SELF-SIMILAR FLUID DYNAMIC LIMITS FOR THE BROADWELL SYSTEM

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Abstract. This report discusses a new approach for the resolution of the fluid dynamic limit for the Broadwell system of the kinetic theory of gases, appropriate in the case of Riemann, Maxwellian data. Since the formal limiting system is expected to have self-similar solutions, we are motivated to replace the Knudsen number ε in the Broadwell model so that the resulting model admits self-similar solutions in $\xi = x/t$ and then let ε go to zero. The limiting procedure is justified and the resulting limit is a solution of the Riemann problem for the fluid dynamic limit equations. A class of Riemann data for which this program can be carried out is exhibited. Furthermore it is shown that for the Carleman model the complete program can be done successfully for arbitrary Riemann data.

§0. Introduction

The Broadwell system of discrete kinetic theory is given by the system of partial differential equations

$$\begin{aligned}
\frac{\partial f_1}{\partial t} + c\frac{\partial f_1}{\partial x} &= \sigma(f_3 f_4 + f_5 f_6 - 2 f_1 f_2)\,,\\
\frac{\partial f_2}{\partial t} - c\frac{\partial f_2}{\partial x} &= \sigma(f_3 f_4 + f_5 f_6 - 2 f_1 f_2)\,,\\
\frac{\partial f_3}{\partial t} + c\frac{\partial f_3}{\partial y} &= \sigma(f_1 f_2 + f_5 f_6 - 2 f_3 f_4)\,,\\
\frac{\partial f_4}{\partial t} - c\frac{\partial f_4}{\partial y} &= \sigma(f_1 f_2 + f_5 f_6 - 2 f_3 f_4)\,,\\
\frac{\partial f_5}{\partial t} + c\frac{\partial f_5}{\partial z} &= \sigma(f_1 f_2 + f_3 f_4 - 2 f_5 f_6)\,,\\
\frac{\partial f_6}{\partial t} - c\frac{\partial f_6}{\partial z} &= \sigma(f_1 f_2 + f_3 f_4 - 2 f_5 f_6)\,.
\end{aligned} \tag{0.1}$$

The model describes a gas of particles with identical masses moving along three perpendicular coordinate axes with the same speed c. Results of a particular collision have the same probability and only binary collisions are considered. The functions $f_i = f_i(x, y, t)$, $i = 1, \dots, 6$ denote the densities of particles moving in the six allowed directions; $\sigma/2c$ is the cross section for binary collisions.

For flows which are independent of y, z and for which $f_3 = f_4 = f_5 = f_6$ the above six velocity Broadwell model reduces to the simpler form

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon}\Big(f_3^2 - f_1 f_2\Big) \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon}\Big(f_3^2 - f_1 f_2\Big) \\ \frac{\partial f_3}{\partial t} &= \frac{1}{2\varepsilon}\Big(f_1 f_2 - f_3^2\Big) \end{aligned} \tag{0.2}$$

where for simplicity we have set $c = 1$ and $\sigma = \frac{1}{2\varepsilon}$; ε the Knudsen number or "mean free path" of the gas. As the "mean free path" is the distance between successive collisions, a small mean free path means the gas becomes less rarefied and a "macroscropic" description of the gas based on fluid dynamic Euler and/or Navier-Stokes equations should become meaningful.

The problem of rigorously passing to the fluid dynamic limit has a long history. Here we give a quick summary of relevant results. Additional references may be found in the book of Cercignani [4] for work on the Boltzmann equation and the review paper of Platkowski and Illner [12] with regard to research on discrete velocity models in the kinetic theory of gases.

First within the realm of discrete velocity models the Carleman model does allow for rigorous passage to the fluid dynamic limit. This was done in the work of Kurtz [11]. But as the Carleman model does not conserve momentum it is perhaps a poor test case.

For the Broadwell model the basic result is due to Broadwell himself [1]. One begins by rewriting the system (0.2) as

$$\begin{aligned} \frac{\partial}{\partial t}(f_1 + f_2 + 4f_3) + \frac{\partial}{\partial x}(f_1 - f_2) = 0\,, \\ \frac{\partial}{\partial t}(f_1 - f_2) + \frac{\partial}{\partial x}(f_1 + f_2) = 0\,, \\ \frac{\partial f_3}{\partial t} = \frac{1}{2\varepsilon}(f_1 f_2 - f_3^2)\,. \end{aligned} \tag{0.3}$$

Next one makes the ansatz of travelling wave solutions $f_1 = f_1(\theta), f_2 = f_2(\theta),\ \ f_3 = f_3(\theta),\ \ \theta = \frac{x-st}{\varepsilon}$ where s will be the speed of the wave. Substitution of this ansatz into (0.3) yields the system of ordinary differential equations

$$\begin{aligned} -s(f_1+f_2+4f_3)' + (f_1-f_2)' = 0\,, \\ -s(f_1-f_2)' + (f_1+f_2)' = 0\,, \\ -sf_3' = \frac{1}{2}(f_1f_2 - f_3^2)\,. \end{aligned} \tag{0.4}$$

We pose downstream and upstream positive, constant data $(f_1, f_2, f_3) \to \left(f_1^{\pm}, f_2^{\pm}, f_3^{\pm}\right)$ as $\theta \to \pm\infty$ which of course is consistent with (0.4) if and only if the data are Maxwellians, i.e. $f_1^{\pm} f_2^{\pm} = f_3^{\pm 2}$. Since s is a constant $(0.4)_1$, $(0.4)_2$ may be integrated from $-\infty$ to θ to yield:

$$-s(f_1+f_2+4f_3) + (f_1-f_2) = -s(f_1^- + f_2^- + 4f_3^-) + (f_1^- - f_2^-)\,,$$

$$-s(f_1-f_2) + (f_1+f_2) = -s(f_1^- - f_2^-) + (f_1^- + f_2^-)\,.$$

These two equations determine $f_1,\ f_2$ as functions of $f_3(\theta)$ and s. Substitution of these functions into $(0.4)_3$ will yield an autonomous scalar ordinary differential equation for f_3 with precisely two equilibrium points at $f_3^{\pm} = (f_1^{\pm} f_2^{\pm})^{1/2}$. Since such boundary value problems must possess solutions it follows that a travelling wave solution exists. The value of s is found by integrating (0.4) from $-\infty$ to $+\infty$:

$$\begin{aligned} -s\Big(f_1^+ + f_2^+ + 4(f_1^+f_2^+)^{1/2}\Big) + (f_1^+ - f_2^+) \\ = -s\Big(f_1^- + f_2^- + 4(f_1^-f_2^-)^{1/2}\Big) + (f_1^- - f_2^-)\,, \\ -s(f_1^+ - f_2^+) + (f_1^+ + f_2^+) = -s(f_1^- - f_2^-) + (f_1^- + f_2^-)\,, \end{aligned} \tag{0.5}$$

and solving the system (0.5) for s. These are just the Rankine-Hugoniot jump conditions.

Once the existence of a travelling wave is established, we let $\varepsilon \to 0+$ and obtain

$$(f_1, f_2, f_3) \to \Big(f_1^-, f_2^-, (f_1^-f_2^-)^{1/2}\Big) \text{ if } x < st\,;\ \Big(f_1^+, f_2^+, (f_1^+f_2^+)^{1/2}\Big) \text{ if } x > st\,.$$

The limit function is a weak solution of the limiting fluid dynamic conservation laws

$$\begin{aligned} \frac{\partial}{\partial t}\Big(f_1 + f_2 + 4(f_1 f_2)^{1/2}\Big) + \frac{\partial}{\partial x}(f_1 - f_2) = 0\,, \\ \frac{\partial}{\partial t}(f_1 - f_2) + \frac{\partial}{\partial x}(f_1 + f_2) = 0\,. \end{aligned} \tag{0.6}$$

This is because the limit function is piecewise constant possessing a jump discontinuity across the shock $x = st$ and across $x = st$ the limit function satisfies the jump condition (0.5). In fact the limit function is a solution to the *Riemann problem* (0.6) with piecewise constant initial data

$$\begin{aligned} f_1 = f_1^- \,(x < 0)\,, \quad f_1 = f_1^+ \,(x > 0)\,; \\ f_2 = f_2^- \,(x < 0)\,, \quad f_2 = f_2^+ \,(x > 0)\,. \end{aligned} \tag{0.7}$$

Here we have used the usual definition of weak solution: a pair of bounded measurable functions $f_1,\ f_2$ on $(-\infty,\infty)\times[0,\infty)$ is called *a weak solution* of the Riemann initial value problem (0.6), (0.7) provided that

$$\int_0^\infty \int_{-\infty}^\infty (f_1 + f_2 + 4(f_1 f_2)^{1/2})\,\frac{\partial\psi}{\partial t} + (f_1 - f_2)\,\frac{\partial\psi}{\partial x}\,dxdt = 0\,,$$

$$\int_0^\infty \int_{-\infty}^\infty (f_1 - f_2)\,\frac{\partial\psi}{\partial t} + (f_1 + f_2)\,\frac{\partial\psi}{\partial x}\,dxdt = 0\,,$$

for all C^∞ functions ϕ, ψ with compact support in $t > 0$, $-\infty < x < \infty$ and

$$\lim_{t\to 0+} f_j(x,t) = \begin{cases} f_j^+, & x > 0 \\ f_j^-, & x < 0 \end{cases}$$

Introduction of the macroscopic variables $\rho = f_1 + 4(f_1 f_2)^{1/2} + f_2,\ \rho u = f_1 - f_2$ shows that the fluid limit equations (0.6), (0.7) can be written in the form of the Euler equations

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho g(u)) = 0\,, \end{aligned} \tag{0.8}$$

with $g(u) := \frac{1}{3}\Big[2(1+3u^2)^{1/2} - 1\Big]$ and Riemann initial data

$$\begin{aligned} \rho = \rho^- \,(x < 0),\ \rho = \rho^+ \,(x > 0)\,; \\ \rho u = \rho^- u^- \,(x < 0),\ \rho u = \rho^+ u^+ \,(x > 0)\,; \\ \rho^\pm \,:= f_1^\pm + 4(f_1^\pm f_1^\pm)^{1/2} + f_2^\pm\,, \\ \rho^\pm u^\pm \,:= f_1^\pm - f_2^\pm\,. \end{aligned} \tag{0.9}$$

It should be noted that the Boltzmann equation also possesses a travelling wave solution for Maxwellian data which are close (Caflisch and Nicolaenko [2]). The data in [2] must also be consistent with relevant fluid dynamic jump conditions which are the Rankine-Hugoniot jump conditions for a shock wave in an ideal fluid.

In summary we see for *Riemann data satisfying the Rankine-Hugoniot jump conditions associated with the fluid dynamic limit equations* (0.6) (or equivalently (0.8)) passage to the fluid dynamic limit for the Broadwell model can be achieved (and with a smallness assumption on the variation of the data for the Boltzmann equation also).

What can be said regarding the fluid dynamic limit for arbitrary data or for that matter even the more restricted case of *arbitrary Riemann data*?

For the case of smooth data Inoue and Nishida [10] have shown that one can pass to the fluid dynamic limit for the Broadwell system on a sufficiently small time interval to yield a smooth solution of the fluid dynamic limit equations. They show *compactness* of the parametrized sequence $\{f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon\}$ satisfying the Broadwell system in a space that allows passage to the fluid dynamic limit. This result does not cover the case when the solutions of the limit equations have shocks.

In a work of complementary nature concerning again solutions without shocks, Caflisch and Papanicolaou [3] show that a given smooth solution to the limit equations can be *approximated* by a solution to the Broadwell system when ε is small. The *approximation* program was recently carried out at the level of solutions with shocks by Xin [15]. He shows that given a piecewise smooth solution (with finitely many noninteracting shocks) of the fluid dynamic-limit equations, there exist solutions to the Broadwell system which converge asymptotically to the fluid dynamical solution as $\varepsilon \to 0+$. Conceptually the *approximation* program presupposes knowledge of a smooth solution in [3] or an admissible solution in [15] to the underlying limit conservation laws (0.6) and is intended as a method to solve the Broadwell system (0.2) based on solutions to (0.6). By contrast the *compactness* method attempts to construct solutions of (0.6) as limits of solutions of (0.2).

In the research discussed here we continue in the spirit of the *compactness* issue. The goal is to extend the success of Broadwell's original travelling wave idea to more general Riemann data, not necessarily consistent with the Rankine-Hugoniot jump conditions. The idea is based on the following observation: Any system of conservation laws

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = 0\,,$$

$U : (-\infty, \infty) \times (0, \infty) \to \mathbb{R}^N$, $F, G : \mathbb{R}^N \to \mathbb{R}^N$, with Riemann data

$$U(x,0) = \begin{cases} U^-, & x < 0 \\ U^+, & x > 0 \end{cases} \tag{0.10}$$

must possess space-time dilation invariance: For any positive constant $\alpha > 0$, the change of variable $(x,t) \to (\alpha x, \alpha t)$ preserves both the equations and the initial data. Hence solutions of Riemann problems are expected to depend only on the similarity variable $\xi = \frac{x}{t}$, that is $U(x,t) = U(\xi)$.

For example, if one was attempting to solve the Riemann problem for a system of conservation laws by the artificial viscosity method

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = \varepsilon \frac{\partial^2 U}{\partial x^2} \tag{0.11}$$

one might first consider substitution of the ansatz $U(x,t) = U(\xi)$ into (0.11). But unfortunately (0.11) does not possess space-time dilational invariance. For this reason Dafermos [5] suggested a new type of "viscous" limit problem

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = \varepsilon t \frac{\partial^2 U}{\partial x^2} \tag{0.12}$$

which does possess space-time dilation invariance. Substitution of $U(x,t) = U(\xi)$ into (0.12) yields the system of ordinary differential equations

$$-\xi F(U(\xi))' + G(U(\xi))' = \varepsilon U'' \tag{0.13}$$

and (0.10) implies boundary conditions

$$U(-\infty) = U^{-}\,,\; U(+\infty) = U^{+}\,. \tag{0.14}$$

In papers [5], [6] Dafermos and DiPerna showed that for $N = 2$ a large class of Riemann problems for hyperbolic conservation laws may be solved as limits of solutions of (0.13), (0.14) as $\varepsilon \to 0+$. The program has been continued in the work of Slemrod [13], Fan [8], and Slemrod and Tzavaras [14].

In the same spirit we easily recognize that the Broadwell system does not possess space-time dilational invariance. Hence we are motivated to consider an artificial Broadwell system

$$\begin{aligned}
\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2)\,,\\
\frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2)\,,\\
\frac{\partial f_3}{\partial t} &= \frac{1}{2\varepsilon t}(f_1 f_2 - f_3^2)\,,
\end{aligned} \tag{0.15}$$

which does possess the desired space-time dilational invariance. (Of course the same observation is true for the Boltzmann equation and any of the standard discrete velocity models in the kinetic theory of gases.)

We make the ansatz $f_1(x,t) = f_1(\xi)$, $f_2(x,t) = f_2(\xi)$, $f_3(x,t) = f_3(\xi)$, with $\xi = \frac{x}{t}$, and substitute into (0.15) to obtain the system of non-autonomous ordinary differential equations

$$\begin{aligned}
-(\xi - 1) f_1'(\xi) &= (f_3^2 - f_1 f_2)/\varepsilon\,,\\
-(\xi + 1) f_2'(\xi) &= (f_3^2 - f_1 f_2)/\varepsilon\,,\\
-\xi f_3' &= (f_1 f_2 - f_3^2)/2\varepsilon\,.
\end{aligned} \tag{0.16}$$

Henceforth, we will use the notation $f = (f_1, f_2, f_3)$.

