(x,t)\otimes\eta'(\xi)>
$$

where

$$
\begin{aligned}
m^\varepsilon &= \frac{1}{2a\varepsilon}\Big(1\!\!1(\hat{u}(au+v),\xi) - 1\!\!1(\bar{u}(au-v),\xi)\Big)(v-f(u)) \\
&= \frac{1}{2a\varepsilon}\big(1\!\!1(au+v, a\xi+f(\xi)) - 1\!\!1(au-v, a\xi-f(\xi))\big)(v-f(u))
\end{aligned}
$$

Following the argument in section 2.1, it is easy to derive (2.45).

We turn to properties of m^ε. Consider the case $v > f(u)$ and note that since $\bar{u}$, $\hat{u}$ are increasing, we have

$$
\hat{u}(au-v) < \bar{u}(au-f(u)) = u = \hat{u}(au+f(u)) < \bar{u}(au+v) \tag{2.46}
$$

By contrast, when $v < f(u)$ all inequalities are reversed. Since the function $1\!\!1(\cdot,\xi)$ is increasing, it follows easily that $m^\varepsilon \geq 0$.

For $\theta(x,t,\xi) \in C_0$ with $|\theta| \leq 1$, we have

$$
\begin{aligned}
|<m^\varepsilon,\theta>| &= \Big|\int_{x,t,\xi} \frac{1}{2a\varepsilon}\Big(1\!\!1(\hat{u}(au+v),\xi) - 1\!\!1(\bar{u}(au-v),\xi)\Big)(v-f(u))\theta dxdtd\xi\Big| \\
&\leq \int_{x,t} |v-f(u)|\frac{1}{2a\varepsilon}\Big(\int_\xi |1\!\!1(\hat{u}(au+v),\xi) - 1\!\!1(\bar{u}(au-v),\xi)|d\xi\Big)dxdt \\
&= \int_{x,t} \frac{1}{2a\varepsilon}|v-f(u)|\,|\hat{u}(au+v) - \bar{u}(au-v)|dxdt
\end{aligned}
$$

The last term is bounded, because by (2.46), (2.37)

$$|\hat{u}(au+v) - \bar{u}(au-v)| \leq O(1)|v - f(u)|$$
$$\int_{x,t} |v - f(u)|^2 dxdt \leq O(\varepsilon)$$

Corollary 2 implies m^ε is bounded in measures. $\square$

We obtain compactness of approximate solutions by using the averaging lemma:

Corollary 6 *Let f be globally Lipschitz and satisfy*

$$meas\{\xi \in \mathbb{R} \mid f'(\xi) = \alpha\} = 0, \quad \forall \alpha \in \mathbb{R}. \tag{2.47}$$

Let a be selected so that (2.33) holds, and let u^ε be a family of solutions to (2.32) generated by data subject to the uniform bounds (2.44)*. Then u^ε converges to u in $L^p_{loc}(\mathbb{R} \times \mathbb{R}^+)$, $1 < p < 2$, and u is an entropy solution of (2.34).*

Proof. Let

$$I = \frac{1}{2a}[1\!\!1(au+v, as+f(s)) - 1\!\!1(au+f(u), as+f(s))](a+f'(s))$$

$$J = \frac{1}{2a}[1\!\!1(au-v, as-f(s)) - 1\!\!1(au-f(u), as-f(s))](a-f'(s))$$

and note that

$$\frac{1}{2a}[1\!\!1(au+f(u), as+f(s))(a+f'(s)) + 1\!\!1(au-f(u), as-f(s))(a-f'(s))] = 1\!\!1(u,s)$$

$$\frac{1}{2}[1\!\!1(au+f(u), as+f(s))(a+f'(s)) - 1\!\!1(au-f(u), as-f(s))(a-f'(s))] = f'(s)1\!\!1(u,s)$$

The approximate transport equation (2.45) can be rewritten as

$$\partial_t 1\!\!1(u^\varepsilon, s) + f'(s)\, \partial_x 1\!\!1(u^\varepsilon, s) = -\partial_t(I^\varepsilon + J^\varepsilon) - \partial_x(aI^\varepsilon - aJ^\varepsilon) + \partial_s m^\varepsilon \tag{2.48}$$

