L^1 stability for the one-dimensional Broadwell model of a discrete velocity gas

Seung-Yeal Ha[1] and Athanasios E. Tzavaras[2]

[1] Department of Mathematics, University of Wisconsin-Madison, *ha@math.wisc.edu*,
[2] Department of Mathematics, University of Wisconsin-Madison, *tzavaras@math.wisc.edu*

Summary. We prove L^1 stability for the one-dimensional Broadwell model for a discrete velocity gas. For initial data in $L^1(R) \cap L^\infty_+(R)$ with small mass, we show that bounded mild solutions are L^1-stable. For this, we employ a nonlinear functional $\mathcal{H}(t)$ that is equivalent to the L^1 distance between two mild solutions and non-increasing in time t.

1 Introduction

The celebrated Boltzmann equation describes the evolution of the density function $f(x, v, t)$ for a dilute gas:

$$\partial_t f(x,v,t) + v \cdot \partial_x f(x,v,t) = Q(f,f)(x,v,t). \tag{1}$$

The transport operator $\partial_t f(x,v,t) + v \cdot \partial_x f(x,v,t)$ describes the free motion of non-interacting particles while the integral operator $Q(f,f)$ accounts for binary collisions between particles. The discretization of the velocity space in the Boltzmann equation (1) allows to replace the integro-differential equation by a system of semilinear hyperbolic equations. The paradigm for one-dimensional discrete Boltzmann equations is the one-dimensional Broadwell model:

$$\begin{aligned} &\partial_t f_1 - \partial_x f_1 = f_2^2 - f_1 f_3, \\ &\partial_t f_2 = -\frac{1}{2}(f_2^2 - f_1 f_3), \\ &\partial_t f_3 + \partial_x f_3 = f_2^2 - f_1 f_3. \end{aligned} \tag{2}$$

Here, f_1, f_2 and f_3 are densities of particles moving with velocities $-1, 0$ and 1 respectively. This system was proposed by Broadwell [5] as one of the simplest models for the description of a dilute gas, with molecules moving with finite speeds. Let $Q_i(f,f)$ denote the collision terms appearing in the equation for the f_i,

$$Q_1(f,f) = -2Q_2(f,f) = Q_3(f,f) \equiv f_2^2 - f_1 f_3.$$

The definition of solution in the mild sense can be stated as follows.

Definition 1. *The function* $f = (f_1, f_2, f_3) \in C([0,T]; (L^1(R))^3 \cap (L^\infty(R))^3)$ *is a mild solution for (2) with initial data* $f^0 \in (L^1(R))^3 \cap (L^\infty(R))^3$ *if and only if, for* $t \in [0,T]$*, a.e* $x \in R$ *and* $i = 1,2,3$*,* $f(x,t)$ *satisfies the integral equation*

$$f_i(x,t) = f_i^0(x - v_i t)) + \int_0^t Q_i(f,f)(x - v_i(t-s), s)ds,$$

where $v_1 = -1$, $v_2 = 0$, and $v_3 = 1$. Global existence of mild solutions for (2) is proved by Nishida and Mimura [12] for small L^1-initial data, and by Tartar [15, 16], Bony [3] for one-dimensional discrete Boltzmann equations with large L^1 data. Large time behavior of mild solutions was studied in Beale [1, 2]. We refer for further references to the survey article by Platkowski and Illner [13].

For discrete velocity Boltzmann equations the kinetic function f is expected to be positive, and the mass is invariant in time. The L^1 topology is thus a natural framework for the analysis. Our stability analysis will monitor the L^1 norm, it is based on appropriate nonlinear functionals and is motivated by the Liu-Yang's functional [11] utilized in stability for hyperbolic systems of conservation laws. In the sequel, we denote the weighted L^1 distance by $||\cdot||$. For a mild solution $f = (f_i)_{i=1}^3$, $||f||$ is defined by

$$||f(\cdot,t)|| \equiv ||f_1(\cdot,t)||_{L^1(R)} + 4||f_2(\cdot,t)||_{L^1(R)} + ||f_3(\cdot,t)||_{L^1(R)}.$$

