ELASTIC AS LIMIT OF VISCOELASTIC RESPONSE,

IN A CONTEXT OF SELF-SIMILAR VISCOUS LIMITS

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Abstract. We study the equations of one-dimensional isothermal elastic response as the small viscosity limit of the equations of viscoelasticity, in a context of self-similar viscous limits for Riemann data. No size restrictions on the data or genuine-nonlinearity assumptions are imposed. The limiting procedure is justified and a solution of the Riemann problem for the equations of elasticity is obtained. The emerging solution is composed of two wave fans, each consisting of rarefactions, shocks and contact discontinuities, separated by constant states. At shocks the self-similar viscous solution has the internal structure of traveling waves, and an admissibility criterion identified by Wendroff [W] is fullfilled.

Research partially supported by the National Science Foundation under Grant DMS-9209049 and the Office of Naval Research under Contract N00014-93-0015.

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1. INTRODUCTION

The equations describing one-dimensional, isothermal motions of elastic materials in Lagrangean coordinates are

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x \sigma(u) &= 0 \ . \end{aligned} \tag{E}$$

Under the assumption

$$\sigma_u(u) > 0 \tag{H}$$

they form a pair of strictly hyperbolic conservation laws with characteristic speeds $\lambda_1(u) = -\sqrt{\sigma_u(u)}$, $\lambda_2(u) = +\sqrt{\sigma_u(u)}$. As a model for nonlinear elastic response (E) provides an appropriate description for both longitudinal and shearing motions. In the former case u stands for the longitudinal strain (or specific volume) and takes strictly positive values, $\sigma(u)$ is the (extensive or compressive) normal stress and v is the velocity. In the latter case u describes the shear strain, taking now values on the whole real line, $\sigma(u)$ stands for the shear stress and v for the velocity in the shearing direction.

It is well known that, due to wave breaking phenomena pertaining to nonlinear elastic response, (E) does not in general admit globally defined smooth solutions. One approach for obtaining admissible weak solutions is to view elastic as the limiting case of viscoelastic response and attempt to construct solutions of (E) as limits of solutions of associated viscoelastic models, such as

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x \sigma(u) &= \varepsilon \ \partial_x (k(u) \partial_x v) \ , \end{aligned} \tag{VE}$$

as the viscosity $\varepsilon \to 0$; above $k(u) = 1/u\ ,\ u \in (0, \infty)$, for longitudinal motions while $k(u) = 1,\ u \in (-\infty, \infty)$, for shearing motions. The question of viscous limits for systems of conservation laws has received attention both at the level of general systems as well as for specific problems. Despite recent advances via the method of compensated compactness (*e.g.* DiPerna [Dp]) and the work of Goodman and Xin [GX], the understanding of the limiting process for solutions emanating from fixed initial data remains incomplete even in the case of Riemann data

$$u(x,0) = \begin{cases} u_- , & x < 0 \\ u_+ , & x > 0 \end{cases} , \qquad v(x,0) = \begin{cases} v_- , & x < 0 \\ v_+ , & x > 0 \end{cases} . \tag{1.1}$$

In a notable exception, for data corresponding to a single shock in (E), Hoff and Liu [HL] obtain detailed information on the qualitative properties of the viscous solution and justify the inviscid limit. For data corresponding to a rarefaction wave the inviscid limit is justified by Xin [X].

Due to the invariance under dilations of coordinates $(x,t) \to (ax, at)$, $a > 0$, the Riemann problem for (E) admits solutions of the form $\big(u(\frac{x}{t}),\ v(\frac{x}{t})\big)$, functions of the single variable $\xi = \frac{x}{t}$. These are constructed as weak solutions on $(-\infty, \infty)$ of the problem

$$-\xi \begin{pmatrix} u \\ v \end{pmatrix}' - \begin{pmatrix} v \\ \sigma(u) \end{pmatrix}' = 0 \tag{1.2}$$

$$u(\pm\infty) = u_\pm \quad , \quad v(\pm\infty) = v_\pm \tag{1.3}$$

subject to appropriate admissibility criteria at shocks (Tupciev [Tu$_1$], Wendroff [W], Liu [L] for general strictly hyperbolic systems).

Our objective is to justify $(1.2 - 1.3)$ as the $\varepsilon \to 0$ limit of solutions to the boundary value problem

$$-\xi \begin{pmatrix} u \\ v \end{pmatrix}' - \begin{pmatrix} v \\ \sigma(u) \end{pmatrix}' = \varepsilon \begin{pmatrix} 0 \\ (k(u)\, v')' \end{pmatrix} , \tag{1.4}$$

with data (1.3), and to investigate the admissibility restrictions imposed by the limiting process. The problem $(1.4 - 1.3)$ is denoted by $(\mathcal{P}_\varepsilon)$. It is studied with the intent to reveal aspects of the more complex relationship between (VE) and (E).