Since we wish $f_j(x,t) \to f_j^{\pm}$ for $x \gtrless 0$ as $t \to 0+$ we impose boundary data

$$f(-\infty) = f^-, \; f(+\infty) = f^+, \; j = 1, 2, 3 \tag{0.17}$$

where $f^- = (f_1^-, f_2^-, f_3^-)$ and $f^+ = (f_1^+, f_2^+, f_3^+)$ are Maxwellian states: $f_1^{\pm} f_2^{\pm} = f_3^{\pm 2}$.

System (0.16), (0.17) is considerably harder to analyze than system (0.4) obtained from the travelling wave ansatz. The reasons are obvious: (i) (0.16) is non-autonomous in the similarity variable ξ and (ii) (0.16) does not possess any first integrals that allow to reduce the number of dependent variables. It does however posssess a simplification. Since $f^{\pm}$ are equilibria, we must have $f(\xi) = f^-$ on $(-\infty, -1)$ and $f(\xi) = f^+$ on $(1, \infty)$. The boundary conditions at $\xi = \pm\infty$ are replaced by

$$f(-1) = f^-, \quad f(+1) = f^+, \tag{0.18}$$

and (0.16) need only be considered on $-1 < \xi < 1$.

The goal now is twofold. First, to construct solutions of the boundary value problem (0.16), (0.18) for $\varepsilon > 0$ fixed. Second, to show that as $\varepsilon \to 0+$ the solutions f^ε have a limit which is a weak solution to the Riemann problem (0.6), (0.7) for the fluid limit system. Remarkably the second part of this program is easier than the first. We exhibit here only a class of Maxwellian data f^-, f^+ for which solutions f^ε exist to (0.16), (0.18) for all $\varepsilon > 0$. However, for any data $f^{\pm}$ for which solutions f^ε exist for all $\varepsilon > 0$, one can extract a convergent subsequence $f^\varepsilon \to f$ a.e. in $(-\infty, \infty)$. The limiting function f is a local Maxwellian, $f_3 = (f_1 f_2)^{1/2}$ a.e., and a weak solution of the Riemann problem (0.6), (0.7). The proof is based on total variation estimates and use of Helly's theorem.

We note that the approach given here has some resemblance to a recent paper of F.

Golse [9]. In his paper Golse makes the self-similar ansatz for the Broadwell system

$$f_j(x,t) = F_j(\xi)/t \quad , \quad j = 1,2,3 \tag{0.19}$$

where again $\xi = \frac{x}{t}$. Substitution of (0.19) into (0.2) yields the system of ordinary differential equations

$$\begin{aligned} -[(\xi-1)F_1(\xi)]' &= (F_3^2 - F_1F_2)/\varepsilon \ , \\ -[(\xi+1)F_2(\xi)]' &= (F_3^2 - F_1F_2)/\varepsilon \ , \\ -[\xi F_3(\xi)]' &= (F_1F_2 - F_3^2)/2\varepsilon \ , \end{aligned} \tag{0.20}$$

which differs from (0.16) in the fact that the left hand side of (0.16) has differentiation followed by multiplication while (0.20) has the reverse. System (0.20) then has the same property as (0.4) of possessing two first integrals. Golse exploits this property to show the existence of a solution F_1, F_2, F_3 of (0.20) analytic on $-1 < \xi < 1$. The importance of the result is that it displays explicitly the large time $O\left(\frac{1}{t}\right)$ behavior of a class of solutions to the Broadwell system (0.2). The solutions f_1, f_2, f_3 of course do not possess space-time dilational invariance and do not appear relevant to solving the Riemann problem for the limit fluid dynamic system (0.6), (0.7).

The paper is divided into five sections after this one. Section 1 reformulates the nonlinear boundary value problem (0.16), (0.18) and provides information on the behavior of solutions to (0.16), (0.18). Section 2 provides the main result on the fluid dynamic limit problem: if for fixed Maxwellian initial data $f = f^-\,(x < 0)$, $f = f^+\,(x > 0)$ (0.16), (0.18) possesses a solution for all values of the Knudsen number $\varepsilon > 0$ then the sequence of solutions $\{f^\varepsilon(\xi)\}$ of (0.16), (0.18) possesses a subsequence which converges bounded a.e. in $-\infty < x < \infty$, $t > 0$ to a solution of the fluid dynamic Riemann problem (0.6), (0.7) (or equivalently (0.8), (0.9)). Section 3 provides a class of data for which (0.16), (0.18) does indeed possess solutions for all values of the Knudsen number $\varepsilon > 0$. The proof of existence for all $\varepsilon > 0$ is based on the contraction mapping principle and a continuation argument. Section 4 applies the program originally described for the Broadwell system to the simpler Carleman model of the kinetic theory of gases. For the Carleman model we show that for all Riemann, Maxwellian data the fluid dynamic limit can be achieved. Finally Section 5 provides an appendix describing the behavior of non-homogeneous singular first order equations; this appendix is used in Sections 3 and 4.

§1. The nonlinear boundary value problem.

In this section we derive various properties of solutions to the nonlinear boundary value problem (0.16), (0.18), which for convenience we call $(\mathcal{P}_\varepsilon)$.

Since the underlying differential equations are singular, it is worthwhile clarifying what is meant by a solution. The weak form of (0.16) takes the form:

$$-(\xi-1)f_1\big|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} f_1(\tau)d\tau = \frac{1}{\varepsilon}\int_{\xi_1}^{\xi_2} Q(f(\tau))d\tau\,, \tag{1.1}$$

$$-(\xi+1)f_2\big|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} f_2(\tau)d\tau = \frac{1}{\varepsilon}\int_{\xi_1}^{\xi_2} Q(f(\tau))d\tau\,, \tag{1.2}$$

$$-\xi f_3\big|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} f_3(\tau)d\tau = -\frac{1}{2\varepsilon}\int_{\xi_1}^{\xi_2} Q(f(\tau))d\tau\,, \tag{1.3}$$

for $\xi_1, \xi_2 \,\epsilon\, [-1, 1]$, with the notation

$$Q(f) = f_3^2 - f_1 f_2 \tag{1.4}$$

used throughout for the collision operator.

Definition. The triplet $f(\xi) = (\,f_1(\xi)\,,\, f_2(\xi)\,,\, f_3(\xi)\,)$ defined for $\xi \,\epsilon\, [-1, 1]$ is a *solution* of the boundary-value problem $(\mathcal{P}_\varepsilon)$ if $f_1\,,\, f_2\,,\, f_3 \,\epsilon\, C[-1, 1]$ satisfy the integral relations $(1.1-1.3)$ for $\xi_1, \xi_2 \,\epsilon\, [-1, 1]$, and the boundary conditions $f_j(\pm 1) = f_j^{\pm 1}\,,\ j = 1\,,\,2\,,\,3$.

It follows immediately that such a solution enjoys the properties:

(i) $f_1 \,\epsilon\, C^1[-1, 1)\,, \quad f_2 \,\epsilon\, C^1(-1, 1]\,, \quad f_3 \,\epsilon\, C^1([-1, 0) \cup (0, 1])$,

(ii) equations (0.16) are satisfied for $-1 < \xi < 0$ and $0 < \xi < 1$,

(iii) the boundary conditions (0.18) are satisfied at $\xi = -1$ and $\xi = 1$.

Also, solutions of $(\mathcal{P}_\varepsilon)$ satisfy the balance of mass, balance of momentum, and entropy production equations

$$-\xi(f_1 + f_2 + 4f_3)' + (f_1 - f_2)' = 0\,, \tag{1.5}$$

$$-\xi(f_1 - f_2)' + (f_1 + f_2)' = 0\,, \tag{1.6}$$

$$\begin{aligned} -\xi(f_1\,\ell n f_1 + f_2\,\ell n f_2 + 4f_3\,\ell n f_3)' + (f_1\,\ell n f_1 - f_2\,\ell n f_2)' \\ = -\frac{1}{\varepsilon}(f_3^2 - f_1 f_2)(\ell n f_3^2 - \ell n f_1 f_2)\,, \end{aligned} \tag{1.7}$$

for $\xi\,\epsilon\,(-1,0)$ and $\xi\,\epsilon\,(0,1)$. The last equation is obtained upon multiplying (0.16a) by $(\ell n f_1 + 1)$, (0.16b) by $(\ell n f_2 + 1)$, (0.16c) by $4(\ell n f_3 + 1)$ and adding the resulting identities. The weak form of $(1.5 - 1.7)$ reads

$$-\xi(f_1 + f_2 + 4f_3)|_{\xi_1}^{\xi_2} + (f_1 - f_2)|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} (f_1 + f_2 + 4f_3)d\xi = 0\,, \tag{1.8}$$

$$-\xi(f_1 - f_2)|_{\xi_1}^{\xi_2} + (f_1 + f_2)|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} (f_1 - f_2)d\xi = 0\,, \tag{1.9}$$

$$\begin{aligned} -\xi(f_1\ell n f_1 + f_2\ell n f_2 + 4f_3\ell n f_3)|_{\xi_1}^{\xi_2} + (f_1\ell n f_1 - f_2\ell n f_2)|_{\xi_1}^{\xi_2} + \\ \int_{\xi_1}^{\xi_2} (f_1\ell n f_1 + f_2\ell n f_2 + 4f_3\ell n f_3)d\xi = -\frac{1}{\varepsilon}\int_{\xi_1}^{\xi_2} (f_3^2 - f_1 f_2)(\ell n f_3^2 - \ell n f_1 f_2)d\xi \end{aligned} \tag{1.10}$$

for $\xi_1, \xi_2\,\epsilon\,[-1,1]$.

The first lemma plays a key role in the analysis, as it will provide control on the shapes of solutions of $(\mathcal{P}_\varepsilon)$.

Lemma 1.1. Let $f = (f_1, f_2, f_3)$ be a continuous on $[-1,1]$ solution of (0.16). Then

(i) $Q(f(\xi))$ does not change sign on the intervals $(-1,0)$ or $(0,1)$.

(ii) $Q(f(-1)) = Q(f(0)) = Q(f(+1)) = 0$.

Proof. From (0.16) and (1.4), it follows that $Q(f)$ satisfies the differential equation

$$\frac{d}{d\xi}Q(f) = \frac{1}{\varepsilon}\Big(\frac{f_1(\xi)}{\xi+1} + \frac{f_2(\xi)}{\xi-1} + \frac{f_3(\xi)}{\xi}\Big)\, Q(f)\,.$$

By uniqueness for ordinary differential equations, if $Q(f(\xi_0)) = 0$ at some point $\xi_0 \,\epsilon\, (-1,0)$ then $Q(f)$ vanishes on the whole interval $(-1,0)$. The same holds on the interval $(0,1)$, hence (i).

In the integral relation (1.3) set $\xi_2 = \xi \neq 0$ and $\xi_1 = 0$. Then

$$f_3(\xi) = \frac{1}{\xi}\int_0^{\xi} f_3(\tau)d\tau + \frac{1}{2\varepsilon\xi}\int_0^{\xi} Q(f(\tau))d\tau\,.$$

Since $f_3(\xi)$, $Q(f(\xi))$ are continuous we let $\xi \to 0$ and find $f_3(0) = f_3(0) + \frac{1}{2\varepsilon}Q(f(0))$, i.e. $Q(f(0)) = 0$. Similarly, using (1.1) and (1.2) we obtain $Q(f(-1)) = Q(f(1)) = 0$. ∎

For the remainder of the section $f = (f_1, f_2, f_3)$ will stand for a solution of $(\mathcal{P}_\varepsilon)$ defined on $[-1,1]$ and corresponding to some $\varepsilon > 0$ and positive Maxwellian data

$$Q(f^-) = Q(f^+) = 0\,, \quad f_1^{\pm}\,,\, f_2^{\pm}\,,\, f_3^{\pm} > 0\,. \tag{M}$$

The components of f enjoy the regularity : $f_1 \,\epsilon\, C[-1,1]\cap C^1[-1,1)$, $f_2 \,\epsilon\, C[-1,1]\cap C^1(-1,1]$ and $f_3 \,\epsilon\, C[-1,1]\cap C^1([-1,0)\cup(0,1])$. Lemma 1.1 together with the form of the equations (0.16) impose restrictions on the shapes of the functions f_j, allowing only for the following possibilities:

- C_1: $Q(f) > 0$ for $-1 < \xi < 0$ and $0 < \xi < 1$: In this case f_1 is increasing on $(-1,1)$, f_2 is decreasing on $(-1,1)$, and f_3 is decreasing on $(-1,0)$ and increasing on $(0,1)$.
- C_2: $Q(f) < 0$ for $-1 < \xi < 0$ and $Q(f) > 0$ for $0 < \xi < 1$: In this case f_1 is decreasing on $(-1,0)$ and increasing on $(0,1)$, f_2 is increasing on $(-1,0)$ and decreasing on $(0,1)$, f_3 is increasing on $(-1,1)$.

C$_3$: $Q(f) < 0$ for $-1 < \xi < 0$ and $0 < \xi < 1$: In this case f_1 is decreasing on $(-1, 1)$, f_2 is increasing on $(-1, 1)$, and f_3 is increasing on $(-1, 0)$ and decreasing on $(0, 1)$.

C$_4$: $Q(f) > 0$ for $-1 < \xi < 0$ and $Q(f) < 0$ for $0 < \xi < 1$: In this case f_1 is increasing on $(-1, 0)$ and decreasing on $(0, 1)$, f_2 is decreasing on $(-1, 0)$ and increasing on $(0, 1)$, f_3 is decreasing on $(-1, 1)$.

C$_5$: $Q(f) = 0$ for $-1 < \xi < 0$ and/or $Q(f) = 0$ for $0 < \xi < 1$. In this case f_1, f_2, f_3 are constant on the region where $Q(f) = 0$ and f_1, f_2, f_3 have the behavior indicated in Cases 1 – 4 where $Q(f) > 0$ or $Q(f) < 0$.

The next lemma provides L^∞ and total variation bounds for solutions of $(\mathcal{P}_\varepsilon)$.

Lemma 1.2. For data satisfying (M), the functions f_1, f_2, f_3 are positive on $[-1, 1]$. Moreover, f_1, f_2, f_3 are uniformly bounded from above and below by positive constants, and of uniformly bounded total variation on $[-1, 1]$. The bounds depend on the data f^+ and f^- but not on ε.

Proof. The cases C$_1$-C$_5$ are analyzed separately:

C$_1$: Here, monotonicity implies $0 < f_1^- \le f_1 \le f_1^+$, $0 < f_2^+ \le f_2 \le f_2^-$. Since $Q \ge 0$ on $[-1, 1]$, it follows that f_3 does not vanish and

$$0 < (f_1^- f_2^+)^{1/2} < (f_1 f_2)^{1/2} \le f_3 \le \max\{f_3^-, f_3^+\} \qquad \text{on } [-1, 1]\,.$$

C$_2$: The shapes of the f_i's in this case dictate

$$0 < f_3^- \le f_3 \le f_3^+\,, \quad 0 < \min\{f_2^-, f_2^+\} \le f_2 \le f_2(0)\,,$$

$$f_1(0) \le f_1 \le \max\{f_1^-, f_1^+\} \qquad \text{on } [-1, 1]\,.$$

Thus f_2 and f_3 are positive. Lemma 1.1 part (ii) states that Q vanishes at $\xi = 0$, that is $f_1(0)\, f_2(0) = f_3^2(0)$, hence f_1 is also positive. Next, use the balance of total mass obtained from (1.8)

$$-\xi(f_1+f_2+4\,f_3)|_{\xi=-1}^{\xi+1}+(f_1-f_2)|_{\xi=-1}^{\xi=+1}+\int_{-1}^{1}(f_1+f_2+4f_3)d\xi=0\,,$$

to compute the total mass

$$\int_{-1}^{1}(f_1+f_2+4f_3)\,d\xi=2\,f_1^-+2\,f_2^++4(f_3^-+f_3^+)\,.$$

Use again (1.8) but this time on $-1\le\xi\le 0$ to arrive at

$$f_2(0)=\int_{-1}^{0}(f_1+f_2+4f_3)d\xi-2f_1^--4f_3^-+f_1(0)\le C\,,$$

where the constant C depends only on $f^\pm$ but is independent of ε. Since $f_1(0)=f_3^2(0)/f_2(0)$, we conclude that f_1 and f_2 are bounded from above and below by positive constants depending only on the data.