One obtains by a direct calculation

$$\frac{d}{dt}||f(\cdot,t) - \bar{f}(\cdot,t)|| \le \bar{c}\mu||f(\cdot,t) - \bar{f}(\cdot,t)||,$$

where $\mu = \sup_{t\ge 0,i} ||f_i(\cdot,t)||_{L^1(R)} + \sup_{t\ge 0,i} ||\bar{f}_i(\cdot,t)||_{L^1(R)}$ and $\bar{c}$ is a positive constant. As long as the solutions are uniformly bounded in x and t (which is the case for general classes of one-dimensional discrete velocity Boltzmann models [1, 2, 3]), we have local in time L^1 stability:

$$||f(\cdot,t) - \bar{f}(\cdot,t)|| \le e^{\bar{c}\mu t}||f^0(\cdot) - \bar{f}^0(\cdot)||, \tag{3}$$

The objective of this article is to obtain uniform L^1 stability for (2) by introducing a nonlinear functional which effectively accounts for dispersive effects of the evolution. The nonlinear functional $\mathcal{H}(t) \equiv \mathcal{H}[f(\cdot,t), \bar{f}(\cdot,t)]$ is defined as a linear combination of two subfunctionals $\mathcal{L}(t)$ and $\mathcal{Q}_d(t)$. The functional $\mathcal{L}(t)$ measures the L^1 distance between two mild solutions. Complementing this, the functional $\mathcal{Q}_d(t)$ measures the nonlinear coupling between particles with different velocities. In Section 3, we show that the functional $\mathcal{H}(t)$ satisfies the following two key properties:

1. $\frac{1}{C_0}||f(\cdot,t) - \bar{f}(\cdot,t)|| \le \mathcal{H}(t) \le C_0||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(R)}$,
2. $\mathcal{H}(t)$ is non-increasing in time t, i.e., $\mathcal{H}(t) \le \mathcal{H}(0), \quad t \ge 0$,

where f and $\bar{f}$ are mild solutions of (2) emanating from data f_0 and $\bar{f}_0$ respectively, and C_0 is a positive constant independent of time. From these properties L^1 stability follows:

$$||f(\cdot,t) - \bar{f}(\cdot,t)|| \le C_0 \mathcal{H}(t) \le C_0 \mathcal{H}(0) \le C_0^2 ||f^0(\cdot) - \bar{f}^0(\cdot)||.$$

The time-decay estimate of the functional $\mathcal{H}(t)$ is based on the following conditions:

- Strict hyperbolicity of the system (2).
- Smallness of initial data in L^1.
- Conservation of mass $\sum_{i=1}^{3} Q_i(f,f)(x,t) = 0$, for given $(x,t) \in R \times R^+$.

Remark 1. As in [3, 4], we do not use the H-theorem in our L^1-analysis. Since our initial data are a small perturbation of a vacuum state and the system is strictly hyperbolic, the particles will eventually decouple so that time-asymptotically the solutions tend to non-interacting states.

For a class of discrete velocity Boltzmann equations with transversal interaction terms (which do not include the Broadwell model), L^1 stability was obtained by Tartar [16].

The main theorem is as follows.

Theorem 1. *Let f and $\bar{f} \in C(R_+; (L^1(R))^3 \cap (L^\infty_+(R))^3)$ be two mild solutions of (2) corresponding to initial data f^0 and $\bar{f}^0 \in (L^1(R))^3 \cap (L^\infty_+(R))^3$ such that $||f^0|| + ||\bar{f}^0|| < 2$. Then, we have*

$$||f(\cdot,t) - \bar{f}(\cdot,t)|| \le C||f^0(\cdot) - \bar{f}^0(\cdot)||,$$

where C is a positive constant independent of time t.

This paper is organized as follows. In Section 2, we briefly review the basics of discrete Boltzmann equations and global existence results for mild solutions to the system (2). In Section 3, we explicitly construct the nonlinear functional for the one-dimensional Broadwell model, we study its time-evolution eventually proving L^1 stability for the one-dimensional Broadwell model. For more general one-dimensional models, of the type pursued in [2], L^1 stability analysis will be addressed in the forthcoming paper [6].