The study of self-similar viscous limits was proposed in the articles by Kalasnikov [Ka] and Dafermos [D$_1$]. It is motivated by introducing an artificial regularization

$$\begin{aligned} \partial_t u - \partial_x v &= 0 \\ \partial_t v - \partial_x \sigma(u) &= \varepsilon\, t\ \partial_x (k(u)\, \partial_x v) \end{aligned} \tag{1.5}$$

that preserves the invariance under dilations of coordinates. Problem $(1.5 - 1.1)$ admits as solutions functions of the single variable $\xi = x/t$, constructed by solving $(\mathcal{P}_\varepsilon)$. The approach leads to the study of nonautonomous boundary value problems such as $(\mathcal{P}_\varepsilon)$, and involves variation estimates for effecting the $\varepsilon \to 0$ limit to solve the Riemann problem. The procedure has been carried out for various systems of two conservation laws, both hyperbolic [D$_1$, D$_2$, DDp, KKr, STz] as well as of mixed type [S$_1$, Fa]. The two regularizations have been directly compared for Burgers' equation [S$_2$].

We adopt the following hypotheses on constitutive functions. For shearing motions $\sigma(u)$ is a C^2-function defined on the real line that satisfies (H) and has the behavior

$$\sigma(u) \to -\infty \quad \text{as} \quad u \to -\infty\,, \qquad \sigma(u) \to \infty \quad \text{as} \quad u \to \infty\,. \tag{$(H)_S$}$$

For longitudinal motions $\sigma(u)$ is C^2-function on $(0,\infty)$ satisfying (H) and

$$\sigma(u) \to -\infty \quad \text{as} \quad u \to 0+\,, \qquad \sigma(u) \to \infty \quad \text{as} \quad u \to \infty\,. \tag{$(H)_L$}$$

The latter guarantees that infinite compression is associated with infinite compressive stress. It also requires that infinite extension is associated with infinite extensive stress, what excludes the case of gases. The precise form of $k(u)$ has no influence on our analysis as long as $k(u)$ is a strictly positive smooth function; in the models under study $k(u) = 1$ for shearing motions while $k(u) = 1/u$ for longitudinal motions. The Riemann data $u_\pm$, $v_\pm$ are taken arbitrary for shearing motions, but satisfy $u_+\,,\ u_- > 0$ for longitudinal motions. We emphasize that no size restrictions are imposed on the Riemann data. Also no genuine nonlinearity assumption is required for (E) and thus $\sigma(u)$ is allowed to have inflection points.

The goals set in this article are (i) to construct solutions of the boundary value problem $(\mathcal{P}_\varepsilon)$, $\varepsilon > 0$ fixed, (ii) to pass to the limit $\varepsilon \to 0$ and solve the Riemann problem, (iii) to investigate the structure of the emerging solution. Existence follows from an application of the Leray-Schauder degree theory to a construction scheme that captures the interplay between the "parabolic" and "hyperbolic" terms in (1.4). Based on a-priori estimates developed in Section 2, the construction is carried out in Section 3. Following that in Section 4, the passage to the $\varepsilon \to 0$ limit is performed, in a framework of uniform in ε bounds for the total variation and making use of Helly's theorem. The outcome of the first two steps can be summarized as follows: Under hypotheses $(H)_S$ or $(H)_L$, given any data $(u_\pm, v_\pm)$, $\varepsilon > 0$ the boundary value problem $(\mathcal{P}_\varepsilon)$ admits a solution. Given a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of such solutions there exists a subsequence $\{(u_{\varepsilon_n}, v_{\varepsilon_n})\}$, with $\varepsilon_n \to 0$, such that $u_{\varepsilon_n} \to u$, $v_{\varepsilon_n} \to v$ pointwise on the reals. The function (u, v) is of bounded variation and a weak solution of the Riemann problem $(1.2 - 1.3)$.

The structure of the limit function (u, v) is undertaken in Section 5. The equations of elasticity (E) are covered as a special case in Dafermos [D$_2$], on systems of two conservation laws with a full viscosity matrix. Although several ideas from [D$_2$] are used, this work is based on a special property associated with the full viscosity matrix which is not applicable here. Instead, we follow the strategy used in Tzavaras [Tz] for the Broadwell model. One starts with certain representation formulas expressing the derivatives $(u'_{\varepsilon_n}, v'_{\varepsilon_n})$ as averaging processes and shows that there are finite signed Borel measures λ, ν so that

$$(u'_{\varepsilon_n}, v'_{\varepsilon_n}) \rightharpoonup (\lambda, \nu) \quad \text{weak}-\star \text{ in measures}.$$

The key ingredient is then Proposition 5.1, which shows that on the support of λ, ν a function g related to the antiderivatives of the eigenvalues is minimized. That, in turn, yields a characterization of the behavior of the wave speeds $\lambda_i(u(\xi))$ at places where the solution is nonconstant. As it turns out, (u, v) is composed of two wave fans separated by a constant state. Each wave fan is associated with one of the characteristic fields of (E) and consists of either a single rarefaction, or a single shock, or an alternating sequence of rarefactions and shocks such that each shock adjacent to a rarefaction on one side is a contact on that side. At shocks the Rankine-Hugoniot and a weak form of the Lax shock conditions are satisfied.