But, we have, for any compact subset K of $\mathbb{R} \times \mathbb{R}^+$,

$$\begin{aligned}
&\int_K \int_s |I(x,t,s)|^2 dsdxdt \\
&\quad \leq \quad C \int_K \int_\xi |1\!\!1(au+v,\xi) - 1\!\!1(au+f(u),\xi)|^2 d\xi dxdt \\
&\quad = \quad C \int_K |f(u) - v| dxdt \\
&\quad \leq \quad C|K|^{\frac{1}{2}} \Big(\int_K |f(u) - v|^2 dxdt \Big)^{\frac{1}{2}} = O(\varepsilon^{\frac{1}{2}})
\end{aligned}$$

So, $I, J \to 0$ in $L^2_{loc}(\mathbb{R}_x \times \mathbb{R}^+_t; L^2(\mathbb{R}_s))$, the latter coming from a similar argument. By the averaging lemma, we conclude that $u^\varepsilon \to u$ in $L^p_{loc}, p < 2$. Next, we pass to the limit $\varepsilon \to 0$ in (2.48). Along a further subsequence $m^\varepsilon \rightharpoonup m$ weak-$\star$ in measures; it follows

$$\partial_t \mathbb{1}(u,s) + f'(s)\partial_x \mathbb{1}(u,s) = \partial_s m \qquad \text{in} \quad D'_{x,t,s}, \tag{2.49}$$

and u is the unique entropy solution of (2.34). □

3 Diffusion-Dispersion Approximation

In this section, we study the diffusive-dispersive approximation

$$\begin{aligned} \partial_t u + \mathrm{div} F(u) &= \varepsilon \sum_{j=1}^{d} \partial_{x_j x_j} u + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u, \quad x \in \mathbb{R}^d,\ t \geq 0, \\ u(x,0) &= u_0^\varepsilon(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{3.1}$$

where $F(u) = (F_1(u), \ldots, F_d(u))$ and $a_j = F'_j$. We will show that if the initial data u_0^ε converge weakly to some limit u_0 and $\delta = O(\varepsilon^2)$, then solutions of (3.1) converge towards a weak solution of the scalar conservation law (1.1). Our analysis is based on a kinetic decomposition and the averaging lemma, and it can handle the case $\delta = O(\varepsilon^2)$ (where convergence is expected but not to an entropy solution). We begin with some estimates on smooth solutions u^ε of (3.1) from [19], [10].

Lemma 7 *Suppose that F_j, $j = 1, \cdots, d$, are globally Lipschitz ($|F'_j| \leq A_j$) and assume the data satisfy the uniform bounds*

$$\|u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon \sum_{j=1}^{d} \|\partial_{x_j} u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C. \tag{3.2}$$

Then we have

$$u^\varepsilon(x,t) \quad \in_b \quad L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d)) \tag{3.3}$$

$$2\varepsilon \sum_{j=1}^{d} (\partial_{x_j} u^\varepsilon(x,t))^2 \quad \in_b \quad L^1(\mathbb{R}^d \times \mathbb{R}^+) \tag{3.4}$$

$$\sum_{j=1}^{d} \varepsilon^3 (\partial_{x_j x_j} u^\varepsilon(x,t))^2 \quad \in_b \quad L^1(\mathbb{R}^d \times \mathbb{R}^+) \tag{3.5}$$

Proof. Let $u(x,t) = u^{\varepsilon,\delta}(x,t)$. To derive the L^2 estimate (3.3), we multiply the equation (3.1) by u and get

$$\begin{aligned}\partial_t(\frac{1}{2}|u|^2) + \sum_{j=1}^d \partial_{x_j} Q_j(u) &= \varepsilon \sum_{j=1}^d \Big(\partial_{x_j}(uu_{x_j}) - |u_{x_j}|^2\Big) \\ &\quad + \delta \sum_{j=1}^d \partial_{x_j}\big(uu_{x_jx_j} - \frac{1}{2}|u_{x_j}|^2\big)\,,\end{aligned}\tag{3.6}$$

where $Q_j' = uF_j'$. Integrating over $\mathbb{R}^d \times (0,t)$, we get

$$\int_{\mathbb{R}^d} |u(t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^d |u_{x_j}|^2 dx dt = \int_{\mathbb{R}^d} |u_0|^2 dx.\tag{3.7}$$