2 Preliminaries

In this section, we review the basics of one-dimensional discrete velocity Boltzmann equations (see [2, 3, 7] and [10, Appendix]),

$$\partial_t f_i + v_i \partial_x f_i = \sum_{j,k,l} (A_{ij}^{kl} f_k f_l - A_{kl}^{ij} f_i f_j), \tag{4}$$

where the collision coefficients A_{ij}^{kl} are positive and satisfy the symmetry and micro-reversibility conditions:

$$A_{ij}^{kl} = A_{ij}^{lk} = A_{ji}^{kl}, \quad A_{ij}^{kl} = A_{kl}^{ij}. \tag{5}$$

Such models are derived by considering a three-dimensional discrete velocity model and then restricting to motions that are one dimensional (*e.g.* plane waves). The three-dimensional model is equipped with microscopic conservation of mass, momentum and energy between the pre-collisional and post-collisional velocities. When restricting to one-dimensional motions, the projections of the pre-collisional velocities (in the propagation direction) v_i, v_j and the projections of the post-collisional velocities v_k, v_l satisfy microscopic conservation laws of mass and momentum,

$$v_i + v_j = v_k + v_l, \tag{6}$$

From the viewpoint that the one-dimensional discrete model (4) is a special case of the three-dimensional discrete model, the one-dimensional discrete model does not have to satisfy microscopic conservation of energy. By the same reason, the system does not need to be strictly hyperbolic even though the original three-dimensional system may well be strictly hyperbolic. By assuming the same initial data for particles moving with the same (projections of) velocities and suitable congruence conditions on the transitional probabilities A_{ij}^{kl}, Beale [2] transforms a subclass of the discrete velocity models (4) to

$$\partial_t f_i + v_i \partial_x f_i = \sum_{j,k} B_i^{jk} f_j f_k\,, \tag{7}$$

subject to certain structural assumptions on the interaction coefficients B_i^{jk}. In particular, this class encompasses the model (2). In Section 3, we will discuss stability for (2) while the more general model (7) will be discussed in [6].

Returning to (4), we briefly review the properties of the collision operator:

$$Q_i(f,f) \equiv \sum_{j,k,l} A_{ij}^{kl}(f_k f_l - f_i f_j).$$

Let ϕ be any measurable function. Then,

$$\partial_t \left(\sum_i \phi(v_i) f_i \right) + \partial_x \left(\sum_i v_i \phi(v_i) f_i \right) = \sum_{i,j,k,l} \phi(v_i) A_{ij}^{kl}(f_k f_l - f_i f_j).$$

As in the theory of the Boltzmann equation, the properties (5) imply that the right hand side can be rearranged so that

$$\text{R.H.S.} = \frac{1}{4} \sum_{i,j,k,l} A_{ij}^{kl}(\phi(v_i) + \phi(v_j) - \phi(v_k) - \phi(v_l))(f_k f_l - f_i f_j). \tag{8}$$

Any choice of collisional invariants will then produce an associated macroscopic conservation law. For a model satisfying microscopic mass and momentum conservation, we have macroscopic conservation of mass, momentum:

$$\frac{d}{dt}\sum_{i=1}^{n}\int_R f_i(x,t)dx = 0, \qquad \frac{d}{dt}\sum_{i=1}^{n}\int_R v_i f_i(x,t)dx = 0.$$

The entropy $H(t)$ is defined by

$$H(t) = \sum_{i=1}^{n}\int_R f_i(x,t)\log(f_i(x,t))dx,$$

and satisfies an analog of the Boltzmann H-theorem,

$$\begin{aligned}
&\partial_t\left(\sum_i f_i\log f_i\right) + \partial_x\left(\sum_i v_i f_i\log f_i\right) = \sum_{i,j,k,l} A_{ij}^{kl}\log f_i(f_k f_l - f_i f_j)\\
&= \frac{1}{4}\sum_{i,j,k,l} A_{ij}^{kl}(\log f_i + \log f_j - \log f_k - \log f_l)(f_k f_l - f_i f_j)\\
&= -\frac{1}{4}\sum_{i,j,k,l} A_{ij}^{kl}\log\left(\frac{f_k f_l}{f_i f_j}\right)(f_k f_l - f_i f_j) \le 0, \qquad (9)
\end{aligned}$$

where we have used $A_{ij}^{kl} \ge 0$, (8) and $(\log x - \log y)(x-y) \ge 0$.