In Section 6, we fix a point ξ of discontinuity, with left and right limits $(u(\xi-), v(\xi-))$, $(u(\xi+), v(\xi+))$, and discuss the relation between self-similar limits and traveling wave solutions for the viscous equations. We refer to [Tu$_2$, D$_2$, Fa, Tz] for investigations of this issue in other contexts. It turns out that $(u_{\varepsilon_n}, v_{\varepsilon_n})$ have the internal structure of a traveling wave solution of (VE), and that discontinuities emerging via self-similar viscous limits satisfy for $u(\xi-) < u(\xi+)$ and $\xi > 0$

$$[\,\sigma(U) - \sigma(u(\xi-))\,] - \xi^2\,[\,U - u(\xi-)\,] \geq 0\,, \qquad U \in (u(\xi-),\, u(\xi+))\,. \tag{1.6}$$

(≤ 0 for $\xi < 0$). Moreover, if (1.6) holds as a strict inequality there is a single shock profile connecting $u(\xi-)$ to $u(\xi+)$. On the other hand, if (1.6) holds as an equality at a finite number of points u_j, $j = 1, ..., m$, there will be a chain of traveling waves connecting consecutively $u(\xi-)$ to u_1, each of the points u_j to the next, and u_m to $u(\xi+)$. This relation between self-similar limits and traveling waves was conjectured by Tupciev [Tu$_2$]. Condition (1.6) can be motivated by an analysis of traveling waves and was identified by Wendroff [W] as a criterion to single out admissible shocks for the equations of elasticity.

2. CONNECTING TRAJECTORIES I - A PRIORI ESTIMATES

Consider the nonlinear boundary-value problem $(\mathcal{P}_\varepsilon)$

$$
\begin{aligned}
&-\xi u' - v' = 0 \\
&-\xi v' - \sigma(u)' = \varepsilon (k(u) v')'
\end{aligned}
\tag{2.1}
$$

$$
u(\pm\infty) = u_\pm \quad , \quad v(\pm\infty) = v_\pm \, , \tag{2.2}
$$

on $-\infty < \xi < \infty$ for $\varepsilon > 0$ fixed. Our objective is to construct solutions $(u(\xi), v(\xi))$ of $(\mathcal{P}_\varepsilon)$, that is trajectories connecting the end states (u_-, v_-) and (u_+, v_+). Both cases of longitudinal and shearing motions are studied simultaneously. For longitudinal motions the strain u is expected to turn out positive, $\sigma(u)$ is a C^2-function on $(0, \infty)$ that satisfies (H) and $(H)_L$, while $k(u) = \frac{1}{u}$. For shearing motions u takes values on the reals, $\sigma(u)$ is a C^2-function on $(-\infty, \infty)$ that satisfies (H), $(H)_s$, and $k(u) = 1$. The data $u_\pm$, $v_\pm$ are arbitrary for shearing motions, but satisfy $u_+, u_- > 0$ for longitudinal motions.

2.1 Definition and Regularity Properties for Solutions.

The system of nonautonomous differential equations (2.1) is singular at $\xi = 0$. Before outlining a construction scheme it is expedient to clarify the meaning and to analyze the regularity of solutions. Consider the weak form of (2.1)

$$
\int (\zeta u + v) \varphi' d\zeta + \int u \varphi d\zeta = 0 \tag{2.3}
$$

$$
\int (\zeta v + \sigma(u)) \psi' d\zeta + \varepsilon \int k(u) v' \psi' d\zeta + \int v \psi d\zeta = 0 \, , \tag{2.4}
$$

where $\varphi, \psi \in C_c^1(\mathbb{R})$, continuously differentiable functions with compact support.

Definition. The pair $(u(\xi), v(\xi))$ with $u \in L^2_{\ell oc}(\mathbb{R})$, $v \in H^1_{\ell oc}(\mathbb{R})$, $\sigma(u) \in L^2_{\ell oc}(\mathbb{R})$, $k(u)v' \in L^2_{\ell oc}(\mathbb{R})$ is a solution of $(\mathcal{P}_\varepsilon)$ if (u, v) satisfy $(2.3 - 2.4)$, for any $\varphi, \psi \in C^1_c(\mathbb{R})$, and $\lim\limits_{\xi \to \pm\infty} u(\xi) = u_\pm$, $\lim\limits_{\xi \to \pm\infty} v(\xi) = v_\pm$.