To estimate $u_{x_kx_k}, k = 1, \cdots, d$, we differentiate the equation (3.1) with respect to the variable x_k and then multiply by u_{x_k} . The right hand side is identical to the preceding case with u replaced by u_{x_k}. The flux term is handled via the identity $(u_{x_k}^2 F_j')_{x_j} = 2u_{x_k} u_{x_kx_j} F_j' + (u_{x_k})^2 u_{x_j} F_j''$, and yields

$$\begin{aligned}&\partial_t(\frac{1}{2}|u_{x_k}|^2) + \sum_{j=1}^d \Big(\partial_{x_j}(u_{x_k}^2 F_j'(u)) - u_{x_k} u_{x_kx_j} F_j'(u)\Big) \\ &= \varepsilon \sum_{j=1}^d \Big(\partial_{x_j}(u_{x_k} u_{x_kx_j}) - |u_{x_kx_j}|^2\Big) + \delta \sum_{j=1}^d \partial_{x_j}\big(u_{x_k} u_{x_kx_jx_j} - \frac{1}{2}|u_{x_kx_j}|^2\big)\,,\end{aligned}\tag{3.8}$$

After integrating over $\mathbb{R}^d \times (0,t)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d} |u_{x_k}|^2 dx + 2\varepsilon \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} |u_{x_jx_k}|^2 dx dt \\ &\le \int_{\mathbb{R}^d} |u_{0x_k}|^2 + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} 2\|F_j'\|_\infty |u_{x_jx_k}||u_{x_k}| dx dt \\ &\le \int_{\mathbb{R}^d} |u_{0x_k}|^2 dx + \frac{1}{\varepsilon}\Big(\sum_{j=1}^d \|F_j'\|_\infty^2\Big) \int_0^t \int_{\mathbb{R}^d} |u_{x_k}|^2 dx dt + \varepsilon \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} |u_{x_jx_k}|^2 dx dt.\end{aligned}$$

Using (3.7) and (3.2), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^d} \varepsilon^2 |u_{x_k}(t)|^2 dx + \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \varepsilon^3 |u_{x_jx_k}|^2 dx dt \\ &\le \int_{\mathbb{R}^d} \varepsilon^2 |u_{0x_k}|^2 dx + \Big(\sum_{j=1}^d \|F_j'\|_\infty^2\Big) \int_0^t \int_{\mathbb{R}^d} \varepsilon |u_{x_k}|^2 dx dt \le O(1)\,.\end{aligned}$$

□

Next, we state the main theorem of this section:

Theorem 8 *Suppose the data satisfy (3.2), and F_j are globally Lipschitz functions that satisfy the nondegeneracy condition (2.17)(or (2.16)).*

(i) If $\delta = O(\varepsilon^2)$, then solutions u^ε of (3.1) converge along a subsequence to a function u in $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^+)$, $1 < p < 2$; the limiting u is a weak solution of (1.1).

(ii) If $\delta = o(\varepsilon^2)$, then u is the unique Kruzhkov entropy solution of (1.1).

Proof. Let $\delta = O(\varepsilon^2)$ and denote by $u^\varepsilon = u^{\varepsilon,\delta}(x,t)$. We multiply (3.1) by $\eta'(u^\varepsilon)$ and obtain

$$\begin{aligned}
&\partial_t \eta(u^\varepsilon) + \mathrm{div} q(u^\varepsilon) \\
&\quad = \varepsilon \sum_{j=1}^d \partial_{x_j}(\eta'(u^\varepsilon)\partial_{x_j} u^\varepsilon) - \varepsilon \eta''(u^\varepsilon)\sum_{j=1}^d (\partial_{x_j} u^\varepsilon)^2 \\
&\quad + \delta \sum_{j=1}^d \partial_{x_j}(\eta'(u^\varepsilon)\partial_{x_j x_j} u^\varepsilon) - \delta \eta''(u^\varepsilon)\sum_{j=1}^d (\partial_{x_j} u^\varepsilon)(\partial_{x_j x_j} u^\varepsilon).
\end{aligned} \tag{3.9}$$