A paradigm, for discrete velocity models is the Broadwell model. This corresponds to a specific interaction mechanism [5], and it reads

$$\begin{aligned}
\partial_t f_1^+ + \partial_x f_1^+ &= \frac{1}{2}(f_2^+ f_2^- + f_3^+ f_3^-) - f_1^+ f_1^-,\\
\partial_t f_1^- - \partial_x f_1^- &= \frac{1}{2}(f_2^+ f_2^- + f_3^+ f_3^-) - f_1^+ f_1^-,\\
\partial_t f_2^+ + \partial_y f_2^+ &= \frac{1}{2}(f_1^+ f_1^- + f_3^+ f_3^-) - f_2^+ f_2^-,\\
\partial_t f_2^- - \partial_y f_2^- &= \frac{1}{2}(f_1^+ f_1^- + f_3^+ f_3^-) - f_2^+ f_2^-,\\
\partial_t f_3^+ + \partial_z f_3^+ &= \frac{1}{2}(f_1^+ f_1^- + f_2^+ f_2^-) - f_3^+ f_3^-,\\
\partial_t f_3^- - \partial_z f_3^- &= \frac{1}{2}(f_1^+ f_1^- + f_2^+ f_2^-) - f_3^+ f_3^-,
\end{aligned}$$

Here $f_1^\pm$, $f_2^\pm$ and $f_3^\pm$ are densities of particles moving with velocities ± 1 in the direction of x, y, and z axis respectively. If we consider one-dimensional motions of particles, independent of the y and z coordinates, and motivated by symmetry considerations, set

$$f_1^+ = f_1, \quad f_1^- = f_3, \quad f_2^+ = f_2^- = f_3^+ = f_3^- = f_2.$$

the above system reduces to the one-dimensional Broadwell model (2). For (2) we have the following global existence theorem (see [2, 3] and [12] and [15] for earlier variants).

Theorem 2. *For f^0 in $(L^1(R))^3 \cap (L^\infty_+(R))^3$ there exist a unique globally defined nonnegative solution of (2) in the sense of Definition 1.*

3 L^1 stability

In this section, we construct an explicit nonlinear functional $\mathcal{H}(t)$ which is equivalent to the L^1 distance and non-increasing in time along solutions.

Let f and $\bar{f}$ be two solutions of the Broadwell model which for the time are taken to be C^1. We use the notation $f(x)$, $f(y)$ for the evaluation of f at the points (x,t), (y,t) respectively; the t dependence is mostly suppressed. From (2), we derive the conservations for the partial masses,

$$\partial_t(2f_2 + f_3)(x) + \partial_x f_3(x) = 0. \tag{10}$$

$$\partial_t(f_1 + 2f_2)(y) - \partial_y f_1(y) = 0, \tag{11}$$

In the sequel, we omit the t-dependence and use the simple notation

$$\delta_i(x,t) = \mathrm{sgn}(f_i(x,t) - \bar{f}_i(x,t))$$

for the sign of the difference. We also define a potential of interaction functional $Q(t)$,

$$Q(t) \equiv \int_{R^2} \mathbf{1}_{x<y}[2f_2(x) + f_3(x)][f_1(y) + 2f_2(y)]dydx,$$

whose definition is motivated in the following lemma. This lemma appears in Tartar [17] where it is attributed to Varadhan.