C_3: Here, $0<f_1^+\le f_1\le f_1^-$, $0<f_2^-\le f_2\le f_2^+$. Lemma 1.1 part (ii) implies $Q(f(0))=0$; hence, $f_3(0)$ is bounded by $(f_1^-f_2^+)^{1/2}$ and

$$0<\min\{f_3^-,f_3^+\}\le f_3\le(f_1^-\,f_2^+)^{1/2}\,.$$

C_4: The same argument as given in Case 2 applies here.

C_5: If $f=(f_1,f_2,f_3)$ is constant on either $-1<\xi<0$ or $0<\xi<1$ the monotonicity of f in the region where f is non-constant immediately yields the desired bounds.

The above arguments show the stated uniform bounds. Since f_1,f_2,f_3 may have at most one maximum or minimum on $[-1,1]$, it follows that $f=(f_1,f_2,f_3)$ possesses uniformly bounded total variation as well. ∎

Next, we examine the behavior of f near the singular points with the goal to estimate the modulii of continuity of the components f_j. To this end we use representation formulas and estimates for a singular linear ordinary differential equation that are derived in the Appendix. Naturally, it is not expected that any Hölder or derivative bounds for the f_j's will be independent of ε. To differentiate the dependences, for the rest of the section C_ε will stand for constants that depend on $f^\pm$ and ε, while constants that depend on the data but are ε-independent are denoted by C.

It was shown in Lemma 1.2 that

$$0 < \frac{1}{C} \le f_j(\xi) \le C \qquad \xi \,\epsilon\, [-1,1]\,, \quad j = 1,\,2,\,3; \tag{1.11}$$

hence, as a consequence of (0.16),

$$\begin{aligned} |\, f_1'(\xi)\,| &\le \frac{C}{\varepsilon\,(1-\xi)} && \xi \,\epsilon\, [-1,1) \\ |\, f_2'(\xi)\,| &\le \frac{C}{\varepsilon\,(\xi+1)} && \xi \,\epsilon\, (-1,1] \\ |\, f_3'(\xi)\,| &\le \frac{C}{\varepsilon\,|\xi|} && \xi \,\epsilon\, [-1,1]\backslash\{0\}\,. \end{aligned} \tag{1.12}$$

Relations (1.12) provide derivative bounds away from the singular points $\xi = -1$, 0 and 1 respectively; they are improved in the following lemma.

Lemma 1.3. Let $f = (f_1, f_2, f_3)$ be a solution of $(\mathcal{P}_\varepsilon)$ with $\varepsilon > 0$, corresponding to data satisfying (M). There is p_ε, $1 < p_\varepsilon \le \infty$, such that f_1', f_2', f_3' are p_ε-integrable on $[-1,1]$. The exponent p_ε depends on the data $f^\pm$ and ε, it is increasing as ε decreases, and there is $\varepsilon_0 = \varepsilon_0(f^\pm) > 0$ such that for $\varepsilon < \varepsilon_0$ the exponent $p_\varepsilon = \infty$. Moreover, the L^{p_ε}-norms of f_1', f_2', f_3' are bounded by constants that depend on $f^\pm$ and ε.

Proof. First, we work with the component f_3 for $\xi > 0$ in a neighborhood of 0. Recall that $f_3 \,\epsilon\, C[-1,1]$ satisfies

$$\begin{aligned} -\xi f_3' + \frac{1}{2\varepsilon} f_3 f_3 &= \frac{1}{2\varepsilon} f_1 f_2 \qquad \text{on} \quad (0, 1/2]\,, \\ f_3^2(0) - f_1(0) f_2(0) &= 0 \end{aligned} \tag{1.13}$$

With the purpose to use the results in the Appendix, note that $p = f_3$ satisfies an equation of the form (5.1) on $(0,1/2]$ with $\phi = \frac{1}{2\varepsilon} f_3$ and $h = \frac{1}{2\varepsilon} f_1 f_2$. Hypothesis (H_1) requires

$$\phi(0) = \frac{1}{2\varepsilon} f_3(0) > 0\,,$$

which is valid in our case, and

$$\frac{1}{2\varepsilon} \int_0^{1/2} \left| \frac{f_3(\zeta) - f_3(0)}{\zeta} \right| d\zeta \, < \, \infty \tag{1.14}$$

which we proceed to show.

In the interval $(0,1)$ either $Q(f) > 0$, or $Q(f) < 0$, or $Q(f) = 0$. For concreteness, the analysis is presented for the case $Q > 0$. (The case $Q < 0$ is treated similarly while the case $Q = 0$ is trivial). Note that $Q > 0$ implies f_3 increasing, $f_3 > (f_1 f_2)^{1/2}$ and $f_3 > f_3(0)$ on $(0,1)$. From (1.13) we obtain the identity

$$\begin{aligned} f_3(1/2) - f_3(0) &= \frac{1}{2\varepsilon} \int_0^{1/2} \frac{Q(f(\zeta))}{\zeta} \, d\zeta \\ &= \frac{1}{2\varepsilon} \int_0^{1/2} \left(\frac{f_3(\zeta) - \sqrt{f_1(\zeta)\, f_2(\zeta)}}{\zeta} \right) \left(f_3(\zeta) + \sqrt{f_1(\zeta)\, f_2(\zeta)} \,\right) d\zeta\,, \end{aligned}$$

which, together with (1.11), yields

$$0 < \int_0^{1/2} \frac{f_3(\zeta) - \sqrt{f_1(\zeta)\, f_2(\zeta)}}{\zeta} \, d\zeta \, \le \varepsilon\, C$$

Using the fundamental theorem of calculus, (1.11) and (1.12), we deduce

$$\begin{aligned} 0 &< \int_0^{1/2} \frac{f_3(\zeta) - f_3(0)}{\zeta} \, d\zeta \\ &= \int_0^{1/2} \frac{f_3(\zeta) - \sqrt{f_1(\zeta)\, f_2(\zeta)}}{\zeta} \, d\zeta + \int_0^{1/2} \frac{\sqrt{f_1(\zeta)\, f_2(\zeta)} - \sqrt{f_1(0)\, f_2(0)}}{\zeta} \, d\zeta \\ &= \int_0^{1/2} \frac{f_3(\zeta) - \sqrt{f_1(\zeta)\, f_2(\zeta)}}{\zeta} \, d\zeta + \int_0^{1/2} \left(\int_0^1 \frac{f_1'(t\zeta)\, f_2(t\zeta) + f_1(t\zeta)\, f_2'(t\zeta)}{2\sqrt{f_1(t\zeta)\, f_2(t\zeta)}} \, dt \right) d\zeta \\ &\le \varepsilon\, C + \frac{C}{\varepsilon}\,. \end{aligned}$$

Thus (1.14) follows and Hypothesis (H_1) is satisfied.

Observe next that (5.6) reads in the case of (1.13):

$$\begin{aligned}
&[f_3(\xi)-f_3(0)] + f_3(0)\ \xi^{f_3(0)/2\varepsilon} \int_\xi^{1/2} \Big(\frac{f_3(s)-f_3(0)}{2\,\varepsilon\, s}\Big)\ s^{-f_3(0)/2\varepsilon}\ e^{\omega(\xi)-\omega(s)}\,ds \\
&\quad = [f_3(1/2)-f_3(0)]\ (2\xi)^{f_3(0)/2\varepsilon}\ e^{\omega(\xi)-\omega(1/2)} \\
&\quad + \xi^{f_3(0)/2\varepsilon} \int_\xi^{1/2} \frac{f_1(s)f_2(s)-f_1(0)f_2(0)}{2\,\varepsilon\, s}\ s^{-f_3(0)/2\varepsilon}\ e^{\omega(\xi)-\omega(s)}\,ds \qquad (1.15)
\end{aligned}$$

where

$$\omega(\xi)-\omega(s) = \int_s^\xi \frac{f_3(\zeta)-f_3(0)}{2\,\varepsilon\,\zeta}\,d\zeta\,.$$

The two first terms in (1.15) have the same sign. Thus using (1.15), together with (1.11), (1.12) and (1.14), we find

$$\begin{aligned}
0 < f_3(\xi)-f_3(0) &< C_\varepsilon\,\xi^{f_3(0)/2\varepsilon} + C_\varepsilon\,\xi^{f_3(0)/2\varepsilon}\int_\xi^{1/2} s^{-f_3(0)/2\varepsilon}\,ds \\
&\le \begin{cases} C_\varepsilon(\xi+\xi^{f_3(0)/2\varepsilon}) & \text{if } f_3(0)\neq 2\varepsilon \\ C_\varepsilon\,\xi\,(1+|ln\xi|) & \text{if } f_3(0)=2\varepsilon \end{cases} \qquad (1.16)
\end{aligned}$$

Equation (1.13) implies with the help of (1.11), (1.12) and (1.16) that for $\xi\,\epsilon\,[0,1/2]$

$$\begin{aligned}
0 \le |f_3'(\xi)| &= \left|\frac{1}{2\varepsilon\xi}\big[(f_3^2-f_3^2(0))-(f_1f_2-f_1(0)f_2(0))\big]\right| \\
&\le \begin{cases} C_\varepsilon(1+|\xi|^{\frac{f_3(0)}{2\varepsilon}-1}) & \text{if } f_3(0)\neq 2\varepsilon \\ C_\varepsilon\,(1+|ln\xi|) & \text{if } f_3(0)=2\varepsilon \end{cases} \qquad (1.17)
\end{aligned}$$

This improves $(1.12)_3$ for $\xi>0$ in a neighborhood of 0. Since $f_3(-\xi)$ satisfies an equation of the form (1.13) as well, the same analysis shows that (1.17) in fact holds in a full neighborhood of 0 and ultimately, upon combining with $(1.12)_3$, for $\xi\,\epsilon\,[-1,1]$. From (1.17) we estimate the L^p-norm of f_3' on the interval $[-1,1]$

(i) If $f_3(0)>2\varepsilon$ then $||f_3'||_{L^\infty}\le C_\varepsilon$.

(ii) If $f_3(0)\le 2\varepsilon$ then for $p_\varepsilon < 1/(1-\frac{f_3(0)}{2\varepsilon})$ we have $||f_3'||_{L^{p_\varepsilon}}\le C_\varepsilon$.

Next we turn to the component f_1. If we set $p(\xi) = f_1(1-\xi)$ then p satisfies an equation of the form (5.1)

$$-\xi p'(\xi) + \frac{1}{\varepsilon} f_2(1-\xi)\, p(\xi) = \frac{1}{\varepsilon} f_3^2(1-\xi) \qquad \text{on} \quad (\,0, 1/2\,]\,,$$

$$p(0) = f_1^+ = f_3^2(1)/f_2(1)$$

Hypotheses $(H_1 - H_2)$ are satisfied with $\beta = \gamma = 1$ and (5.12), (5.16) together with (1.11), (1.12) yield

$$|f_1'(\xi)| \leq \begin{cases} C_\varepsilon(1 + |1-\xi|^{\frac{f_2^+}{\varepsilon} - 1}) & \text{if } f_2^+ \neq \varepsilon \\ C_\varepsilon\,(1 + |ln(1-\xi)|) & \text{if } f_2^+ = \varepsilon \end{cases} \qquad \text{for} \quad \xi \,\epsilon\, [-1, 1] \tag{1.18}$$

In turn, (1.18) implies that $||f_1'||_{L^{p_\varepsilon}} \leq C_\varepsilon$ for $p_\varepsilon = \infty$ in case $f_2^+ > \varepsilon$, and for any $p_\varepsilon < 1/(1 - \frac{f_2^+}{\varepsilon})$ in case $f_2^+ \leq \varepsilon$.

Finally, a similar argument, working with the function $p(\xi) = f_2(-1+\xi)$, shows that

$$|f_2'(\xi)| \leq \begin{cases} C_\varepsilon(1 + |1+\xi|^{\frac{f_1^-}{\varepsilon} - 1}) & \text{if } f_1^- \neq \varepsilon \\ C_\varepsilon\,(1 + |ln(1+\xi)|) & \text{if } f_1^- = \varepsilon \end{cases} \qquad \text{for} \quad \xi \,\epsilon\, [-1, 1] \tag{1.19}$$

and that $||f_2'||_{L^{p_\varepsilon}} \leq C_\varepsilon$ for $p_\varepsilon = \infty$ in case $f_1^- > \varepsilon$, and for any $p_\varepsilon < 1/(1 - \frac{f_1^-}{\varepsilon})$ in case $f_1^- \leq \varepsilon$.

We emphasize that for $\varepsilon < \min\{f_1^-,\, f_2^+,\, f_3(0)/2\}$ the derivatives $f_j'(\xi)$ are uniformly bounded on $[-1, 1]$. Since $f_3(0)$ is bounded from below by (1.11), there is a threshold $\varepsilon_0 > 0$ (depending only on $f^\pm$) so that for $\varepsilon < \varepsilon_0$ the norms $||f_j'||_{L^\infty} \leq C_\varepsilon$, $j = 1, 2, 3$. Naturally, the bounds blow up as $\varepsilon \to 0$. ∎

A useful consequence of the derivative bounds, the Sobolev-type inequality

$$|\, f_j(\xi) - f_j(\zeta)\,| = \Big|\int_\zeta^\xi f_j'(s)\,ds\,\Big| \leq ||f_j'||_{L^{p_\varepsilon}}\; |\xi - \zeta|^{1 - \frac{1}{p_\varepsilon}}$$

and Lemma 1.2 is the following corollary:

Corollary 1.4. The functions $f_1,\ f_2,\ f_3$ are Hölder continuous with exponent $\alpha_\varepsilon = 1 - \frac{1}{p_\varepsilon}$ ($\alpha_\varepsilon = 1$ if $p_\varepsilon = \infty$). Moreover, the α_ε-Hölder norms of $f_1,\ f_2,\ f_3$ are bounded by constants depending on $f^\pm$ and ε.

We close this section with a compactness lemma.

Lemma 1.5. Let $\varepsilon > 0$ be fixed and let $\{f^k\} = \{(f_1^k,\ f_2^k\ , f_3^k)\}$ be a sequence of solutions to the nonlinear boundary value problem $(\mathcal{P}_\varepsilon)$ taking on Maxwellian data $f^{\pm k}$ at $\xi = \pm 1$. Assume the data are uniformly bounded from above and below independently of k, i.e. there exist constants $0 < \delta, M < \infty$ so that $\delta \le f^{+k},\ f^{-k} \le M$. Then there exist a subsequence $\{f^{k'}\}$ of $\{f^k\}$ and a continuous function f, such that $f^{k'} \to f$ uniformly on $[-1, 1]$, and f satisfies $(\mathcal{P}_\varepsilon)$ and admits positive Maxwellian boundary data.

Proof. From Corollary 1.4 we see that for each fixed j the sequences $\{f_j^k\}_{k\ge 1}$ lie in bounded sets of $C^\alpha[-1, 1]$. Since on $[-1, 1]$ the embedding $C^\alpha \to C^\beta$ ($\beta < \alpha$) is compact, there is a subsequence $\{f^{k'}\}$ and functions $f_j \,\epsilon\, C^\alpha$ such that $f_j^{k'} \to f_j$ in C^β. In particular, $f_j^{\pm k'} \to f_j^\pm$ and since the data of the sequence correspond to positive Maxwellians, the boundary values $f(\pm 1) = f^\pm$ will be positive Maxwellians as well. Passing to the limit $k' \to \infty$ in (1.1-1.3) we see that f is a solution of $(\mathcal{P}_\varepsilon)$ (in the sense of the definition) and thus enjoys all the additional regularity properties that were proved in this section. ∎

§2. The fluid dynamic limit.