Let $\varphi(x,t) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and let $\eta \in C_c^\infty(\mathbb{R})$ be viewed as a test function. By introducing the indicator function $1\!\!1(u^\varepsilon, \xi)$, we have

$$\begin{aligned}
&-\int_{x,t,\xi} \Big(1\!\!1(u^\varepsilon,\xi)\partial_t \varphi(x,t) + \sum_{j=1}^d F_j'(\xi) 1\!\!1(u^\varepsilon,\xi)\partial_{x_j}\varphi(x,t)\Big)\eta'(\xi)\,d\xi dx dt \\
&= -\int_{x,t}\sum_{j=1}^d \Big(\varepsilon \partial_{x_j} u^\varepsilon + \delta \partial_{x_j x_j} u^\varepsilon\Big)\eta'(u^\varepsilon)\partial_{x_j}\varphi(x,t)\, dx dt \\
&\quad - \int_{x,t} \eta''(u^\varepsilon)\Big(\varepsilon \sum_{j=1}^d (\partial_{x_j} u^\varepsilon)^2 + \delta \sum_{j=1}^d (\partial_{x_j} u^\varepsilon)(\partial_{x_j x_j} u^\varepsilon)\Big)\varphi(x,t)\, dx dt
\end{aligned} \tag{3.10}$$

Let $\chi^\varepsilon = 1\!\!1(u^\varepsilon, \xi)$,

$$H^\varepsilon(x,t) = \varepsilon \partial_{x_j} u^\varepsilon + \delta \partial_{x_j x_j} u^\varepsilon$$
$$G^\varepsilon(x,t) = \varepsilon \sum_{j=1}^d (\partial_{x_j} u^\varepsilon)^2 + \delta \sum_{j=1}^d (\partial_{x_j} u^\varepsilon)(\partial_{x_j x_j} u^\varepsilon)$$

From arguments as in the case of relaxation approximations, it follows that

$$\begin{aligned}
\partial_t \chi^\varepsilon + a(\xi)\cdot \nabla \chi^\varepsilon &= \sum_{j=1}^d \partial_{x_j}\Big(H^\varepsilon(x,t)\delta(u^\varepsilon - \xi)\Big) + \partial_\xi\Big(G^\varepsilon(x,t)\delta(u^\varepsilon-\xi)\Big) \\
&=: \sum_{j=1}^d \partial_{x_j}\pi_j^\varepsilon + \partial_\xi k^\varepsilon \qquad \text{in} \quad D'_{x,t,\xi}.
\end{aligned} \tag{3.11}$$

We estimate first the terms π_j^ε. Let $\theta(x,t,\xi) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. Using the estimates in lemma 7 and the hypothesis (2.27), we see that for $\delta = O(\varepsilon^2)$

$$
\begin{aligned}
| < H^\varepsilon \delta(u^\varepsilon - \xi), \theta(x,t,\xi) > | &= | \int_{x,t} (\varepsilon \partial_{x_j} u^\varepsilon + \delta \partial_{x_j x_j} u^\varepsilon) \theta(x,t,u^\varepsilon(x,t)) dx dt | \\
&\le \| \varepsilon \partial_{x_j} u^\varepsilon + \delta \partial_{x_j x_j} u^\varepsilon \|_{L^2_{x,t}} \cdot \| \theta(x,t,u^\varepsilon(x,t)) \|_{L^2_{x,t}} \\
&\le C \varepsilon^{1/2} \left(\| \varepsilon^{1/2} \partial_{x_j} u^\varepsilon \|_{L^2_{x,t}} + \| \varepsilon^{3/2} \partial_{x_j x_j} u^\varepsilon \|_{L^2_{x,t}} \right) \| \theta \|_{L^2_{x,t}(H^1_\xi)} \\
&\le C \varepsilon^{1/2} \| \theta \|_{L^2_{x,t}(H^1_\xi)} .
\end{aligned}
$$

This shows that $\pi_j^\varepsilon \to 0$ in $L^2_{x,t}(H_\xi^{-1})$ as $\varepsilon \to 0$, or in other words

$$
\pi_j^\varepsilon = \bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon \qquad \text{with } \bar{g}_j^\varepsilon,\ g_j^\varepsilon \to 0 \text{ in } L^2_{x,t,\xi}.
$$