Proposition 1. *Along solutions f of (2), we have*

$$\frac{dQ(t)}{dt} = -2\int_R \big(f_1f_3 + f_1f_2 + f_2f_3\big)(x,t)\,dx.$$

Proof. We multiply (10) by $(f_1 + 2f_2)(y)$ and (11) by $(2f_2 + f_3)(x)$. Adding and multiplying the resulting identity by $\mathbf{1}_{x<y}$, we arrive at the identity

$$\begin{aligned}
&\partial_t\Big[(f_1 + 2f_2)(y)\,(2f_2 + f_3)(x)\,\mathbf{1}_{x<y}\Big] \\
&\quad + \mathrm{div}_{(x,y)}\Big[\Big(f_3(x)(f_1 + 2f_2)(y), -(2f_2 + f_3)(x)f_1(y)\Big)\mathbf{1}_{x<y}\Big] \\
&\qquad + \delta(x-y)\Big(f_3(x)(f_1 + 2f_2)(y) + f_1(y)(2f_2 + f_3)(x)\Big) = 0
\end{aligned}$$

Note that the last term is positive and provides some decay by dispersion. Integrating the above equation over R^2, we obtain

$$\frac{dQ(t)}{dt} = -2\int_R [f_1(x)f_3(x) + f_1(x)f_2(x) + f_2(x)f_3(x)]dx.$$

This completes the proof. $\square$

Next, we define certain nonlinear functionals and study their time-variations.

$$
\begin{aligned}
\mathcal{L}(t) &\equiv \int_R \Big(|f_1-\bar f_1| + 4|f_2-\bar f_2| + |f_3-\bar f_3|\Big)(x)dx,\\
\mathcal{Q}_d(t) &\equiv \int_{R^2} \mathbf{1}_{x<y}\Big(\big(2f_2+2\bar f_2+f_3+\bar f_3\big)(x)\big(|f_1-\bar f_1|+2|f_2-\bar f_2|\big)(y)\Big)dxdy\\
&\quad + \int_{R^2} \mathbf{1}_{x<y}\Big(\big(2|f_2-\bar f_2|+|f_3-\bar f_3|\big)(x)\big(f_1+\bar f_1+2f_2+2\bar f_2\big)(y)\Big)dxdy,\\
\mathcal{H}(t) &\equiv \mathcal{L}(t) + K\mathcal{Q}_d(t),
\end{aligned}
$$

where K is positive constant to be determined later. We also define the instantaneous interaction productions as follows.

$$\Lambda(f,\bar f)(t) = \Lambda_s(f,\bar f)(t) + \Lambda_d(f,\bar f)(t), \tag{12}$$

$$\Lambda_s(f,\bar f) \equiv \int_R \left(2 - \frac{\delta_1(x)}{\delta_2(x)} - \frac{\delta_3(x)}{\delta_2(x)}\right)|f_2(x)-\bar f_2(x)|(f_2(x)+\bar f_2(x))dx \tag{13}$$

$$\Lambda_d(f,\bar f)(t) \equiv \int_R \sum_{m=1}^{3} |f_m(x)-\bar f_m(x)| \sum_{n\neq m}(f_n(x)+\bar f_n(x))dx. \tag{14}$$

Note that the above functionals are all positive and that for positive and bounded solutions $\mathcal{H}(t)$ is equivalent to the L^1 norm. Furthermore, $\mathcal{L}(t)$ denotes the weighted L^1 distance (and the total mass at time t) while $\mathcal{Q}_d(t)$ represents the potential of interactions between particles.

Next, we study the time-evolution of these functionals.

Lemma 1. *Let f and $\bar f$ be two solutions of (2) corresponding to initial data f^0 and $\bar f^0$ with $||f^0+\bar f^0|| < 2$. Then K can be selected so that the functionals satisfy the Lyapunov type estimates:*

$$
\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} &\le -\Lambda_s(f,\bar f)(t) + \Lambda_d(f,\bar f)(t),\\
\frac{d\mathcal{Q}_d(t)}{dt} &\le (-2+||f+\bar f||)\Lambda_d(f,\bar f)(t),\\
\frac{d\mathcal{H}(t)}{dt} &\le -C_1\Lambda(f,\bar f)(t),
\end{aligned}
$$

where C_1 is a positive constant independent of time t.