Lemma 2.1. *Let (u, v) be a solution of $(\mathcal{P}_\varepsilon)$. Then*

(i) *ξu, v are continuous on $\mathbb{R}$, v' continuous on $\mathbb{R} - \{0\}$*

(ii) $\lim\limits_{\xi \to 0\pm} \Big(\sigma(u(\xi)) + \varepsilon k(u(\xi))v'(\xi)\Big)$ *exist and are equal*

(iii) *The functions (u, v) satisfy*

$$\Big[\xi u(\xi) + v(\xi)\Big]_a^b - \int_a^b u(\zeta)d\zeta = 0 \tag{2.5}$$

$$\Big[\xi v(\xi) + \sigma(u(\xi)) + \varepsilon k(u(\xi))v'(\xi)\Big]_a^b - \int_a^b v(\zeta)d\zeta = 0 \tag{2.6}$$

for any $a, b \in \mathbb{R}$. In (2.5), (2.6) the appropriate right or left limits are understood whenever a or b are equal to 0.

Proof. It follows from (2.3) that v, $\xi u \in H^1_{\ell oc}(\mathbb{R})$ and thus v, ξu are continuous. Moreover, $u \in L^2_{\ell oc}(\mathbb{R})$, $\xi u \in H^1_{\ell oc}(\mathbb{R})$ are easily seen to imply $\xi u(\xi) \to 0$ as $\xi \to 0$.

Fix now $a, b \in \mathbb{R}$ with $a, b \neq 0$, $a < b$ and consider (2.4) with

$$\psi_n(\xi) = \begin{cases} 0 & -\infty < \xi \leq a - \dfrac{1}{n} \\ n[\xi - (a - \dfrac{1}{n})] & a - \dfrac{1}{n} \leq \xi \leq a \\ 1 & a \leq \xi \leq b \\ -n[\xi - (b + \dfrac{1}{n})] & b \leq \xi \leq b + \dfrac{1}{n} \\ 0 & b + \dfrac{1}{n} \leq \xi < +\infty \, . \end{cases}$$

As $\psi_n \notin C^1_c(\mathbb{R})$ it cannot be directly used as a test function. However, using approximation by C^1_c functions, (2.4) can be easily extended to hold for ψ_n and yields

$$n \int_{a-\frac{1}{n}}^{n} (\zeta v + \sigma(u) + \varepsilon k(u)v')d\zeta - n \int_b^{b+\frac{1}{n}} (\zeta v + \sigma(u) + \epsilon k(u)v')d\zeta$$
$$+ \int_{a-\frac{1}{n}}^{b+\frac{1}{n}} v\psi_n d\zeta = 0 \, .$$

Letting $n \to \infty$ shows that (2.6) holds for a.e. $a, b \in \mathbb{R}$. Since v, ξu are continuous, v' is also continuous on $\mathbb{R} - \{0\}$ and (2.6) holds for any $a < b$ with a, $b \neq 0$. Letting now $a \to 0-$ or $b \to 0+$ establishes (ii). A similar argument shows (2.5). ∎

In summary, it follows from (2.5) and (2.6) that $u \in C^1(\mathbb{R} - \{0\})$, $v \in C^2(\mathbb{R} - \{0\})$ satisfy the differential equations

$$\begin{aligned} -\xi u' - v' &= 0 \\ -\xi v' - \sigma(u)' &= \varepsilon\,(k(u)\,v')' \end{aligned} \tag{2.7}$$

for $\xi \neq 0$, and the jump relations

$$\begin{aligned} v(0-) &= v(0+) \\ \sigma(u(0-)) + \varepsilon k(u(0-))v'(0-) &= \sigma(u(0+)) + \varepsilon k(u(0+))v'(0+)\,. \end{aligned} \tag{2.8}$$

Next we analyze the behavior of (u, v) in the neighborhood of $\xi = 0$ and $\xi = \pm\infty$. From (2.7) we obtain

$$\varepsilon(k(u)v')' + \frac{\xi^2 - \sigma_u(u)}{\xi} v' = 0\,, \qquad \xi \neq 0\,, \tag{2.9}$$

which in turn yields

$$\varepsilon \frac{d}{d\xi}\Big[k(u)v' \exp\Big\{\frac{1}{\varepsilon}\int^{\xi} \frac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta k(u(\zeta))} d\zeta\Big\}\Big] = 0\,,$$

and upon integrating

$$v'(\xi) = \begin{cases} \dfrac{k(u(\alpha_+))v'(\alpha_+)}{k(u(\xi))} \exp\Big\{-\dfrac{1}{\varepsilon}\displaystyle\int_{\alpha_+}^{\xi} \dfrac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta k(u(\zeta))} d\zeta\Big\}\,, & \xi > 0 \\ \dfrac{k(u(\alpha_-))v'(\alpha_-)}{k(u(\xi))} \exp\Big\{-\dfrac{1}{\varepsilon}\displaystyle\int_{\alpha_-}^{\xi} \dfrac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta k(u(\zeta))} d\zeta\Big\}\,, & \xi < 0 \end{cases} \tag{2.10}$$

$$u'(\xi) = -\frac{1}{\xi} v'(\xi) \quad , \quad \xi \neq 0\,, \tag{2.11}$$

where α_-, α_+ are any constants with $\alpha_- < 0$ and $\alpha_+ > 0$. The above formulas suggest that the behavior of v', u' near $\xi = 0$ and $\xi = \pm\infty$ is determined by the behavior of $u(\xi)$ as $\xi \to 0$ and as $\xi \to \pm\infty$, respectively.