Let $f^\pm$ be fixed positive Maxwellians and let $\{f^\varepsilon\}_{\varepsilon > 0}$ be a family of solutions of $(\mathcal{P}_\varepsilon)$ admitting $f^\pm$ as boundary data. The members of the family are extended to the whole real line, by setting $f^\varepsilon(\xi) = f^-$ for $\xi \le -1$ and $f^\varepsilon(\xi) = f^+$ for $\xi \ge 1$. The extended functions are again denoted by f^ε. In this section we show that the family of the extended functions possesses a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ which converges pointwise to a solution f of the fluid dynamic Riemann problem.

Theorem 2.1. *Let $\{f^\varepsilon\}_{\varepsilon>0}$ be a family of solutions of $(\mathcal{P}_\varepsilon)$ corresponding to data $f^\pm$ satisfying (M). There exists a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and a positive, bounded function f of bounded variation such that $f^{\varepsilon_n} \to f$ pointwise on the reals. The function f is a local Maxwellian, that is $f_3^2 = f_1 f_2$ for a.e. ξ, and satisfies the balance of mass and momentum equations*

$$-\xi\frac{d}{d\xi}(f_1 + f_2 + 4(f_1 f_2)^{1/2}) + \frac{d}{d\xi}(f_1 - f_2) = 0\,, \tag{2.1}$$

$$-\xi\frac{d}{d\xi}(f_1 - f_2) + \frac{d}{d\xi}(f_1 + f_2) = 0\,, \tag{2.2}$$

in the sense of distributions and in the sense of measures.

Proof. Since $Q(f^\pm) = 0$ the functions f^ε satisfy the equations (1.1-1.3) for any ξ_1, ξ_2 in the reals. Lemma 1.2 in conjunction with Helly's theorem implies that $\{f^\varepsilon\}$ possesses a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ which converges pointwise on $[-1,1]$ to a function $f = (f_1, f_2, f_3)$ with positive, bounded components of bounded variation. The functions f^{ε_n} admit constant values $f^\pm$ outside $[-1,1]$; therefore, first $f^{\varepsilon_n} \to f$ pointwise on $(-\infty, \infty)$, and second $f(\xi) = f^-$ for $\xi \le -1$ and $f(\xi) = f^+$ for $\xi \ge 1$.

Let now $\xi_1, \xi_2 \,\epsilon\, [-1,1]$. Equation (1.10) for the functions f^{ε_n}, with the help of the inequality

$$(a-b)\,(\ell n a - \ell n b) \ge 4(a^{1/2} - b^{1/2})^2\,, \qquad a,\, b > 0\,,$$

and the uniform bounds (1.11), gives

$$\begin{aligned}-\varepsilon_n C &\le -\int_{\xi_1}^{\xi_2} ({f_3^{\varepsilon_n}}^2 - f_1^{\varepsilon_n} f_2^{\varepsilon_n})(\ell n\, {f_3^{\varepsilon_n}}^2 - \ell n f_1^{\varepsilon_n} f_2^{\varepsilon_n})\, d\xi \\ &\le -4\int_{\xi_1}^{\xi_2} (f_3^{\varepsilon_n} - (f_1^{\varepsilon_n} f_2^{\varepsilon_n})^{1/2})^2\, d\xi \le 0\,.\end{aligned}$$

Passing to the limit $\varepsilon_n \to 0$, we find from the dominated convergence theorem

$$\int_{\xi_1}^{\xi_2} (f_3 - (f_1 f_2)^{1/2})^2 \, d\xi = 0 \tag{2.3}$$

and thus $f_3 = (f_1 f_2)^{1/2}$ a.e. in $[-1, 1]$.

Let $\varphi(\xi)$ be a C^∞ function with compact support in $(-\infty, \infty)$. Equation (1.5) gives

$$\int_{-\infty}^{\infty} (f_1^{\varepsilon_n} + f_2^{\varepsilon_n} + 4 f_3^{\varepsilon_n}) \, (\xi \, \varphi(\xi))' - (f_1^{\varepsilon_n} - f_2^{\varepsilon_n}) \, \varphi'(\xi) \, d\xi = 0 \, .$$

Letting $\varepsilon_n \to 0$ and using once again the dominated convergence theorem, we obtain

$$\int_{-\infty}^{\infty} (f_1 + f_2 + 4(f_1 f_2)^{1/2}) \, (\xi \, \varphi(\xi))' - (f_1 - f_2) \, \varphi'(\xi) \, d\xi = 0 \, . \tag{2.4}$$

Thus (2.1) holds in the sense of distributions and, because f_1, f_2 are of bounded variation, it also holds in the sense of measures. Passing to the limit $\varepsilon_n \to 0$ in (1.6), we show (2.2) and complete the proof. ∎

With f_1, f_2 as above, define

$$F_1(x,t) = f_1(\frac{x}{t}) \, , \quad F_2(x,t) = f_2(\frac{x}{t}) \, , \qquad (x,t) \, \epsilon \, (-\infty, \infty) \times (0, \infty) \tag{2.5}$$

Clearly $\lim_{t \to 0} F_j(x,t) = f_j^-$ for $x < 0$, f_j^+ for $x > 0$, $j = 1, 2$. Further, a solution (F_1, F_2) of the form (2.5) is a weak solution of (0.6) on $(-\infty, \infty) \times (0, \infty)$ if and only if (f_1, f_2) is a weak solution of (2.1-2.2) on $(-\infty, \infty)$.

The equivalence follows from a transformation among test functions (Dafermos [7]). Indeed, let $\psi(x,t)$ be a C^∞ function with compact support on $(-\infty, \infty) \times (0, \infty)$ and define

$$\varphi(\xi) = \int_0^\infty \psi(\xi t, t) \, t \, dt \tag{2.6}$$

The resulting function $\varphi \, \epsilon \, C^\infty$ and has compact support on $(-\infty, \infty)$. Conversely, any test function φ may be represented in the form (2.6) by choosing $\psi = \varphi(x/t) a(t)$, with $a(t)$

a fixed C^∞ function compactly supported in $(0,\infty)$ and satisfying $\int_0^\infty a(t)\,t\,dt = 1$. For solutions of the type (2.5), the weak formulation for the first (say) equation in (0.6) takes the form

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty (F_1 + F_2 + 4(F_1F_2)^{1/2})\,\psi_t(x,t) + (F_1 - F_2)\,\psi_x(x,t)\,dxdt \\
&= \int_{-\infty}^\infty (f_1 + f_2 + 4(f_1f_2)^{1/2})\,\Big(\int_0^\infty \psi_t(\xi t,t)\,t\,dt\Big) + (f_1 - f_2)\,\Big(\int_0^\infty \psi_x(\xi,t)\,t\,dt\Big)\,d\xi \\
&= \int_{-\infty}^\infty (f_1 + f_2 + 4(f_1f_2)^{1/2})\,\big(-\xi\,\varphi(\xi)\big)' + (f_1 - f_2)\,\varphi'(\xi)\,d\xi
\end{aligned}$$

and the equivalence follows from the chain of identities.

§3. Existence theory for the nonlinear boundary value problem $(\mathcal{P}_\varepsilon)$

The scope of this section is to develop an existence theory for the nonlinear boundary value problem $(\mathcal{P}_\varepsilon)$, consisting of (0.16) and (0.18), for $\varepsilon > 0$ fixed. Throughout, the boundary data $f^\pm$ are assumed to be positive Maxwellian states satisfying (M).

First, some convenient notation is introduced. Let

$$b(f) = (-f_2, -f_1, 2f_3) \tag{3.1}$$

and define the matrices

$$B(f) = \begin{bmatrix} -f_2 & -f_1 & 2f_3 \\ -f_2 & -f_1 & 2f_3 \\ \frac{1}{2}f_2 & \frac{1}{2}f_1 & -f_3 \end{bmatrix} \tag{3.2}$$

$$A(\xi) = \begin{bmatrix} -(\xi-1) & 0 & 0 \\ 0 & -(\xi+1) & 0 \\ 0 & 0 & -\xi \end{bmatrix} \quad , \qquad -1 < \xi < 1\,. \tag{3.3}$$

Then $Q(f) = \frac{1}{2}b(f)\cdot f$, and (0.16) may be written in the form

$$A(\xi)f' = \frac{1}{2\varepsilon}B(f)f\,. \tag{3.4}$$

Next, let $F = (F_1, F_2, F_3)$ be a given solution of $(\mathcal{P}_\varepsilon)$ defined on $[-1,1]$ and taking boundary values $F(\pm 1) = F^\pm$ positive Maxwellian states. The analysis in Section 1 implies that the functions F_j are Hölder continuous with exponent α_ε; moreover, for $\varepsilon < \varepsilon_0(F^\pm)$ they are Lipschitz continuous. Any nearby solution f can be written as

$$f = F + p ,$$

with $p = (p_1, p_2, p_3)$ a perturbation. Since

$$Q(f) = Q(F + p) = Q(F) + b(F) \cdot p + Q(p) , \tag{3.5}$$

p is a continuous function on $[-1, 1]$ that satisfies the equations

$$A(\xi)p' = \frac{1}{\varepsilon} B(F(\xi))p + \frac{1}{2\varepsilon} B(p)p , \tag{3.6}$$

for $\xi \in (-1, 0)$ and $\xi \in (0, 1)$, and the boundary conditions

$$p(\pm 1) = p^\pm := f^\pm - F^\pm . \tag{3.7}$$

Since $f^\pm$ and $F^\pm$ are Maxwellian states, (3.5) imposes the restriction

$$b(F^\pm) \cdot p^\pm + Q(p^\pm) = 0 \tag{3.8}$$

on the data $p^\pm$, and that in turn implies

$$B(F^\pm)p^\pm + \frac{1}{2} B(p^\pm)p^\pm = 0 . \tag{3.9}$$

Our strategy for proving existence is to use a continuation argument on the set of data $f^\pm$ for which $(\mathcal{P}_\varepsilon)$ admits a solution. Any constant Maxwellian is a trivial solution of $(\mathcal{P}_\varepsilon)$, corresponding to data with $f^+ = f^-$, and can serve as a point of departure for the continuation argument. The key ingredients are: Lemma 1.5 and showing that for prescribed $F(\xi)$ and $f^\pm$ with $f^\pm - F^\pm$ sufficiently small a solution of (3.6-3.7) exists. This second objective is pursued here.

We seek for solutions of (3.6-3.7) as fixed points of the map that carries a continuous function $P = (P_1, P_2, P_3)$ with boundary values $P(\pm 1) = p^{\pm}$, to the solution $p = (p_1, p_2, p_3)$ of the boundary value problem

$$\begin{aligned} A(\xi)p' &= \frac{1}{\varepsilon}B(F(\xi))p + \frac{1}{2\varepsilon}B(P(\xi))P(\xi) \\ p(\pm 1) &= p^{\pm} \; . \end{aligned} \tag{3.10}$$

The study of this map is based on properties of linear boundary value problems near the regular singular points $\xi = 0, \ \pm 1$, that are developed in the following subsection.

§3.1. The linearized boundary value problem

Consider the singular, linear, boundary value problem

$$A(\xi)p' = \frac{1}{\varepsilon}B(F(\xi))p + \frac{1}{\varepsilon}G(\xi) \quad , \quad -1 < \xi < 1 \; , \tag{3.11}$$

$$p(\pm 1) = p^{\pm} \; . \tag{3.12}$$

The matrix $B(F(\xi))$ is defined in (3.2) and has Hölder continuous entries on $[-1, 1]$ with exponent α_ε, while $G = (g_1, g_2, g_3)$ is a continuous vector valued function on $[-1, 1]$. The boundary data $p^{\pm}$ are assumed to satisfy

$$\begin{aligned} b(F^+) \cdot p^+ + g_1(+1) &= 0 \\ b(F^-) \cdot p^- + g_2(-1) &= 0 \; . \end{aligned} \tag{3.13}$$

Note that the boundary value problem (3.10) fits into this framework with $G = \frac{1}{2}B(P)P$, and that relations (3.13) reflect the restrictions imposed by (3.8).

Our goal is to study the solvability of (3.11-3.12) subject to (3.13), and to establish estimates for the solution p in terms of the data G and $p^{\pm}$. For the estimates we use the sup-norm $\|\cdot\|$ for scalar or vector-valued functions (depending on the context). The results are summarized in the following Fredholm-alternative type of theorem.

Theorem 3.1. *Assume the boundary data $p^\pm$ satisfy* (3.13). *Then,*

(i) *the boundary-value problem* (3.11-3.12) *admits a unique continuous solution on* $[-1,1]$ *for any data $p^\pm$ compatible with* (3.13), *if and only if the only continuous solution of the homogeneous problem*

$$\begin{aligned} A(\xi)p' &= \frac{1}{\varepsilon}B(F(\xi))p \\ p(\pm 1) &= 0 \end{aligned} \tag{3.14}$$

is the trivial solution $p = 0$.

(ii) *If the only continuous solution of* (3.14) *is the trivial solution, then the solution p of* (3.11-3.12) *satisfies*

$$\sup_{\xi\in[-1,1]} |p(\xi)| \le C(\sup_{\xi\in[-1,1]} |G(\xi)| + |p^+| + |p^-|) \ . \tag{3.15}$$

Proof. The proof is long and will be established in four steps. The main obstacle is the presence of singularities at $\xi = -1,\ 0$ and 1.

The first two steps are preparatory in nature and concern the behavior of solutions in the neighborhood of *one* singular point. It is expedient to consider the auxilliary system

$$\alpha(\xi)\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}' = \frac{1}{\varepsilon}\beta(\xi)\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \frac{1}{\varepsilon}p_3\gamma(\xi) + \frac{1}{\varepsilon}H(\xi) \tag{3.16}$$

$$-\xi p_3' = -\frac{1}{\varepsilon}\varphi(\xi)p_3 + \frac{1}{\varepsilon}\delta(\xi)\cdot(p_1,p_2) + \frac{1}{\varepsilon}h(\xi)\ , \tag{3.17}$$

taken on the interval $[0,\frac{1}{2}]$ under the list of hypotheses: α is a nonsingular continuous matrix on $[0,\frac{1}{2}]$, β a continuous matrix, $\gamma = (\gamma_1,\gamma_2)$, $\delta = (\delta_1,\delta_2)$ and $H = (h_1,h_2)$ continuous vector-valued functions, h is a continuous scalar-valued function, and $\varphi > 0$ is a Hölder continuous, positive, scalar-valued function.

Step 1. Behavior of solutions near a singular point.

It is well known that for a linear homogeneous system of nonsingular ordinary differential equations the solution operator is an isomorphism in the Euclidean space. We proceed to study the analog of this result for a singular system with the structure of (3.16-3.17).

In preparation, consider (3.16-3.17) supplemented with the initial-boundary conditions

$$p_1(0) = p_{10} \ , \ p_2(0) = p_{20} \ , \ p_3(a) = p_{3a} \ , \tag{3.18}$$

where the last condition is applied at some intermediate point $a \in [0, \frac{1}{2}]$. We show:

Theorem 3.2. *It a is sufficiently small, there exists a unique continuous solution of* (3.16-3.17) *that admits the data* (3.18). *It satisfies*

$$\varphi(0)p_3(0) - \delta(0) \cdot (p_1(0), p_2(0)) = h(0) \ . \tag{3.19}$$

Proof. We apply the contraction mapping theorem to an equivalent integral formulation of (3.16-3.18).