Proof. First, we derive the equations for the differences $|f_i(x,t)-\bar f_i(x,t)|$, $1\le i\le 3$, in the form

$$\partial_t|f_1-\bar f_1| - \partial_x|f_1-\bar f_1| = \frac{\delta_1}{\delta_2}|f_2-\bar f_2|(f_2+\bar f_2) - |f_1-\bar f_1|\frac{f_3+\bar f_3}{2}$$

$$
\begin{aligned}
&\qquad -\frac{\delta_1}{\delta_3}|f_3-\bar f_3|\frac{f_1+\bar f_1}{2},\\
\partial_t|f_2-\bar f_2| &= -\frac{1}{2}|f_2-\bar f_2|(f_2+\bar f_2)+\frac{\delta_2}{2\delta_1}|f_1-\bar f_1|\frac{f_3+\bar f_3}{2}\\
&\qquad +\frac{\delta_2}{2\delta_3}|f_3-\bar f_3|\frac{f_1+\bar f_1}{2}\\
\partial_t|f_3-\bar f_3|+\partial_x|f_3-\bar f_3| &= \frac{\delta_3}{\delta_2}|f_2-\bar f_2|(f_2+\bar f_2)-\frac{\delta_3}{\delta_1}|f_1-\bar f_1|\frac{f_3+\bar f_3}{2}\\
&\qquad -|f_3-\bar f_3|\frac{f_1+\bar f_1}{2}.
\end{aligned}
\tag{15}
$$

Step 1: We consider the functionals separately. From (15) we have

$$
\begin{aligned}
&\partial_t\Big(|f_1-\bar f_1|+4|f_2-\bar f_2|+|f_3-\bar f_3|\Big)+\partial_x\big(|f_3-\bar f_3|-|f_1-\bar f_1|\big)\\
&\qquad +\big(2-\frac{\delta_1}{\delta_2}-\frac{\delta_3}{\delta_2}\big)|f_2-\bar f_2|(f_2+\bar f_2)\\
&= \big(2\frac{\delta_2}{\delta_1}-\frac{\delta_3}{\delta_1}-1\big)|f_1-\bar f_1|\frac{f_3+\bar f_3}{2}+\big(2\frac{\delta_2}{\delta_3}-\frac{\delta_1}{\delta_3}-1\big)|f_3-\bar f_3|\frac{f_1+\bar f_1}{2}
\end{aligned}
\tag{16}
$$

By the definitions of $\mathcal{L}(t)$, Λ_s and Λ_d, we have

$$
\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} &\le -\Lambda_s(f,\bar f)(t)+\int_R |f_1-\bar f_1|(f_3+\bar f_3)+|f_3-\bar f_3|(f_1+\bar f_1)\,dx\\
&\le -\Lambda_s(f,\bar f)(t)+\Lambda_d(f,\bar f)(t)\,.
\end{aligned}
\tag{17}
$$

Step 2: By a direct yet cumbersome calculation, we obtain from (10), (11) and (15) the identity