To explore this point further, assume that u is bounded

$$\sup_{-\infty<\xi<\infty} |u(\xi)| \leq M \tag{$(M)_S$}$$

in the case of shearing motions, or that u is bounded from above and bounded away from zero from below

$$0 < \delta \leq u(\xi) \leq M \quad , \quad \xi \in (-\infty, \infty) \tag{$(M)_L$}$$

in the case of longitudinal motions. Then hypothesis (H) and the form of $k(u)$ imply

$$0 < k_0 \leq k(u(\xi)) \leq K_0 \tag{2.12}$$

$$0 < a_0 \leq \sigma_u(u(\xi)) \leq A_0 \ , \tag{2.13}$$

where the constants k_0, K_0, a_0 and A_0 depend on δ and M.

Lemma 2.2. *Let (u, v) be a solution of* (2.7) *that satisfies $(M)_S$ for shearing motions or $(M)_L$ for longitudinal motions, so that* (2.12) *and* (2.13) *hold. Let $\lambda_\pm = \pm\sqrt{a_0}$, $\Lambda_\pm = \pm\sqrt{A_0}$. Then for $0 < \xi < \alpha_+ < \lambda_+$*

$$|v'(\xi)| \leq k(u(\alpha_+))|v'(\alpha_+)| \frac{1}{k_0} \Big(\frac{\xi}{\alpha_+}\Big)^{\frac{\lambda_+^2 - \alpha_+^2}{\varepsilon K_0}} \ ; \tag{2.14}$$

for $\Lambda_+ < \alpha_+ < \xi$

$$|v'(\xi)| \leq k(u(\alpha_+))|v'(\alpha_+)| \frac{1}{k_0} \exp\left\{ -\frac{\alpha_+^2 - \Lambda_+^2}{2\varepsilon K_0} \left(\Big(\frac{\xi}{\alpha_+}\Big)^2 - 1 \right) \right\} ; \tag{2.15}$$

for $\lambda_- < \alpha_- < \xi < 0$

$$|v'(\xi)| \leq k(u(\alpha_-))|v'(\alpha_-)| \frac{1}{k_0} \Big(\frac{\xi}{\alpha_-}\Big)^{\frac{\lambda_-^2 - \alpha_-^2}{\varepsilon K_0}} \ ; \tag{2.16}$$

and for $\xi < \alpha_- < \Lambda_-$

$$|v'(\xi)| \leq k(u(\alpha_-))|v'(\alpha_-)| \frac{1}{k_0} \exp\left\{ -\frac{\alpha_-^2 - \Lambda_-^2}{2\varepsilon K_0} \left(\Big(\frac{\xi}{\alpha_-}\Big)^2 - 1 \right) \right\} . \tag{2.17}$$

Proof. We show (2.14) and (2.15); the proof of (2.16) and (2.17) is similar.

Fix $0 < \xi < \alpha_+ < \lambda_+$. Then (2.12), (2.13) imply for $0 < \zeta < \alpha_+$

$$\frac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta} \le \frac{\zeta^2 - a_0}{\zeta} \le -(\lambda_+^2 - \alpha_+^2)\frac{1}{\zeta} < 0$$

and thus

$$\exp\Big\{\frac{1}{\varepsilon}\int_\xi^{\alpha_+} \frac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta k(u(\zeta))} d\zeta\Big\} \le \exp\Big\{-\frac{\lambda_+^2 - \alpha_+^2}{\varepsilon K_0}\int_\xi^{\alpha_+}\frac{d\zeta}{\zeta}\Big\}$$
$$= \Big(\frac{\xi}{\alpha_+}\Big)^{\frac{\lambda_+^2-\alpha_+^2}{\varepsilon K_0}} .$$

Hence, (2.14) follows from (2.10).