Let $\Psi(\xi)$ be a fundamental matrix solution for

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}' = \frac{1}{\varepsilon}\alpha^{-1}(\xi)\beta(\xi)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

with $\Psi(0) = I$. The variation of constants formula applied to (3.16) implies

$$\begin{aligned} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \Psi(\xi)\begin{pmatrix} p_{10} \\ p_{20} \end{pmatrix} &+ \frac{1}{\varepsilon}\Psi(\xi)\int_0^\xi p_3(s)\Psi^{-1}(s)\alpha^{-1}(s)\gamma(s)ds \\ &+ \frac{1}{\varepsilon}\Psi(\xi)\int_0^\xi \Psi^{-1}(s)\alpha^{-1}(s)H(s)ds \ . \end{aligned} \tag{3.20}$$

The representation formula (5.5) in the Appendix, applied to (3.17), yields

$$\begin{aligned} p_3 = &p_{3a}\left(\frac{\xi}{a}\right)^{\frac{1}{\varepsilon}\varphi(0)} e^{w(\xi)-w(a)} \\ &+ \frac{1}{\varepsilon}\xi^{\frac{1}{\varepsilon}\varphi(0)}\int_\xi^a \delta(s)\cdot(p_1, p_2)\, s^{-\frac{1}{\varepsilon}\varphi(0)-1}\, e^{w(\xi)-w(s)}ds \\ &+ \frac{1}{\varepsilon}\xi^{\frac{1}{\varepsilon}\varphi(0)}\int_\xi^a h(s)\, s^{-\frac{1}{\varepsilon}\varphi(0)-1}\, e^{w(\xi)-w(s)}\, ds \ , \end{aligned} \tag{3.21}$$

where in this case

$$w(\xi) = \frac{1}{\varepsilon} \int_0^\xi \frac{\varphi(\zeta) - \varphi(0)}{\zeta} d\zeta \; . \tag{3.22}$$

The integral equations (3.20-3.21) provide an equivalent formulation for the problem consisting of (3.16-3.18).

Next, set

$$V(\xi) = \Psi(\xi) \int_0^\xi \Psi^{-1}(s)\alpha^{-1}(s)H(s)ds$$
$$v(\xi) = \xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^a h(s)\, s^{-\frac{1}{\varepsilon}\varphi(0)-1} e^{w(\xi)-w(s)} ds$$

and define the maps T and S, on spaces of continuous functions, as follows:

(i) The map T carries the pair of continuous functions (P_1, P_2) to p_3 given by

$$p_3(\xi) = p_{3a}(\frac{\xi}{a})^{\frac{1}{\varepsilon}\varphi(0)} e^{w(\xi)-w(a)} +$$
$$+ \frac{1}{\varepsilon}\xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^a \delta(s)\cdot(P_1(s), P_2(s))\, s^{-\frac{1}{\varepsilon}\varphi(0)-1} e^{w(\xi)-w(s)} ds + \frac{1}{\varepsilon} v(\xi) \; .$$

If (P_1, P_2), $(\overline{P}_1, \overline{P}_2)$ are two pairs of continuous functions and $p_3 = T(P_1, P_2)$, $\overline{p}_3 = T(\overline{P}_1, \overline{P}_2)$ their respective images, then

$$(p_3 - \overline{p}_3)(\xi) = \frac{1}{\varepsilon}\xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^a \delta(s)\cdot\Big((P_1 - \overline{P}_1)(s), (P_2 - \overline{P}_2)(s)\Big)\, s^{-\frac{1}{\varepsilon}\varphi(0)-1}\, e^{w(\xi)-w(s)}\, ds$$

and a derivation in the spirit of Lemma 5.1 indicates that the map T satisfies the estimate

$$\|p_3 - \overline{p}_3\| \le C(\|P_1 - \overline{P}_1\| + \|P_2 - \overline{P}_2\| \; , \tag{3.23}$$

where $\|\cdot\|$ stands for the $C[0, a]$ sup-norm and the constant C is independent of a.

(ii) The map S carries the continuous function P_3 to the pair (p_1, p_2) defined by

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \Psi(\xi)\begin{pmatrix} p_{10} \\ p_{20} \end{pmatrix} + \frac{1}{\varepsilon}\Psi(\xi) \int_0^\xi P_3(s)\Psi^{-1}(s)\alpha^{-1}(s)\gamma(s)ds + \frac{1}{\varepsilon}V(\xi) \; .$$

If now P_3 and $\overline{P}_3$ are two continuous functions and $(p_1, p_2) = S(P_3)$, $(\overline{p}_1, \overline{p}_2) = S(\overline{P}_3)$ their respective images, then

$$\begin{pmatrix} p_1 - \overline{p}_1 \\ p_2 - \overline{p}_2 \end{pmatrix} = \frac{1}{\varepsilon}\Psi(\xi)\int_0^{\xi}(P_3 - \overline{P}_3)(s)\Psi^{-1}(s)\alpha^{-1}(s)\gamma(s)ds \ ,$$

and it follows easily that the map S satisfies the estimate

$$\|p_1 - \overline{p}_1\| + \|p_2 - \overline{p}_2\| \leq Ca\|P_3 - \overline{P}_3\| \tag{3.24}$$

with the constant C independent of a.

Combining (3.23) with (3.24) we see that the composite map

$$S \circ T : C[0, a] \times C[0, a] \to C[0, a] \times C[0, a]$$

is a contraction, provided that a is sufficiently small. Therefore, $S \circ T$ admits a unique fixed point which is a solution of (3.20-3.21). Finally, (3.19) follows from relation (5.7) in the Appendix. ∎

Consider now the homogeneous system of (3.16-3.17) with $H = 0$, $h = 0$. Let $\phi_{(1)}$, $\phi_{(2)}$ and $\phi_{(3)}$ be the solutions corresponding to the choices of data (p_{10}, p_{20}, p_{3a}) equal to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Then $\phi_{(1)}$, $\phi_{(2)}$, $\phi_{(3)}$ form a fundamental set of solutions for the homogeneous system (3.16-3.17). They also enjoy the properties:

(a) For $\xi \neq 0$ the vectors $\phi_{(1)}(\xi)$, $\phi_{(2)}(\xi)$, $\phi_{(3)}(\xi)$ are linearly independent.

(b) For $\xi = 0$ the vectors $\phi_{(1)}(0)$, $\phi_{(2)}(0)$, $\phi_{(3)}(0)$ are linearly dependent and span the two-dimensional space defined by (3.19) for $h = 0$.

The general form of a fundamental matrix for the homogeneous (3.16-3.17) is given by

$$[\phi_{(1)}\ \phi_{(2)}\ \phi_{(3)}]S \ ,$$

where each ϕ_j is a column vector and S is a nonsingular constant matrix. From the form of the general fundamental matrix, it follows that any fundamental set enjoys properties (a) and (b).

Step 2. An a-priori estimate

Let now $p = (p_1, p_2, p_3)$ be a solution of (3.16-3.17) on $(0, \frac{1}{2}]$ (not necessarily defined at $\xi = 0$), and introduce the notation $r_j = p_j(\frac{1}{2}),\ j = 1, 2, 3$. We show

Lemma 3.3. The solution p can be extended to a continuous function on $[0, \frac{1}{2}]$ that satisfies (3.19) and the estimate

$$\|p\| \le C\Big(\sum_{j=1}^{3} |r_j| + \|H\| + \|h\|\Big) , \tag{3.25}$$

where $\|\cdot\|$ stands for the (vector or scalar) sup-norm on $[0, \frac{1}{2}]$.

Proof. The variation of constants formula yields from (3.16)

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \Psi(\xi)\Psi^{-1}(\frac{1}{2})\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} - \frac{1}{\varepsilon}\Psi(\xi)\int_{\xi}^{1/2} p_3(s)\Psi^{-1}(s)\alpha^{-1}(s)\gamma(s)ds - \frac{1}{\varepsilon}U(\xi) , \tag{3.26}$$

where

$$U(\xi) = \Psi(\xi)\int_{\xi}^{1/2} \Psi^{-1}(s)\alpha^{-1}(s)H(s)ds . \tag{3.27}$$

From (3.26) we obtain

$$|p_1(\xi)| + |p_2(\xi)| \le C\Big(|r_1| + |r_2| + \int_{\xi}^{1/2} |p_3(s)|ds + |U(\xi)|\Big) . \tag{3.28}$$

Next, (3.21) gives for $a = \frac{1}{2}$

$$\begin{aligned} p_3 &= r_3(2\xi)^{\frac{1}{\varepsilon}\varphi(0)}e^{w(\xi)-w(\frac{1}{2})} + \\ &+ \frac{1}{\varepsilon}\xi^{\frac{1}{\varepsilon}\varphi(0)}\int_{\xi}^{\frac{1}{2}} \delta(s)\cdot(p_1, p_2)\, s^{-\frac{1}{\varepsilon}\varphi(0)-1}\, e^{w(\xi)-w(s)}ds + \frac{1}{\varepsilon}u(\xi) , \end{aligned} \tag{3.29}$$

where

$$u(\xi) = \xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^{\frac{1}{2}} h(s)\, s^{-\frac{1}{\varepsilon}\varphi(0)-1} e^{w(\xi)-w(s)} ds \; . \tag{3.30}$$

Using the estimate

$$e^{w(\xi)-w(s)} = e^{\frac{1}{\varepsilon}\int_s^\xi \frac{\varphi(\zeta)-\varphi(0)}{\zeta} d\zeta} \le e^{\frac{1}{\varepsilon}\int_0^{\frac{1}{2}} |\frac{\varphi(\zeta)-\varphi(0)}{\zeta}| d\zeta} \le C \; ,$$

the inequality

$$\xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^{\frac{1}{2}} s^{-\frac{1}{\varepsilon}\varphi(0)-1} ds \le \frac{\varepsilon}{\varphi(0)} \tag{3.31}$$

and (3.28), we deduce

$$\begin{aligned} |p_3(\xi)| &\le C(|r_3| + |u(\xi)|) + C\xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^{\frac{1}{2}} (|p_1(s)| + |p_2(s)|) s^{-\frac{1}{\varepsilon}\varphi(0)-1} ds \\ &\le C\Big(\sum_{j=1}^3 |r_j| + |u(\xi)|\Big) + \\ &+ C\xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^{\frac{1}{2}} \Big(\int_s^{\frac{1}{2}} |p_3(\zeta)| d\zeta\Big) s^{-\frac{1}{\varepsilon}\varphi(0)-1} ds \\ &+ C\xi^{\frac{1}{\varepsilon}\varphi(0)} \int_\xi^{\frac{1}{2}} |U(s)| s^{-\frac{1}{\varepsilon}\varphi(0)-1} ds \\ &\le C\Big(\sum_{j=1}^3 |r_j| + \|U\| + \|u\|\Big) + C \int_\xi^{\frac{1}{2}} |p_3(\zeta)| d\zeta \; . \end{aligned}$$

Gronwall's inequality then implies

$$|p_3(\xi)| \le C\Big(\sum_{j=1}^3 |r_j| + \|U\| + \|u\|\Big) \; , \; 0 < \xi \le \frac{1}{2} \; . \tag{3.32}$$

Therefore p_3 is bounded and thus p_1, p_2 are Lipschitz. Lemma 5.1 then implies that $p_3(\xi)$ has a limit as $\xi \to 0+$ and the limiting function satisfies (3.19). As a matter of fact, Lemma 5.2 implies that p_3 is Hölder continuous on $[0, \frac{1}{2}]$ On account of (3.28) and (3.32), we obtain

$$|p_1(\xi)| + |p_2(\xi)| \le C\Big(\sum_{j=1}^3 |r_j| + \|U\| + \|u\|\Big) \; .$$

Finally, (3.27), (3.30) and (3.31) imply

$$\|U\| \leq C\|H\| \ , \ \|u\| \leq C\|h\|$$

and the proof is complete. ∎

Step 3. The singular system (3.11)

From now on we focus on the system (3.11). First we explore the relation between (3.11) and the auxilliary system (3.16-3.17) studied in Steps 1 and 2. Clearly, the latter captures the local structure of the former in a positive neighborhood of 0 and thus describes the behavior of solutions to (3.11) as $\xi \to 0+$. More is true: (3.16-3.17) captures the local structure of (3.11) in the neighborhood of each of the singular points, not only as $\xi \to 0+$ but also as $\xi \to 0-$, as $\xi \to 1-$ and as $\xi \to -1+$. To see that, consider a solution $(p_1(\xi),\ p_2(\xi),\ p_3(\xi))$ of (3.11) and suppose we are interested in the behavior as $\xi \to 1-$. Performing the change of variables $\xi = 1 - \zeta$, we deduce that the triplet

$$(q_1(\zeta),\ q_2(\zeta),\ q_3(\zeta)) = (p_2(1-\zeta),\ p_3(1-\zeta),\ p_1(1-\zeta))$$

satisfies for $0 < \zeta \leq \frac{1}{2}$ a system of the form (3.16)-3.17) with $\varphi(\zeta) = F_2(1-\zeta) > 0$ and $\alpha(\zeta)$ a nonsingular matrix on $[0, \frac{1}{2}]$. Lemma 3.3 implies that p can be extended to a continuous function as $\xi \to 1-$, that the limiting value satisfies

$$b(F(+1)) \cdot p(+1) + g_1(+1) = 0 \ ,$$

and that p can be estimated by

$$\sup_{\xi \in [\frac{1}{2},1]} |p(\xi)| \leq C\Big(\sum_{j=1}^{3} |p_j(+\frac{1}{2})| + \sup_{\xi \in [\frac{1}{2},1]} |G(\xi)| \Big) \ .$$

Similar statements follow from analyzing the behavior as $\xi \to 0-$ and as $\xi \to -1+$. They are summarized below:

Corollary 3.4. Let p be a solution of (3.11) on $(-1,0)\cup(0,1)$. Then p can be extended to a function that is Hölder continuous on each of the intervals $[-1,0]$ and $[0,1]$ (though not necessarily continuous at $\xi=0$) and which satisfies:

(i) At the singular points the conditions

$$\begin{aligned} b(F(+1))\cdot p(+1)+g_1(+1)&=0\ ,\\ b(F(-1))\cdot p(-1)+g_2(-1)&=0\ ,\\ b(F(0))\cdot p(0\pm)+g_3(0)&=0\ . \end{aligned} \tag{3.33}$$

(ii) The estimates

$$\|p\|_\pm \le C\Big(\sum_{j=1}^{3}|r_{j\pm}|+\|G\|_\pm\Big)\ , \tag{3.34}$$

where $r_{j\pm}=p_j(\pm\frac{1}{2})$, $j=1,2,3$, $\|\cdot\|_+$ stands for the sup-norm on $[0,1]$ and $\|\cdot\|_-$ for the sup-norm on $[-1,0]$.