$$
\begin{aligned}
\frac{d\mathcal{Q}_d(t)}{dt} &= -2\int_R \sum_{m=1}^{3}|f_m-\bar f_m|(x)\left(\sum_{n\neq m}(f_n+\bar f_n)(x)\right)\\
&+\int_{R^2}\mathbf{1}_{x<y}\left(\frac{\delta_1}{\delta_2}(y)-1\right)(f_2+\bar f_2)(y)|f_2-\bar f_2|(y)|(f_3+\bar f_3)(x)\,dxdy\\
&+\int_{R^2}2\mathbf{1}_{x<y}\left(\frac{\delta_1}{\delta_2}(y)-1\right)(f_2+\bar f_2)(y)|f_2-\bar f_2|(y)(f_2+\bar f_2)(x)\,dxdy\\
&+\int_{R^2}\mathbf{1}_{x<y}\left(\frac{\delta_3}{\delta_2}(x)-1\right)(f_2+\bar f_2)(x)|f_2-\bar f_2|(x)|(f_1+\bar f_1)(y)\,dxdy\\
&+\int_{R^2}2\mathbf{1}_{x<y}\left(\frac{\delta_3}{\delta_2}(x)-1\right)(f_2+\bar f_2)(x)|f_2-\bar f_2|(x)|(f_2+\bar f_2)(y)\,dxdy\\
&+\int_{R^2}\mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_1}(y)-1\right)\frac{f_3+\bar f_3}{2}(y)|f_1-\bar f_1|(y)(f_3+\bar f_3)(x)\,dxdy\\
&+\int_{R^2}\mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_1}(y)-1\right)(f_3+\bar f_3)(y)|f_1-\bar f_1|(y)(f_2+\bar f_2)(x)\,dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{R^2} \mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_1}(x)-\frac{\delta_3}{\delta_1}(x)\right)\frac{f_3+\bar f_3}{2}(x)|f_1-\bar f_1|(x)(f_1+\bar f_1)(y)\,dxdy\\
&+\int_{R^2} \mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_1}(x)-\frac{\delta_3}{\delta_1}(x)\right)(f_3+\bar f_3)(x)|f_1-\bar f_1|(x)(f_2+\bar f_2)(y)\,dxdy\\
&+\int_{R^2} \mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_3}(x)-1\right)\frac{f_1+\bar f_1}{2}(x)|f_3-\bar f_3|(x)(f_1+\bar f_1)(y)\,dxdy\\
&+\int_{R^2} \mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_3}(x)-1\right)(f_1+\bar f_1)(x)|f_3-\bar f_3|(x)(f_2+\bar f_2)(y)\,dxdy\\
&+\int_{R^2} \mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_3}(y)-\frac{\delta_1}{\delta_3}(y)\right)\frac{f_1+\bar f_1}{2}(y)|f_3-\bar f_3|(y)(f_3+\bar f_3)(x)\,dxdy\\
&+\int_{R^2} \mathbf{1}_{x<y}\left(\frac{\delta_2}{\delta_3}(y)-\frac{\delta_1}{\delta_3}(y)\right)(f_1+\bar f_1)(y)|f_3-\bar f_3|(y)(f_2+\bar f_2)(x)\,dxdy\\
&\le -2\int_R \sum_{m=1}^{3}|f_m-\bar f_m|(x)\left(\sum_{n\neq m}(f_n+\bar f_n)(x)\right)\\
&+\big(||f_1+\bar f_1||_{L^1(R)}+2||f_2+\bar f_2||_{L^1(R)}\big)\int_R (f_3+\bar f_3)(x)|f_1-\bar f_1|(x)dx\\
&+\big(2||f_2+\bar f_2||_{L^1(R)}+||f_3+\bar f_3||_{L^1(R)}\big)\int_R (f_1+\bar f_1)(x)|f_3-\bar f_3|(x)dx\\
&\le (-2+||f+\bar f||)\Lambda_d(f,\bar f)(t).
\end{aligned}
$$

Step 3: By the definition of $\mathcal{H}(t)$, we have

$$
\begin{aligned}
\frac{d\mathcal{H}(t)}{dt} &= \frac{d\mathcal{L}(t)}{dt}+K\frac{d\mathcal{Q}_d(t)}{dt}\\
&\le -\Lambda_s(f,\bar f)(t)+[1+K(-2+||f+\bar f||)]\Lambda_d(f,\bar f)(t).
\end{aligned}
$$

Since $||f+\bar f||<2$, we can choose K sufficiently large so that

$$1+K(-2+||f+\bar f||)<0.$$

We then have

$$\frac{d\mathcal{H}(t)}{dt}\le -C_1\Lambda(f,\bar f)(t),$$

where C_1 is a positive constant independent of time t. This completes the proof. □

From Lemma 1, we have the following L^1 stability estimate.

Proof of Theorem 1 Let $f_0^{(k)}$ and $\bar f_0^{(k)}$ be smooth approximations of the given initial data f_0 and $\bar f_0$ such that

$$f_0^{(k)}\to f_0,\quad \bar f_0^{(k)}\to \bar f_0 \quad \text{in } L^1(R) \quad \text{as } k\to\infty$$

Then we can construct smooth solutions $f^{(k)}(x,t)$ and $\bar f^{(k)}(x,t)$ corresponding to the data $f_0^{(k)}$ and $\bar f_0^{(k)}$ respectively. It follows from (3) that

$$f^{(k)}(x,t) \to f(x,t), \qquad \bar{f}^{(k)}(x,t) \to \bar{f}(x,t) \quad \text{in } L^1(R) \ \text{ as } k \to \infty.$$

Define the nonlinear functional $\mathcal{H}(t)$ for f and $\bar{f}$ as follows.