Let now $0 < \Lambda_+ < \alpha_+ < \xi$ be fixed and observe that for $\zeta > \alpha_+$

$$\frac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta} \ge \zeta - \frac{A_0}{\zeta} \ge \frac{\alpha_+^2 - \Lambda_+^2}{\alpha_+^2}\zeta > 0 .$$

Hence

$$\exp\Big\{-\frac{1}{\varepsilon}\int_{\alpha_+}^{\xi} \frac{\zeta^2 - \sigma_u(u(\zeta))}{\zeta k(u(\zeta))} d\zeta\Big\} \le \exp\Big\{-\frac{\alpha_+^2 - \Lambda_+^2}{\alpha_+^2\,\varepsilon K_0}\int_{\alpha_+}^{\xi}\zeta d\zeta\Big\}$$
$$= \exp\Bigg\{-\frac{\alpha_+^2 - \Lambda_+^2}{2\varepsilon K_0}\bigg(\Big(\frac{\xi}{\alpha_+}\Big)^2 - 1\bigg)\Bigg\}$$

and (2.15) follows from (2.10). ∎

Lemma 2.2 implies that solutions (u, v) of $(\mathcal{P}_\varepsilon)$ with u bounded have the following properties:

(i) $|v'(\xi)| \to 0$, $|u'(\xi))| \to 0$ as $\xi \to \pm\infty$

(ii) $v'(\xi) = 0(|\xi|^\alpha)$, $u'(\xi) = 0(|\xi|^{\alpha-1})$ as $\xi \to 0\pm$, for some $\alpha > 0$.

Therefore, $v'(0+) = v'(0-) = 0$ and the right and left limits $u(0+)$, $u(0-)$ exist and are finite. By (2.8) and (H),

$$\sigma(u(0-)) = \sigma(u(0+)) , \qquad u(0-) = u(0+) , \tag{2.18}$$

and thus $u \in C(\mathbb{R}) \cap C^1(\mathbb{R} - \{0\})$, while $v \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})$.

Remark. The existence of discontinuous solutions was studied by Hoff [H_1, H_2] first for the equations of viscoelasticity (VE) (or compressible, isentropic Navier-Stokes equations) and later for the full compressible Navier-Stokes equations. Hoff [H_1] identifies the jump relations (2.8) at points of dicontinuity and shows that the magnitudes of the jumps decay exponentially in time, with an exponent inversely proportional to ε. It is instructive to compare these results with the behavior of solutions corresponding to Riemann data for the system (1.5) studied here. Although solutions with jumps are allowed from the weak form of the equations $(2.5-2.6)$ leading to the jump conditions (2.8), the initial jump is resolved instanteneously and yields a continuous solution for any $t > 0$ (*c.f.* Lemma 2.2 and (2.18)). The operator in (1.5) thus misses the initial layer present in the case of (VE). Still, in analogy to the dependence on ε of the decay rate of the jumps in (VE), the regularity of the resulting solution of (1.5) improves as $\varepsilon \to 0$ at a rate inversely proportional to ε.

2.2 The Construction Scheme.

Our approach for constructing solutions of $(\mathcal{P}_\varepsilon)$ is to apply the Leray-Schauder degree theory (eg. Rabinowitz [R]) to a deformation of maps, that preserves the regularity structure outlined in the preceding section. Degree theory has been a successful tool for establishing connecting trajectories in problems involving self-similar viscous limits (Dafermos [D_1], Slemrod [S], Slemrod and Tzavaras [STz]). Due to the absence of diffusion in $(2.1)_1$ a candidate scheme for constructing trajectories should capture the interplay between the "hyperbolic" and "parabolic" effects in system (2.1) and be thus different in spirit than the "parabolic" based schemes used in [D_1], [S] or [STz].

Let $X = C^0(-\infty, \infty)$ be the space of bounded, continuous functions. X equipped with the norm $\|u\| = \sup\limits_{-\infty<\xi<\infty} |u(\xi)|$ is a Banach space. Consider the sets

$$\Omega = \{u \in X : 0 < \frac{\delta}{2} < u(\xi) < M+1\ ,\ -\infty < \xi < \infty\} \tag{2.19$_L$}$$

in the case of longitudinal motions,

$$\Omega = \{u \in X : |u(\xi)| < M+1\ ,\ -\infty < \xi < \infty\} \tag{2.19$_S$}$$

in the case of shearing motions, with $\delta,\ M > 0$ constants to be determined later. In either case Ω is a bounded open set in X. For $\mu \in [0,1]$, $\varepsilon > 0$ fixed and $U \in \bar{\Omega}$ consider

$$\begin{aligned} -\xi u' - v' &= 0 \\ -\xi v' - \sigma_u(U(\xi))u' &= \varepsilon(k(U(\xi))v')' \end{aligned} \qquad -\infty < \xi < \infty \tag{2.20}$$

$$u(\pm\infty) = u_\pm(\mu) := u_- + \mu(u_\pm - u_-)$$
$$v(\pm\infty) = v_\pm(\mu) := v_- + \mu(v_\pm - v_-) \; . \tag{2.21}$$

Note that for longitudinal motions $u_+, \; u_- > 0$ and thus $u_+(\mu), \; u_-(\mu) > 0$ for $\mu \in [0,1]$.