Consider now the homogeneous system (3.11) with $G=0$. Choose a suffiently small and assign data for $(p_1(0),\ p_2(0),\ p_3(a))$ equal to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. From Theorem 3.2, we obtain a fundamental set of solutions $\phi^+_{(1)},\ \phi^+_{(2)},\ \phi^+_{(3)}$ on $[0,1]$. On account of (3.33) the functions $\phi^+_{(j)}$ take boundary values satisfying

$$b(F_+)\cdot\phi^+_{(j)}(+1)=0\ ,\ b(F(0))\cdot\phi^+_{(j)}(0+)=0\ . \tag{3.35}$$

Similarly, by assigning data for $(p_1(0),\ p_2(0),\ p_3(-a))$ equal to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ we construct a fundamental set of solutions $\phi^-_{(1)},\ \phi^-_{(2)}$ and $\phi^-_{(3)}$, respectively, on $[-1,0]$. The boundary values of these functions now satisfy

$$b(F_-)\cdot\phi^-_{(j)}(-1)=0\ ,\ b(F(0))\cdot\phi^-_{(j)}(0-)=0\ . \tag{3.36}$$

By their construction and (3.35), (3.36),

$$\phi^+_{(1)}(0+)=\phi^-_{(1)}(0-)\ ,\ \phi^+_{(2)}(0+)=\phi^-_{(2)}(0-)\ ,\ \phi^+_{(3)}(0+)=\phi^-_{(3)}(0-)=0\ . \tag{3.37}$$

Next, we turn to the inhomogeneous system (3.11) and construct a particular solution ψ^+ on $[0,1]$ by solving (3.11) subject to the initial condition $\psi^+(+\frac{1}{2}) = 0$. Note that the resulting solution ψ^+ is bounded, continuous and takes values at $\xi = 0+, +1$ satisfying (3.33). Similarly, we construct a particular solution ψ^- on $[-1,0]$ which satisfies $\psi^-(-\frac{1}{2}) = 0$ and the conditions (3.33) at $\xi = -1, 0-$. Corollary 3.4 and the construction also dictate

$$\|\psi^+\|_+ \le C\|G\|_+ \ , \ \|\psi^-\|_- \le C\|G\|_- \ . \tag{3.38}$$

The general solution of the inhomogeneous system (3.11) is given by

$$p(\xi) = \begin{cases} a_1\phi^-_{(1)}(\xi) + a_2\phi^-_{(2)}(\xi) + a_3\phi^-_{(3)}(\xi) + \psi^-(\xi) \ , \ -1 \le \xi \le 0 \\ \\ b_1\phi^+_{(1)}(\xi) + b_2\phi^+_{(2)}(\xi) + b_3\phi^+_{(3)}(\xi) + \psi^+(\xi) \ , \ 0 \le \xi \le 1 \end{cases} \tag{3.39}$$

with $a_1, \ a_2, \ a_3, \ b_1, \ b_2, \ b_3$ arbitrary constants.

Step 4. The singular boundary-value problem (3.11-3.12)

The boundary conditions (3.12) imply

$$a_1\phi^-_{(1)}(-1) + a_2\phi^-_{(2)}(-1) + a_3\phi^-_{(3)}(-1) = p_- - \psi^-(-1) \ ,$$

$$b_1\phi^+_{(1)}(+1) + b_2\phi^+_{(2)}(+1) + b_3\phi^+_{(3)}(+1) = p_+ - \psi^+(+1) \ .$$

On the other hand, the requirement of continuity $p(0+) = p(0-) = p_0$ dictates

$$a_1\phi^-_{(1)}(0-) + a_2\phi^-_{(2)}(0-) + a_3\phi^-_{(3)}(0-) = p_0 - \psi^-(0-) \ ,$$

$$b_1\phi^+_{(1)}(0+) + b_2\phi^+_{(2)}(0+) + b_3\phi^+_{(3)}(0+) = p_0 - \psi^+(0+) \ .$$

In view of (3.37), it is equivalent to solve the inhomogeneous algebraic system

$$\begin{aligned} &a_1\phi^-_{(1)}(-1) + a_2\phi^-_{(2)}(-1) + a_3\phi^-_{(3)}(-1) = p_- - \psi^-(-1) \ , \\ &(a_1 - b_1)\phi^-_{(1)}(0-) + (a_2 - b_2)\phi^-_{(2)}(0-) = \psi^+(0+) - \psi^-(0-) \ , \\ &b_1\phi^+_{(1)}(+1) + b_2\phi^+_{(2)}(+1) + b_3\phi^+_{(3)}(+1) = p_+ - \psi^+(+1) \ . \end{aligned} \tag{3.40}$$

This is a system of nine equations in six unknowns. However, because of (3.33) applied to the functions $\psi^+, \ \psi^-$, the relations (3.35-3.36) and the compatibility conditions (3.13), at most six equations are linearly independent. Solvability of (3.40) for any choice of the data $p_{\pm}$ is equivalent to the homogeneous algebraic system having only the trivial solution, which is in turn equivalent to the homogeneous problem (3.14) possessing only the trivial solution $p = 0$ in the class of continuous on $[-1, 1]$ functions. This completes the proof of part (i) of the theorem.

To prove part (ii) observe that, if the homogeneous (3.40) has only the trivial solution, then the solution $(a_1, a_2, a_3, b_1, b_2, b_3)$ of the inhomogeneous (3.40) satisfies the estimate

$$\sum_{j=1}^{3} |a_j| + |b_j| \leq C[|p_- - \psi^-(-1)| + |\psi^+(0+) - \psi^-(0-)| + |p_+ - \psi^+(+1)|] \,. \qquad (3.41)$$

Moreover, evaluating (3.39) at $\xi = \pm\frac{1}{2}$, we deduce

$$\sum_{j=1}^{3} |r_{j+}| + |r_{j-}| \leq C \sum_{j=1}^{3} |a_j| + |b_j| \,. \qquad (3.42)$$

Finally, combining (3.34) with (3.41), (3.42) and (3.38), we arrive at (3.15). ∎

§3.2. The nonlinear boundary-value problem

Next, we study the solvability of (3.6-3.7). Recall that $\varepsilon > 0$ is fixed, $F(\xi)$ is a given solution of $(\mathcal{P}_\varepsilon)$, while the data $p^{\pm}$ satisfy the restriction (3.8). We show:

Theorem 3.5. *If the only continuous solution of* (3.14) *is the trivial solution* $p = 0$, *there exists a positive constant* r *such that for any data satisfying* (3.8) *and* $|p^+| + |p^-| < r$ *the boundary-value problem* (3.6-3.7) *has a solution.*

Proof. Since our proof will rely on the contraction mapping theorem first define the closed, bounded subset of the continuous functions

$$\mathcal{A} = \{P \in C[-1, 1] : P(\pm 1) = p^{\pm} \ , \ \|P\| \leq m\}$$

where $\|\cdot\|$ stands for the sup-norm and m is a positive constant. Consider the map $\mathcal{F}$ that carries $P \in \mathcal{A}$ to the solution p of the boundary-value problem (3.10). In view of the Theorem 3.1 and the compatibility conditions (3.8) the map $\mathcal{F}$ is well defined.

Our goal is to show that upon choosing m and r sufficiently small the map $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ and is a contraction. The resulting fixed point $p \in \mathcal{A}$ will then be the claimed solution.

To accomplish that, observe that (3.15) implies

$$\begin{aligned} \|p\| &\leq C(|p^+| + |p^-| + \|\frac{1}{2}B(P)P\|) \\ &\leq K_1(|p^+| + |p^-| + \|P\|^2) \leq K_1(r + m^2) \,. \end{aligned}$$

On the other hand if P, $\overline{P} \in \mathcal{A}$ and $p = \mathcal{F}(P)$, $\overline{p} = \mathcal{F}(\overline{P})$ then using again (3.15) we conclude

$$\begin{aligned} \|p - \overline{p}\| &\leq C\|\frac{1}{2}B(P)P - \frac{1}{2}B(\overline{P})\overline{P}\| \\ &\leq K_2(\|P\| + \|\overline{P}\|)\|P - \overline{P}\| \leq 2K_2 m\|P - \overline{P}\| \,. \end{aligned}$$

Choosing $m < \min\{\frac{1}{2K_1}, \frac{1}{2K_2}\}$ and then $r < \frac{m}{2K_1}$, we see that $\|p\| \leq m$ and that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a contraction. ∎

§3.3. Nonexistence of eigensolutions

In order to apply Theorem 3.5, we need to check that the homogeneous boundary-value problem (3.14):

$$\begin{aligned} -(\xi - 1)p_1' &= \frac{1}{\varepsilon}b(F(\xi)) \cdot p \\ -(\xi + 1)p_2' &= \frac{1}{\varepsilon}b(F(\xi)) \cdot p \\ -\xi p_3' &= -\frac{1}{2\varepsilon}b(F(\xi)) \cdot p \end{aligned} \tag{3.43}$$

with boundary conditions $p(\pm 1) = 0$, does not admit any nontrivial continuous eigensolutions. We have been unable to do that for a general solution $F(\xi)$ of $(\mathcal{P}_\varepsilon)$. (Note however that such a result is true in the case of the Carleman system; see the next Section.) Nevertheless, it is possible to rule out eigensolutions for certain classes of $F(\xi)$, namely:

Lemma 3.6. In either of the cases:

(a) $F(\xi)$ is a constant Maxwellian state, or

(b) $Q(F(\xi)) \geq 0$ for $\xi \in [-1, 1]$,

the homogeneous problem (3.14) possesses only the trivial solution $p = 0$ in the class of continuous functions.

Proof. The proof is based on an energy identity for solutions of (3.43). Multiplying the first equation in (3.43) by $F_2 p_1$, the second by $F_1 p_2$, the third by $4F_3 p_3$ and adding the resulting equations, we obtain

$$-(\xi - 1)F_2\left(\frac{1}{2}p_1^2\right)' - (\xi + 1)F_1\left(\frac{1}{2}p_2^2\right)' - 4\xi F_3\left(\frac{1}{2}p_3^2\right)' = -\frac{1}{\varepsilon}(b(F) \cdot p)^2 \,.$$

Integrating by parts and using (0.16), we arrive at

$$\begin{aligned} -\xi\Big(F_2\frac{1}{2}p_1^2 + F_1\frac{1}{2}p_2^2 + 2F_3p_3^2\Big)' + \Big(F_2\frac{1}{2}p_1^2 - F_1\frac{1}{2}p_2^2\Big)' + \\ + \frac{1}{\varepsilon}Q(F)\Big(\frac{1-\xi}{(\xi+1)}\frac{1}{2}p_1^2 + \frac{\xi+1}{(1-\xi)}\frac{1}{2}p_2^2 + p_3^2\Big) = -\frac{1}{\varepsilon}(b(F) \cdot p)^2 \,. \end{aligned} \tag{3.44}$$

We integrate (3.44) over $[-1, 1]$ and use the continuity of p at $\xi = 0$ and the boundary conditions $p(\pm 1) = 0$ to conclude

$$\begin{aligned} \int_{-1}^{1} \Big(\frac{1}{2}F_2p_1^2 + \frac{1}{2}F_1p_2^2 + 2F_3p_3^2\Big)d\xi + \\ \frac{1}{\varepsilon}\int_{-1}^{1} Q(F)\Big(\frac{1}{2}\frac{1-\xi}{(\xi+1)}p_1^2 + \frac{1}{2}\frac{1+\xi}{(1-\xi)}p_2^2 + p_3^2\Big) + (b(F) \cdot p)^2 d\xi = 0 \,. \end{aligned}$$

Since F is a solution of $\mathcal{P}_\varepsilon$, the functions F_j are strictly positive. Thus, in either case (a) when $Q(F) = 0$ or (b) when $Q(F) \geq 0$ it follows that $p_1 = p_2 = p_3 = 0$ on $[-1, 1]$. ∎

Remark. Both conditions (a) and (b) are stated in terms of the unknown solution F of $(\mathcal{P}_\varepsilon)$. However, since the shapes of the functions F_j necessarily fall under one of the cases $C_1 - C_5$, it is possible to identify hypotheses on the boundary values $F_j^{\pm}$ implying that either (a) or (b) holds. Specifically:

(i) Condition (a) is equivalent to $F_j^- = F_j^+,\ j = 1, 2, 3.$

(ii) If $F_1^- < F_1^+,\ F_2^- > F_2^+$ and $F_3^- = F_3^+$ then Condition (b) is satisfied.

§3.4. Existence theorems for $(\mathcal{P}_\varepsilon)$

We conclude this Section by stating two existence theorems for $(\mathcal{P}_\varepsilon)$, which follow from the preceding analysis. The first applies to data that are close.

Theorem 3.7. *There is a $r > 0$ so that if f^+, f^- satisfy (M) and $|f^+ - f^-| < r$, the boundary-value problem $(\mathcal{P}_\varepsilon)$ has a solution. The parameter r may depend on ε.*

Proof. Take $F(\xi) = f^-$ and use Lemma 3.6 part (a) and Theorem 3.5. ∎

In this context the region of solvability may depend on ε and thus disintegrate as $\varepsilon \downarrow 0$. We next present a class of data for which this possibility is ruled out.

Theorem 3.8. *Let $f^\pm$ satisfy (M) and $f_1^- < f_1^+,\ f_2^- > f_2^+,\ f_3^- = f_3^+$. For any $\varepsilon > 0$ the boundary-value problem $(\mathcal{P}_\varepsilon)$ has a solution.*

Proof. A continuation argument is used. With the parameter $\mu \in [0, 1]$ let

$$f^-(\mu) = f^- \ ,\ f^+(\mu) = \left(f_1^- + \mu(f_1^+ - f_1^-) \ ,\ \frac{f_1^- f_2^-}{f_1^- + \mu(f_1^+ - f_1^-)} \ ,\ f_3^- \right) .$$

Observe that $Q(f^+(\mu)) = 0,\ f^+(0) = f^-,\ f^+(1) = f^+$ and

$$f_1^-(\mu) < f_1^+(\mu) \ ,\ f_2^-(\mu) > f_2^+(\mu) \ ,\ f_3^-(\mu) = f_3^+(\mu) \ ,\ 0 < \mu \le 1 \ .$$

Define

$$\mathcal{C} = \{\mu \in [0, 1] : (\mathcal{P}_\varepsilon) \text{ with data } f^-(\mu),\ f^+(\mu) \text{ has a solution } \} \ .$$

Clearly $0 \in \mathcal{C}$. Furthermore:

(i) *$\mathcal{C}$ is open.*

0 is an interior point of $\mathcal{C}$ (in the relative topology), by Theorem 3.7. If now $\mu \in (0,1) \cap \mathcal{C}$ let F_μ be the corresponding solution of $(\mathcal{P}_\varepsilon)$ and consider (3.6-3.7) for $F = F_\mu$. By virtue of Lemma 3.6 part (b) and Theorem 3.5 this boundary-value problem has a solution. Therefore, μ is an interior point of $\mathcal{C}$.

(ii) *$\mathcal{C}$ is closed.*

This is an immediate consequence of Lemma 1.5.

We conclude that $\mathcal{C} = [0,1]$ and in particular the boundary-value problem $(\mathcal{P}_\varepsilon)$ with data $f^{\pm}$ has a solution. ∎

We emphasize that while the existence of solutions part requires the extra assumptions $f_1^- \le f_1^+$, $f_2^- \ge f_2^+$, $f_3^- = f_3^+$, the fluid-dynamic limit part (Theorem 2.1) imposes no restrictions on the data, aside from the natural requirement that f^+ and f^- are strictly positive Maxwellians.

It is also interesting to see that the special data in Theorem 3.8 cannot be associated with a nontrivial travelling wave solution as described in §0. The reason is simple. From (0.4) we see a travelling wave would have to satisfy the limiting conditions

$$-s(f_1 + f_2 + 4f_3) + (f_1 - f_2)\Big|_{\theta=-\infty}^{\infty} = 0\,,$$

$$-s(f_1 - f_2) + (f_1 + f_2)\Big|_{\theta=-\infty}^{\infty} = 0\,.$$

For the data of Theorem 3.8 this would imply $f_1^- = f_1^+$, $f_2^- = f_2^+$, $f_3^- = f_3^+$ which yields an everywhere constant solution to (0.4).

§4. The Carleman model

The Carleman model of the kinetic theory of gases consists of the system of semilinear hyperbolic equations

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon}(f_2^2 - f_1^2) \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= -\frac{1}{\varepsilon}(f_2^2 - f_1^2)\ , \end{aligned} \tag{4.1}$$

It describes a system of two kinds of particles moving with velocities $+1$ and -1 respectively and colliding according to a nonphysical collision rule. As a model in kinetic theory it has serious defficiencies; it conserves mass but not momentum. Nevertheless, the Carleman model has generated some interest, because it enjoys part of the structure of the Broadwell system and it allows for rigorous passage to the hydrodynamic limit Kurtz [11]. (See Platkowski and Illner [12] for additional references and a discussion of relevant results). In this section we explore the ideas of self-similar hydrodynamic limits for the case of the Carleman model.