$$\mathcal{H}(t) = \mathcal{H}[f(\cdot,t), \bar{f}(\cdot,t)] \equiv \lim_{k\to\infty} \mathcal{H}[f^{(k)}(\cdot,t), \bar{f}^{(k)}(\cdot,t)].$$

Then by the two key-properties of $\mathcal{H}(t)$ for $f^{(k)}$ and $\bar{f}^{(k)}$, we have

$$||f^{(k)}(\cdot,t) - \bar{f}^{(k)}(\cdot,t)||_{L^1(R)} \le C||f_0^{(k)}(\cdot) - \bar{f}_0^{(k)}(\cdot)||_{L^1(R)},$$

where C is a positive constant independent of time t and k. Letting $k \to \infty$, we have

$$||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(R)} \le C||f_0(\cdot) - \bar{f}_0(\cdot)||_{L^1(R)}.$$

This completes the proof. □

References

1. Beale, J.T. (1985): *Large-time behavior of the Broadwell model of a discrete velocity gas.* Comm. Math. Phys., 102, 217-235.
2. Beale, J. T. (1986): *Large-time behavior of discrete velocity Boltzmann equations.* Comm. Math. Phys., 106, no. 4, 659-678.
3. Bony, J.-M. (1987): *Solutions globales bornées pour les modèles discrete de l'équation de Boltzmann en dimension 1 d'espace.* Actes Journees E.D.P.St. Jean de Monts, n^o XVI.
4. Bony, J.-M. (1991): *Existence globale à données de Cauchy petites pour modèles discrets de l'équation de Boltzmann.* (French) Comm. Partial Differential Equations, 16, no. 4-5, 533-545.
5. Broadwell, J.E. (1964): *Shock structure in a simple discrete velocity gas.* Phy. Fluids, 7, 1243-1247.
6. Ha, S.-Y., Tzavaras, A.E. (2002): *Lyapunov functionals and L^1 stability for discrete velocity Boltzmann equations.* In preparation.
7. Gatignol, R. (1975): *Théorie Cinétique des gaz à Répartition Discrète de vitesses.* (French) Lectures Notes in Physics, vol. 36, Springer-Verlag.
8. Illner, R. (1982): *Global existence results for discrete velocity models fo the Boltzmann equation.* J.Meca. Th. Appl., 1, 611-622.
9. Kawashima, S. (1981): *Global solution of the initial value problem for a discrete velocity model of the Boltzmann equation.* Proc. Japan Acad., 57, 19-24
10. Kawashima, S. (1987): *Large-time behavior of solutions of the discrete Boltzmann equation.* Commun. Math. Phys., 109, 563-589.
11. Liu, T.-P., Yang, T. (1999): *Well posedness theory for hyperbolic conservation laws.* Comm. Pure Appl. Math. 52, no. 12, 1553-1586.
12. Nishida, T., Mimura, M. (1974): *On the Broadwell model of the Boltzmann equation for a simple discrete velocity gas.* Proc. Japan. Acad., 50, 812-817.
13. Platkowski, T., Illner, R. (1988): *Discrete velocity models of the Boltzmann equation: A survey on the mathematical aspects of the theory.* SIAM Review., vol. 30, no. 2.

14. Slemrod, M., (1990): *Large time behavior of the Broadwell model of a discrete velocity gas with specularly reflective boundary conditions.* Arch. Rational Mech. Anal. 111, 323-342.
15. Tartar, L. (1976): *Existence globale pour un système hyperbolique semi-linéaire de la théorie cinétique des gaz.* (French) Séminarie Goulaouic-Schwartz, École Polytech., Palaiseau, no 1.
16. Tartar, L. (1980): *Some existence theorems for semilinear hyperbolic systems in one space variable.* MRC technical summary report, University of Wisconsin.
17. Tartar, L. (1987): *Oscillations and asymptotic behaviour for two semilinear hyperbolic systems.* In Dynamics of infinite-dimensional systems, NATO Adv. Sci. Inst. Ser. F, 37, Springer, Berlin, 341–356.
18. Tzavaras, A. E. (2000): *On the mathematical theory of fluid dynamic limits to conservation laws.* CMS technical summary report $00-06$.