Let $\mathcal{F} : [0,1] \times \bar{\Omega} \to X$ be the map carrying (μ, U) to the first component u_μ of the solution (u_μ, v_μ) of $(2.20 - 2.21)$. Then solutions of $(\mathcal{P}_\varepsilon)$ may be visualized as fixed points of $\mathcal{F}(1,\cdot)$ in Ω. In the following section we show that $\mathcal{F}$ is well defined and apply the Leray-Schauder degree theory to the homotopy $I - \mathcal{F}(\mu,\cdot)$. Towards that it is necessary to estimate the solutions of $u - \mathcal{F}(\mu, u) = 0, \; \mu \in [0,1]$; equivalently, to establish a-priori estimates for solutions of (2.1), (2.21). These are pursued next.

2.3. A-priori Estimates.

For the remainder of the section, let $(u(\xi), v(\xi))$ be a solution of (2.1) on $(-\infty, \infty)$, with smoothness $u \in C(\mathrm{I\!R}) \cap C^1(\mathrm{I\!R} - \{0\}), \; v \in C^1(\mathrm{I\!R}) \cap C^2(\mathrm{I\!R} - \{0\})$, satisfying the boundary conditions (2.21) for some $\mu \in [0,1]$. For longitudinal motions it is also assumed that $u(\xi) > 0$ on $(-\infty, \infty)$. In the sequel C_ε will stand for a generic constant depending on $u_\pm, \; v_\pm$ and ε but independent of μ, while C will denote such constants whenever they are independent of ε.

From (2.10) we see that:

(i) If $v'(\xi_0) = 0$ for some $\xi_0 \in (0, \infty)$ then $v'(\xi) = 0$ on $(0, \infty)$.

(ii) If $v'(\xi_0) = 0$ for some $\xi_0 \in (-\infty, 0)$ then $v'(\xi) = 0$ on $(-\infty, 0)$.

Thus on each of the segments $(-\infty, 0)$ and $(0, \infty)$ either v is strictly monotone or it is a constant state. By (2.11) the same property characterizes u. Consequently we can classify the possible shapes of (u, v) into five distinct categories:

C_1: u is increasing on $(-\infty, \infty)$, v is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

C_2: u is decreasing on $(-\infty, \infty)$, v is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

C_3: u is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, v is increasing on $(-\infty, \infty)$.

C_4: u is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, v is decreasing on $(-\infty, \infty)$.

C_5: (u, v) has the behavior depicted in $C_1 - C_4$ in one of the segments $(-\infty, 0)$ or $(0, \infty)$ and is constant on the other.

The classification of shapes is the main ingredient leading to L^∞- estimates for solutions of the boundary value problems (2.1), (2.21). Such estimates are obvious for solutions of class C_5, so we concentrate on the other cases.

Lemma 2.3. *For solutions (u,v) of class C_1 or C_2:*

$$\min\{u_-, u_+\} \le u(\xi) \le \max\{u_-, u_+\} , \tag{2.22}$$

$$|v(\xi)| \le C , \tag{2.23}$$

for $-\infty < \xi < \infty$. The constant C is independent of μ and ε.

Proof. We present the proof for solutions of class C_1, namely u is increasing from $u_-(\mu)$ to $u_+(\mu)$ and v is increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$. Then (2.22) is clear, and to show (2.23) it suffices to bound $v(0)$ from above.

Since (u,v) is a solution of (2.7) satisfying $(M)_S$ or $(M)_L$, Lemma 2.2 implies $v'(0) = 0$ and $v'(\xi) \to 0$ as $\xi \to \pm\infty$. Integrating $(2.1)_2$ over $(0,\infty)$ we obtain

$$-\int_0^\infty \zeta v'(\zeta) d\zeta = \sigma(u_+(\mu)) - \sigma(u(0)) \le \sigma(u_+) - \sigma(u(0)) ,$$

and thus

$$\begin{aligned} v(1) = v_+(\mu) - \int_1^\infty v'(\zeta) d\zeta &\le v_+(\mu) - \int_1^\infty \zeta v'(\zeta) d\zeta \\ &\le \max\{v_-, v_+\} + \sigma(u_+) - \sigma(u(0)) . \end{aligned}$$

Finally, from $(2.1)_1$,

$$\begin{aligned} v(0) &= v(1) + \int_0^1 \zeta u'(\zeta) d\zeta \\ &\le \max\{v_-, v_+\} + \sigma(u_+) - \sigma(u(0)) + u(1) - u(0) \end{aligned}$$

is bounded independently of μ and ε. ∎

Lemma 2.4. *For solutions (u,v) of class C_3 or C_4,*

$$\min\{v_-, v_+\} \le v(\xi) \le \max\{v_-, v_+\} \tag{2.24}$$

for $-\infty < \xi < \infty$. Moreover:

(i) *Under hypotheses* (H) *and* $(H)_S$ *for shearing motions*

$$|u(\xi)| \le C \tag{2.25$_S$}$$

for $-\infty < \xi < \infty$, *where* C *is independent of* μ *and* ε.