As before, a modified version of the Carleman is considered, admitting solutions of the form $f(\xi) = (f_1(\xi), f_2(\xi))$, with $\xi = \frac{x}{t}$, for Riemann data $f^- = (f_1^-, f_2^-)$, $f^+ = (f_1^+, f_2^+)$. In the variable ξ the system under consideration reads

$$-(\xi - 1)f_1' = \frac{1}{\varepsilon}Q(f) \tag{4.2}$$

$$\begin{aligned} -(\xi + 1)f_2' &= -\frac{1}{\varepsilon}Q(f) \\ f(\pm 1) &= f^{\pm}\ , \end{aligned} \tag{4.2}$$

where $Q(f) = f_2^2 - f_1^2$. The data are assumed positive Maxwellians

$$f_1^{\pm},\ f_2^{\pm} > 0 \quad , \quad Q(f^-) = Q(f^+) = 0\ . \tag{4.4}$$

We refer to this problem as $(\mathcal{R}_\varepsilon)$.

Our goal is to present a case where the analysis is simple and can be carried out completely, with no further restrictions on the data. Most of it parallels the analysis presented for the Broadwell model, and our exposition will be sketchy only emphasizing the differences.

Definition. The pair $f(\xi) = (f_1(\xi), f_2(\xi))$ defined on $[-1,1]$ is a solution of $(\mathcal{R}_\varepsilon)$, if $f_1 \in C[-1,1] \cap C^1[-1,1)$, $f_2 \in C[-1,1] \cap C^1(-1,1]$ satisfy for $\xi_1,\ \xi_2 \in [-1,1]$ the integral equations

$$\begin{aligned} -(\xi-1)f_1\Big|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} f_1(\tau)d\tau &= \frac{1}{\varepsilon}\int_{\xi_1}^{\xi_2} Q(f(\tau))d\tau \\ -(\xi+1)f_2\Big|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} f_2(\tau)d\tau &= -\frac{1}{\varepsilon}\int_{\xi_1}^{\xi_2} Q(f(\tau))d\tau \end{aligned} \tag{4.5}$$

and the boundary conditions $f_j(\pm 1) = f_j^{\pm}$, $j = 1, 2$.

Theorem 4.1. *For each pair of positive Maxwellian data f^-, f^+ and for each $\varepsilon > 0$ the boundary value problem $(\mathcal{R}_\varepsilon)$ has a solution. The solution f is Hölder continuous on $[-1,1]$ with exponent α_ε (Lipshitz continuous for $\varepsilon < \varepsilon_0(f^\pm)$).*

Let now $f^\pm$ be fixed and consider a family $\{f^\varepsilon\}_{\varepsilon>0}$ of solutions to $(\mathcal{R}_\varepsilon)$. The functions f^ε are extended to $(-\infty,\infty)$ by setting $f^\varepsilon = f^-$ on $(-\infty,-1)$ and $f^\varepsilon = f^+$ on $(1,\infty)$. The extended functions are called again f^ε and satisfy (4.5) for $\xi_1,\ \xi_2 \in (-\infty,\infty)$.

Theorem 4.2. *Let $\{f^\varepsilon\}_{\varepsilon>0}$ be a family of extended solutions of $(\mathcal{R}_\varepsilon)$ taking fixed, positive Maxwellian data $f^\pm$. There exists a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and a positive, bounded function $\rho(\xi)$ such that*

$$\begin{pmatrix} f_1^{\varepsilon_n}(\xi) \\ f_2^{\varepsilon_n}(\xi) \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} \rho(\xi) \\ \rho(\xi) \end{pmatrix} \text{ for a.e. } \xi \in (-\infty,\infty)\ . \tag{4.6}$$

The function $\rho(\frac{x}{t})$ is a weak solution of

$$\partial_t \rho = 0\ ,\ \rho(x,0) = \begin{cases} 2f^- & x < 0 \\ 2f^+ & x > 0 \end{cases}\ ,$$

that is

$$\rho(\frac{x}{t}) = \begin{cases} 2f^- & x<0,\ t>0 \\ \\ 2f^+ & x>0,\ t>0 \end{cases} .$$

Proof. The strategy for proving Theorems 4.1 and 4.2 is analogous to the Broadwell case. The proof is decomposed in three parts.

Part 1. A-priori estimates

Let f be a solution of $(\mathcal{R}_\varepsilon)$. Using (4.5) it follows that $Q(f(-1)) = Q(f(+1)) = 0$. Since $Q(f)$ satisfies the differential equation

$$\frac{d}{d\xi}Q(f) = \frac{2}{\varepsilon}\Big(\frac{f_2(\xi)}{\xi+1} - \frac{f_1(\xi)}{1-\xi}\Big)Q(f) \ , \tag{4.7}$$

either $Q(f) = 0$ on $(-1,1)$ or it never vanishes. Hence, the shapes of the components $f_1,\ f_2$ are restricted as follows:

C_1: $Q(f) > 0$ on $(-1,1)$. Then f_1 and f_2 are both increasing on $(-1,1)$.

C_2: $Q(f) < 0$ on $(-1,1)$. Then f_1 and f_2 are both decreasing on $(-1,1)$.

C_3: $Q(f) = 0$ on $(-1,1)$. Then f_1 and f_2 are constant functions.

As a consequence, for $\xi \in [-1,1]$

$$\begin{aligned} &\min\{f_j^-,\ f_j^+\} \le f_j \le \max\{f_j^-,\ f_j^+\} \qquad , \quad j=1,2 \\ &TV f_j = |f_j^+ - f_j^-| \qquad\qquad\qquad , \quad j=1,2 \ . \end{aligned} \tag{4.8}$$

Next, an argument similar to Lemma 1.3 shows that there are expenents $p_\varepsilon,\ 1 < p_\varepsilon \le \infty$, and positive constants C_ε such that

$$\|f_j'\|_{p_\varepsilon} \le C_\varepsilon \quad , \quad j = 1,2 \ . \tag{4.9}$$

The exponent p_ε depends on $f^\pm$ and ε, it is increasing as ε decreases, and there is $\varepsilon_0 = \varepsilon_0(f^\pm) > 0$ so that for $\varepsilon < \varepsilon_0$ the exponent $p_\varepsilon = \infty$. It follows from (4.9) that solutions of $(\mathcal{R}_\varepsilon)$ are Hölder continuous (Lipschitz continuous for $\varepsilon < \varepsilon_0$).

Part 2. The fluid dynamic limit

Let $\{f^\varepsilon\}_{\varepsilon>0}$ be a family of extended solutions to $(\mathcal{R}_\varepsilon)$ corresponding to fixed, positive Maxwellian data $f^\pm$. By (4.8) and Helly's theorem, there exists a subsequence $\{f^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and a function $f = (f_1, f_2)$ with positive, bounded components $f_1,\ f_2$ of bounded variation such that $f^{\varepsilon_n} \to f$ pointwise on $(-\infty, \infty)$. Clearly, $f(\xi) = f^-$ for $\xi \in (-\infty, -1]$ and $f(\xi) = f^+$ for $\xi \in [1, \infty)$.

The members of the sequence $\{f^{\varepsilon_n}\}$ satisfy the identities

$$-\xi\left(f_1^{\varepsilon_n} + f_2^{\varepsilon_n}\right)' + \left(f_1^{\varepsilon_n} - f_2^{\varepsilon_n}\right)' = 0 \tag{4.10}$$

$$\begin{aligned} -\xi\left(f_1^{\varepsilon_n}\ell n f_1^{\varepsilon_n} + f_2^{\varepsilon_n}\ell n f_2^{\varepsilon_n}\right)' + \left(f_1^{\varepsilon_n}\ell n f_1^{\varepsilon_n} - f_2^{\varepsilon_n}\ell n f_2^{\varepsilon_n}\right)' = \\ = -\frac{1}{2\varepsilon}\left({f_1^{\varepsilon_n}}^2 - {f_2^{\varepsilon_n}}^2\right)\left(\ell n {f_1^{\varepsilon_n}}^2 - \ell n {f_2^{\varepsilon_n}}^2\right)\,. \end{aligned} \tag{4.11}$$

Identity (4.11) is obtained upon multiplying the first equation (4.2) by $(1 + \ell n f_1^{\varepsilon_n})$, the second by $(1 + \ell n f_2^{\varepsilon_n})$ and adding the resulting equations.

Fix now $\xi_1,\ \xi_2 \in [-1, 1]$ and combine (4.11) with the uniform bounds (4.8) to obtain

$$\begin{aligned} -\varepsilon_n C &\le -\int_{\xi_1}^{\xi_2}\left({f_1^{\varepsilon_n}}^2 - {f_2^{\varepsilon_n}}^2\right)\left(\ell n {f_1^{\varepsilon_n}}^2 - \ell n {f_2^{\varepsilon_n}}^2\right)d\xi \\ &\le -4\int_{\xi_1}^{\xi_2}\left(f_1^{\varepsilon_n} - f_2^{\varepsilon_n}\right)^2 d\xi \le 0\,, \end{aligned}$$

and upon passing to the limit $\varepsilon_n \downarrow 0$

$$\int_{\xi_1}^{\xi_2}(f_1 - f_2)^2 d\xi = 0\,.$$

We conclude $f_1 = f_2$ for a.e. $\xi \in (-\infty, \infty)$.

Next, set $\rho = f_1 + f_2$. Passing to the limit in (4.10) we see that $-\xi\rho' = 0$ weakly. Therefore $\rho(\frac{x}{t})$ is a weak solujtion of

$$\partial_t \rho = 0 \quad , \quad \rho(x, 0) = \begin{cases} 2f^- & x < 0 \\ 2f^+ & x > 0 \end{cases}\,. \tag{4.12}$$

The solution of (4.12) is given by

$$\rho = \rho(\frac{x}{t}) = \begin{cases} 2f^- & x < 0,\ t > 0\ , \\ \\ 2f^+ & x > 0,\ t > 0\ . \end{cases}$$

Remark. It follows from (4.6) and (4.8) that for any $1 \leq p < \infty$

$$\begin{pmatrix} f_1^{\varepsilon_n} \\ f_2^{\varepsilon_n} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} \rho \\ \rho \end{pmatrix} \text{ in } L^p_{\ell oc}(-\infty, \infty)\ .$$

As a matter of fact, since the limiting function ρ is characterized as the unique solution of (4.12), the above convergence holds over the entire family $\{f^\varepsilon\}_{\varepsilon > 0}$ as $\varepsilon \to 0+$.

Part 3. Existence theory for $(\mathcal{R}_\varepsilon)$.

First, we introduce some notation. Let $b(f) = (-2f_1, 2f_2)$ and define the matrices

$$A(\xi) = \begin{bmatrix} -(\xi - 1) & 0 \\ 0 & -(\xi + 1) \end{bmatrix} \quad , \quad B(f) = \begin{bmatrix} -2f_1 & 2f_2 \\ 2f_1 & -2f_2 \end{bmatrix}\ .$$

Then $Q(f) = \frac{1}{2} b(f) \cdot f$ and (4.2) may be written as

$$A(\xi) f' = \frac{1}{2\varepsilon} B(f) f\ . \tag{4.13}$$

Let $F = (F_1, F_2)$ be a given solution of $(\mathcal{R}_\varepsilon)$ defined on $[-1, 1]$ and taking boundary values $F(\pm 1) = F^\pm$ positive, Maxwellian states. By (4.9) the solution F is Hölder continuous. Any nearby solution f may be written as $f = F + p$, with the perturbation $p = (p_1, p_2)$ a continuous function satisfying

$$\begin{aligned} A(\xi) p' &= \frac{1}{\varepsilon} B(F(\xi)) p + \frac{1}{2\varepsilon} B(p) p \\ p(\pm 1) &= p^\pm \end{aligned} \tag{4.14}$$

and $p^\pm = f^\pm - F^\pm$. Since $f^\pm$ and $F^\pm$ are Maxwellians

$$b(F^\pm) \cdot p^\pm + Q(p^\pm) = 0\ . \tag{4.15}$$

The first step in proving Theorem 4.1 is to show the following analog of Theorem 3.5.

Theorem 4.3. *If the only continuous solution of the homogeneous boundary-value problem*

$$\begin{aligned} A(\xi)p' &= \frac{1}{\varepsilon}B(F(\xi))p \ , \\ p(\pm 1) &= 0 \ , \end{aligned} \tag{4.16}$$

is the trivial solution $p = 0$, *then there exists a positive constant* r *such that for any data satisfying* (4.15) *and* $|p^+| + |p^-| < r$ *the nonlinear boundary-value* (4.14) *has a solution.*

Proof. The proof is long, but it follows the same steps as the proof of Theorem 3.5 being just easier. We only give a rough sketch here.

First, one analyzes the singular linear boundary-value problem

$$\begin{aligned} A(\xi)p' &= \frac{1}{\varepsilon}B(F(\xi))p + \frac{1}{\varepsilon}G(\xi) \ , \\ p(\pm 1) &= p^{\pm} \ , \end{aligned} \tag{4.17}$$

with $p^{\pm}$ subject to the restrictions

$$b(F^+) \cdot p^+ + g_1(+1) = -b(F^-) \cdot p^- + g_2(-1) = 0 \ . \tag{4.18}$$

It turns out that (4.17) admits a unique solution for any data $p^{\pm}$ subject to (4.18) if and only if (4.16) possesses only the trivial solution $p = 0$. Moreover, in that case the solution p of (4.17) can be estimated by

$$\|p\| \leq C(|p^+| + |p^-| + \|G\|) \ , \tag{4.19}$$

where $\|\cdot\|$ stands for the sup-norm on $[-1, 1]$.

Next, one applies the contraction mapping principle to a map carrying P to the solution of (4.17) with $G = \frac{1}{2}B(P)P$. Given the estimates (4.19), the proof of this part is identical to the proof of Theorem 3.5. The resulting fixed point is the solution p of (4.14). ∎

The second step is to show:

Lemma 4.4. The only continuous solution of the homogeneous boundary-value problem (4.16) is the trivial $p = 0$.

Proof. Suppose that $p = (p_1, p_2)$ is a nontrivial eigensolution of

$$\begin{aligned} -(\xi - 1)p_1' &= \frac{1}{\varepsilon} b(F(\xi)) \cdot p \,, \\ -(\xi + 1)p_2' &= -\frac{1}{\varepsilon} b(F(\xi)) \cdot p \,, \end{aligned} \tag{4.20}$$

with $p(\pm 1) = 0$. Then p also satisfies

$$[(1-\xi)p_1 - (\xi+1)p_2]' + (p_1 + p_2) = 0 \,,$$

and integration over any $[a, b] \subset [-1, 1]$ yields

$$(1-b)p_1(b) - (1-a)p_1(a) - (b+1)p_2(b) + (a+1)p_2(a) + \int_a^b (p_1 + p_2) d\xi = 0 \,. \tag{4.21}$$

Consider the curve $\gamma : p = (p_1(\xi), p_2(\xi)),\ \xi \in [-1, 1]$, that describes the solution in the state plane $p_1 - p_2$. Since $(0, 0)$ is an equilibrium, γ only meets the origin at the singular points $\xi = \pm 1$. The vector field in (4.20) vanishes only on the lines $F_1(\xi)p_1 - F_2(\xi)p_2 = 0$ lying in the first and third quadrants. This, together with the form of (4.20), implies that γ may only cross the positive p_2-axis going from the second to the first quadrant, the positive p_1-axis going from the first to the third, the negative p_2-axis from the fourth to the third, and finally the negative p_1-axis from the third to the second. There are the following, not mutually exclusive, possibilities to consider: (i) γ crosses the positive p_2-axis, (ii) γ crosses the negative p_2-axis, and (iii) γ starts and concludes at the origin without crossing the p_2-axis.