Next, consider the term $k^\varepsilon = G^\varepsilon \delta(u^\varepsilon - \xi)$. Observe that

$$
\begin{aligned}
\left| G^\varepsilon - \varepsilon \sum_{j=1}^d |u^\varepsilon_{x_j}|^2 \right| &\le \delta \sum_{j=1}^d |u^\varepsilon_{x_j}| |u^\varepsilon_{x_j x_j}| \\
&\le \frac{\delta}{2\varepsilon} \sum_{j=1}^d |u^\varepsilon_{x_j}|^2 + \frac{\delta \varepsilon}{2} \sum_{j=1}^d |u^\varepsilon_{x_j x_j}|^2
\end{aligned} \tag{3.12}
$$

If $\delta = O(\varepsilon^2)$ the estimates in lemma 7 imply $G^\varepsilon \in_b L^1(\mathbb{R}^d \times \mathbb{R}^+)$. Thus

$$
\begin{aligned}
| < k^\varepsilon, \theta > | &= | < \delta(u^\varepsilon - \xi) G^\varepsilon \, , \, \theta > | \\
&\le \sup_{x,t,\xi} |\theta(x,t,\xi)| \cdot \| G^\varepsilon \|_{L^1(\mathbb{R}^d \times \mathbb{R}^+)} \le C \| \theta \|_{C^0}
\end{aligned}
$$

and as before k^ε is bounded in measures $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and precompact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for $1 \le p < \frac{d+2}{d+1}$.

From now on we proceed as in the case of the relaxation approximation. The function $\chi^\varepsilon = 1\!\!1(u^\varepsilon, \xi)$ satisfies the transport equation

$$
\partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon = \sum_{j=1}^d \partial_{x_j} \left(\bar{g}_j^\varepsilon + \partial_\xi g_j^\varepsilon \right) + \partial_\xi k^\varepsilon \qquad \text{in} \quad D'_{x,t,\xi}, \tag{3.13}
$$

where $\bar{g}_j^\varepsilon$, $g_j^\varepsilon \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. Using the averaging lemma (Theorem 3), we deduce $u^\varepsilon \to u$ in L^p_{loc}, $p < 2$, and almost everywhere and that $u \in L^\infty(L^2)$ satisfies the scalar conservation law.

To pass to the limit in (3.13), note that

$$
\begin{aligned}
&\chi^\varepsilon = 1\!\!1(u^\varepsilon, \xi) \to \chi = 1\!\!1(u, \xi) && \text{a.e. and in } L^p_{loc,x,t}(L^p_\xi),\ 1 \le p < 2 \\
&k^\varepsilon = G^\varepsilon \delta(u^\varepsilon - \xi) \rightharpoonup k && \text{weak-}\star \text{ in } \mathcal{M}_{x,t,\xi}
\end{aligned} \tag{3.14}
$$

and thus χ satisfies

$$\partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_\xi k \qquad \text{in} \quad D'_{x,t,\xi}. \tag{3.15}$$

For $\delta = O(\varepsilon^2)$ the bounded measure k may, in general, be nonpositive. By contrast, for $\delta = o(\varepsilon^2)$ the function $\chi = \mathbb{1}(u, \xi)$ satisfies the kinetic formulation of Lions-Perthame-Tadmor

$$\partial_t \chi + a(\xi) \cdot \nabla \chi = \partial_\xi m$$

with m a positive, bounded measure, and thus u is the unique entropy solution of (1.1) (see [15]). To see that, let m denote the weak-$\star$ limit :

$$\Big(\varepsilon \sum_{j=1}^{d} |u^\varepsilon_{x_j}|^2\Big)\delta(u^\varepsilon - \xi) \rightharpoonup m \quad \text{weak-}\star \text{ in measures.}$$

By lemma 7 and (3.12), we have for $\delta = o(\varepsilon^2)$

$$\Big| G^\varepsilon - \varepsilon \sum_{j=1}^{d} |u^\varepsilon_{x_j}|^2 \Big| \to 0 \quad \text{in } L^1_{x,t}$$

and thus $k = m \geq 0$. $\square$

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