Case (i): Suppose that p crosses the positive p_2-axis at $\xi = a$. Then $p_1(a) = 0,\ p_2(a) > 0$. Also, the curve γ will either cross the positive p_1-axis at some $\xi = b$, or it will wander around in the first quadrant concluding on the origin at $\xi = b = +1$. In either case $p_1(b) \geq 0,\ p_2(b) = 0$ and $p_1(\xi) > 0,\ p_2(\xi) > 0$ for $a < \xi < b$. Identity (4.21) then leads to a contradiction and that excludes case (i). Case (ii) is similarly excluded by applying the same argument to $(-p)$, which is again a solution of (4.20).

We conclude that γ starts and concludes at the origin without crossing the p_1-axis. Without loss of generality we may assume that the curve γ lies on the right half plane; otherwise replace p by $(-p)$ as before. Since $p_1(-1) = p_1(+1) = 0$, the mean value theorem and (4.20) imply there exists $\theta \in (-1, 1)$ such that $F_1(\theta)p_1(\theta) = F_2(\theta)p_2(\theta)$. At θ the curve γ lies in the first quadrant. In view of the form of the vector field in (4.20), the curve γ starts at the origin at $\xi = -1$, lies in the first quadrant thereafter up to at least $\xi = \theta$, and then either escapes to the second quadrant at some point $\xi = b$, or it concludes at the origin at $\xi = b = +1$. In either case, $p_1(b) \geq 0,\ p_2(b) = 0$ and $p_1(\xi) > 0,\ p_2(\xi) > 0$ for $-1 < \xi < b$. Applying (4.21) with $a = -1$ and b as above, we arrive again at a contradiction. Therefore, $p = 0$ on $[-1, 1]$. ∎

The last step in proving Theorem 4.1 is to use a continuation argument. Let $f^- = (f_1^-, f_2^-),\ f^+ = (f_1^+, f_2^+)$ be positive Maxwellians, that is $f_1^-, f_2^-, f_1^+, f_2^+ > 0$ and $f_1^- = f_2^-,\ f_1^+ = f_2^+$. For $\mu \in [0, 1]$, set

$$f^-(\mu) = f^- \ ,\ f^+(\mu) = f^- + \mu(f^+ - f^-)\ ,$$

which are also positive Maxwellians. Consider the set

$$\mathcal{C} = \{\mu \in [0, 1] : (\mathcal{R}_\varepsilon) \text{ has a solution taking data } f^-(\mu),\ f^+(\mu)\}\ .$$

Then $0 \in \mathcal{C}$ and, by virtue of Theorem 4.3 and Lemma 4.4, $\mathcal{C}$ is open.

We proceed to show that $\mathcal{C}$ is closed. Let $\{\mu_n\}$ be a sequence of points in $\mathcal{C}$ such that $\mu_n \to \mu_0$. Suppose that f^{μ_n} are the corresponding solutions of $(\mathcal{R}_\varepsilon)$ admitting data

$f^+(\mu_n)$ and $f^-(\mu_n)$. On account of (4.8) and (4.9), the sequence $\{f^{\mu_n}\}$ is precompact in some Hölder space. Therefore a subsequence converges to a limiting function f^{μ_0}, and a straightforward argument shows that f^{μ_0} is a solution of $(\mathcal{R}_\varepsilon)$ taking data $f^+(\mu_0)$ and $f^-(\mu_0)$. Thus $\mathcal{C}$ is closed.

We conclude that $\mathcal{C} = [0,1]$ and the proof is complete. ∎

§5. Appendix: A linear singular equation.

We record here certain properties of the equation

$$-\xi\, p' + \phi(\xi)\, p = h(\xi) \tag{5.1}$$

that are used in the text to study the behavior of solutions near singular points. The equation is taken in the interval $0 < \xi \leq a$ with $a \leq 1$. The objective is to investigate regularity properties for solutions, and to study the map that carries h to p.

Throughout, the functions ϕ and h are assumed at least continuous on $[0,a]$ with ϕ subject to the restrictions:

$$\phi(0) > 0\,, \qquad \int_0^a \Big|\, \frac{\phi(\zeta) - \phi(0)}{\zeta} \,\Big|\, d\zeta < \infty\,. \tag{H_1}$$

The latter assumption is, for instance, satisfied if the function ϕ is Hölder continuous.

Use of an integrating factor shows the solution of (5.1) on $(0,a]$ is given by

$$p(\xi) = p(a)\, e^{-\int_\xi^a \frac{\phi(\zeta)}{\zeta}\, d\zeta} + e^{-\int_\xi^a \frac{\phi(\zeta)}{\zeta}\, d\zeta} \int_\xi^a \frac{h(s)}{s}\, e^{\int_s^a \frac{\phi(\zeta)}{\zeta}\, d\zeta}\, ds\,. \tag{5.2}$$

Next, certain convenient expressions for the solution are derived. Observe that

$$J(\xi) := e^{-\int_\xi^a \frac{\phi(\zeta)}{\zeta}\, d\zeta} = \big(\frac{\xi}{a}\big)^{\phi(0)}\, e^{\omega(\xi) - \omega(a)}\,, \tag{5.3}$$

where ω is defined in terms of the convergent (due to (H_1)) integral

$$\omega(\xi) := \int_0^\xi \frac{\phi(\zeta) - \phi(0)}{\zeta}\, d\zeta\,. \tag{5.4}$$

On account of (5.3) and (5.4), p may be written in the form

$$p(\xi) = p(a)\,\big(\frac{\xi}{a}\big)^{\phi(0)}\, e^{\omega(\xi)-\omega(a)} + \xi^{\phi(0)}\, e^{\omega(\xi)} \int_\xi^a h(s)\, s^{-\phi(0)-1}\, e^{-\omega(s)}\, ds\,. \tag{5.5}$$

An alternative formula for p is obtained by combining (5.5) with the identity

$$\int_\xi^a s^{-\phi(0)-1}\, e^{-\omega(s)}\, ds = \\ -\frac{1}{\phi(0)}\Big[a^{-\phi(0)}\, e^{-\omega(a)} - \xi^{-\phi(0)}\, e^{-\omega(\xi)} + \int_\xi^a s^{-\phi(0)}\, e^{-\omega(s)}\, \frac{\phi(s)-\phi(0)}{s}\, ds\Big]\,.$$

It reads

$$\begin{aligned} p(\xi) = {} & \frac{h(0)}{\phi(0)} + \big[p(a) - \frac{h(0)}{\phi(0)}\big]\big(\frac{\xi}{a}\big)^{\phi(0)}\, e^{\omega(\xi)-\omega(a)} \\ & + \xi^{\phi(0)} \int_\xi^a \frac{h(s)-h(0)}{s}\, s^{-\phi(0)}\, e^{\omega(\xi)-\omega(s)}\, ds \\ & - \xi^{\phi(0)} \int_\xi^a \frac{h(0)}{\phi(0)}\, \big(\frac{\phi(s)-\phi(0)}{s}\big)\, s^{-\phi(0)}\, e^{\omega(\xi)-\omega(s)}\, ds\,. \end{aligned} \tag{5.6}$$

Formulas (5.5) and (5.6) serve as starting points to produce various estimates for solutions of (5.1).

Lemma 5.1. Let ϕ and h be continuous functions on $[0,a]$ satisfying (H_1). Then

$$\lim_{\xi\to 0} p(\xi) = h(0)/\phi(0)\,, \tag{5.7}$$

p is continuous on $[0,a]$ and satisfies the bound

$$|p(\xi)\, e^{-\omega(\xi)}\,| \le |p(a)\, e^{-\omega(a)}\,| + \frac{1}{\phi(0)} \sup_{0\le s\le a} |h(s)\, e^{-\omega(s)}\,|\,. \tag{5.8}$$

Proof. Hypothesis (H_1) implies that $\omega(\xi)\to 0$ as $\xi\to 0$. Then L'Hôpital's rule shows

$$\lim_{\xi\to 0} \frac{\int_\xi^a h(s)\, s^{-\phi(0)-1}\, e^{-\omega(s)}\, ds}{\xi^{-\phi(0)}} = \frac{1}{\phi(0)} \lim_{\xi\to 0} h(\xi)\, e^{-\omega(\xi)} = h(0)/\phi(0)\,,$$

which together with (5.5) implies (5.7).

Next, combining the estimate

$$|p(\xi)\, e^{-\omega(\xi)}\,| \leq |p(a)\, e^{-\omega(a)}\,| + \sup_{0 \leq s \leq a} |h(s)\, e^{-\omega(s)}|\, \big(\xi^{\phi(0)} \int_{\xi}^{a} s^{-\phi(0)-1}\, ds \big)\,,$$

obtained directly from (5.5), with the formula

$$\xi^{\phi(0)} \int_{\xi}^{a} s^{-\phi(0)-1}\, ds = \frac{1}{\phi(0)} \big(1 - (\xi/a)^{\phi(0)} \big)$$

we deduce (5.8). ∎

The next task is to relate the modulus of continuity of ϕ and h at $\xi = 0$ with the modulus of continuity of the solution. To this end, assume that for some $0 < \beta \leq 1$ and $0 < \gamma \leq 1$ the functions ϕ and h satisfy

$$\begin{aligned} &|\phi(\xi) - \phi(0)| \leq < \phi > \xi^{\beta} \\ &|h(\xi) - h(0)| \leq < h > \xi^{\gamma} \end{aligned} \qquad (H_2)$$

on $[0, a]$; here, the symbols $< \phi >$ and $< h >$ stand for the Hölder constants. We show:

Lemma 5.2. Let ϕ and h satisfy $(H_1 - H_2)$. If $\beta = \gamma = 1$ and $\phi(0) > 1$ then p is Lipshitz and satisfies the bound

$$\big|p(\xi) - p(\zeta)\big| \leq C \; \big(\, a^{-\phi(0)} \big|p(a) - \frac{h(0)}{\phi(0)}\big| + \, < h > \, + \big|\frac{h(0)}{\phi(0)}\big| \, \big) \, |\xi - \zeta| \qquad \xi\,,\,\zeta\, \epsilon\, [0, a]\,. \quad (5.9)$$

Otherwise, p is Hölder continuous with exponent any $0 < \alpha < \alpha_0 := \min\{\beta, \gamma, \phi(0)\}$ and satisfies for $\xi\,,\,\zeta\, \epsilon\, [0, a]$

$$\big|p(\xi) - p(\zeta)\big| \leq C \; \big(\, a^{-\phi(0)} \big|p(a) - \frac{h(0)}{\phi(0)}\big| + \, < h > \, + \big|\frac{h(0)}{\phi(0)}\big| \, \big) \, |\xi - \zeta|^{\alpha}\,. \qquad (5.10)$$

The constant C depends on $< \phi >$, β, γ, α and $\phi(0)$, but is independent of a in the region $0 < a \leq 1$.

Proof. In what follows C will stand for a generic contant exhibiting the aforementioned dependence. First, the terms in the right hand side of (5.6) are estimated. In view of (5.4) and (H_2)

$$|\omega(\xi)-\omega(s)| \le < \phi > \, \Big| \int_s^\xi \zeta^{\beta-1}\, d\zeta \Big| = \frac{<\phi>}{\beta} |\, \xi^\beta - s^\beta \,| \, . \tag{5.11}$$

Since $a \le 1$, (5.6) and (5.11) imply for $0 < \xi \le a$

$$\begin{aligned} \big|p(\xi) - \frac{h(0)}{\phi(0)}\big| &\le \big|p(a) - \frac{h(0)}{\phi(0)}\big| e^{\frac{<\phi>}{\beta}} \, (\frac{\xi}{a})^{\phi(0)} + e^{\frac{<\phi>}{\beta}} \, < h > \, (\, \xi^{\phi(0)} \int_\xi^a s^{\gamma-1-\phi(0)}\, ds\,) \\ &\quad + e^{\frac{<\phi>}{\beta}} \, < \phi > \, \big|\frac{h(0)}{\phi(0)}\big| \, (\, \xi^{\phi(0)} \int_\xi^a s^{\beta-1-\phi(0)}\, ds\,) \\ &\le C\, A\ \xi^{\phi(0)} \big(\, 1 + \int_\xi^1 s^{\gamma-1-\phi(0)} + s^{\beta-1-\phi(0)}\, ds\,\big)\, , \end{aligned}$$

where

$$A := a^{-\phi(0)} \big|p(a) - \frac{h(0)}{\phi(0)}\big| + \, < h > \, + \big|\frac{h(0)}{\phi(0)}\big| \, . \tag{5.12}$$

Set now $\alpha_0 = \min\{\beta, \gamma, \phi(0)\} > 0$ and note that $0 < \alpha_0 \le 1$. Using the formula for the evaluation of the integrals

$$\xi^{\phi(0)} \int_\xi^1 s^{\alpha-1-\phi(0)}\, ds = \begin{cases} \xi^{\phi(0)}\, |ln \xi| & \text{if } \alpha = \phi(0) \\ \frac{1}{|\alpha-\phi(0)|} \, \big|\xi^{\phi(0)} - \xi^\alpha\big| & \text{if } \alpha \ne \phi(0) \end{cases} \tag{5.13}$$

we conclude

$$\big|p(\xi) - \frac{h(0)}{\phi(0)}\big| \le \begin{cases} C\, A\, \xi^{\alpha_0}\, |ln\xi| & \text{if } \alpha_0 = \phi(0) \\ C\, A\, \xi^{\alpha_0} & \text{if } \alpha_0 < \phi(0) \end{cases} , \qquad \text{for } \xi \,\epsilon\, [0,a]. \tag{5.14}$$

The control on the modulus of continuity of $p(\xi)$ at $\xi = 0$ provides bounds on the derivative. To show that, rewrite (5.1) as

$$\begin{aligned} \xi\, p'(\xi) &= \phi(\xi) p(\xi) - h(\xi) \\ &= \frac{h(0)}{\phi(0)} \big(\phi(\xi) - \phi(0)\big) + \phi(0)\, \big(p(\xi) - \frac{h(0)}{\phi(0)}\big) - \big(h(\xi) - h(0)\big) \\ &\qquad + \big(\phi(\xi) - \phi(0)\big)\, \big(p(\xi) - \frac{h(0)}{\phi(0)}\big) \end{aligned} \tag{5.15}$$

and use (H_2) and (5.14) to obtain

$$|p'(\xi)| \leq \begin{cases} C\,A\,\xi^{\alpha_0-1}\,|ln\xi| & \text{if } \alpha_0 = \phi(0) \\ C\,A\,\xi^{\alpha_0-1} & \text{if } \alpha_0 < \phi(0) \end{cases}, \qquad \text{for } \xi \,\epsilon\, (0,a]. \tag{5.16}$$

We emphasize that the constant C in (5.16) depends solely upon $< \phi >$, β, γ and upper bounds for $\phi(0)$. All other dependences are incorporated in the constant A in (5.12).

We can now complete the proof of the lemma:

(i) If $\beta = \gamma = 1 < \phi(0)$ then $\alpha_0 = 1 < \phi(0)$ and (5.9) follows from (5.16).

(ii) In all other cases, fix any $0 < \alpha < \alpha_0 \leq 1$ and let $\xi,\ \zeta \,\epsilon\, (0,a]$. Using Hölder's inequality, we deduce from (5.16)

$$\begin{aligned} |p(\xi) - p(\zeta)| &= \Big|\int_\zeta^\xi p'(s)\,ds\,\Big| \\ &\leq \Big|\int_\zeta^\xi |p'(s)|^{\frac{1}{1-\alpha}}\,ds\,\Big|^{1-\alpha}\,|\xi-\zeta|^\alpha \\ &\leq C\,A\left|\int_0^1 s^{-\frac{1-\alpha_0}{1-\alpha}}\,(1+|lns|)^{\frac{1}{1-\alpha}}\,ds\right|^{1-\alpha}|\xi-\zeta|^\alpha\,. \end{aligned} \tag{5.17}$$

Since $\frac{1-\alpha_0}{1-\alpha} < 1$, the last integral is finite and (5.10) follows from (5.17) and (5.7). ∎

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