(ii) *Under hypotheses* (H) *and* $(H)_L$ *for longitudinal motions*

$$0 < \delta \le u(\xi) \le C \tag{2.25$_L$}$$

for $-\infty < \xi < \infty$, *with* δ *and* C *positive constants independent of* μ *and* ε.

Proof. First we consider solutions (u, v) of class C_3. They satisfy on $(-\infty, \infty)$

$$\begin{gathered} v_-(\mu) \le v(\xi) \le v_+(\mu) \\ \min\{u_-(\mu), u_+(\mu)\} \le u(\xi) \le u(0) \end{gathered} \tag{2.26}$$

and, in view of (2.21), it suffices to show that $u(0)$ is bounded from above.

Integrating $(2.1)_1$ over $(0, \infty)$ we obtain

$$-\int_0^\infty \zeta u'(\zeta) d\zeta = v_+(\mu) - v(0) \le v_+ - v_- \ .$$

In turn, this yields for any $\xi > 0$

$$\begin{aligned} u(\xi) &= u_+(\mu) - \int_\xi^\infty u'(\zeta) d\zeta \\ &\le u_+(\mu) - \frac{1}{\xi} \int_\xi^\infty \zeta u'(\zeta) d\zeta \\ &\le \max\{u_-, u_+\} + \frac{1}{\xi} (v_+ - v_-) \ . \end{aligned} \tag{2.27}$$

Let ξ, θ be two points with $0 < \xi < \theta$. Integrating $(2.1)_2$ over (ξ, θ) leads to

$$\sigma(u(\xi)) + \varepsilon k(u(\xi)) v'(\xi) = \sigma(u(\theta)) + \varepsilon k(u(\theta)) v'(\theta) + \int_\xi^\theta \zeta v'(\zeta) d\zeta \ . \tag{2.28}$$

Choose $\theta \in [1, 2]$ such that $v'(\theta) = v(2) - v(1) \le v_+ - v_-$. Then

$$\begin{aligned} &\sigma(u(\theta)) + \varepsilon k(u(\theta)) v'(\theta) + \int_\xi^\theta \zeta v'(\zeta) d\zeta \\ &\qquad \le \sigma(u(\theta)) + \varepsilon (v_+ - v_-) k(u(\theta)) + 2(v_+ - v_-) \\ &\qquad \le \max_{u_+(\mu) \le q \le u(1)} \Big(\sigma(q) + \varepsilon (v_+ - v_-) k(q) \Big) + 2(v_+ - v_-) \ , \end{aligned}$$

in conjunction with (2.27) and (2.28), implies that

$$\varepsilon\, k(u(\xi))\, v'(\xi) + \sigma(u(\xi)) \leq A\,, \qquad \text{for} \ \ 0 < \xi \leq 1\,, \tag{2.29}$$

with A a constant independent of μ and ε. For solutions of class C_3 it is $v' > 0$ on $(0, \infty)$ and (2.29) gives as $\xi \to 0$

$$\sigma(u(0)) \leq A\,. \tag{2.30}$$

Under hypotheses $(H)_S$ or $(H)_L$, (2.30) implies that $u(0)$ is bounded from above and completes the proof of (2.25).

When $(H)_S$ or $(H)_L$ are violated $\sigma(u)$ increases monotonically to a limiting finite value σ_∞ as $u \to \infty$. If $\sigma_\infty > A$ the above argument still leads to an ε-independent bound for $u(0)$ and provides (2.25) for restricted classes of data. However, if $\sigma_\infty \leq A$ the estimate disintegrates.

Next, we turn to solutions of class C_4. For shearing motions the bound $(2.25)_S$ follows from a similar argument. So consider the case of longitudinal motions for (u, v) of class C_4. Note that

$$0 < \underline{u} := \min\{u_-, u_+\} \leq u_\pm(\mu) \leq \max\{u_-, u_+\} =: \bar{u}\,, \quad v_- > v_+\,,$$

and that

$$\begin{aligned} v_+(\mu) &\leq v(\xi) \leq v_-(\mu) \\ u(0) &\leq u(\xi) \leq \max\{u_+(\mu), u_-(\mu)\} \end{aligned} \tag{2.31}$$

on $(-\infty, \infty)$. It suffices to show that $u(0)$ is bounded from below away from zero.

Observe first that from $(2.1)_1$

$$\int_0^\infty \zeta u'(\zeta) d\zeta = v(0) - v_+(\mu) \leq v_- - v_+\,.$$

Choose n sufficiently large so that $\frac{1}{n}(v_- - v_+) \leq \frac{1}{2}\underline{u}$. Then for $\xi \geq n$

$$\begin{aligned} u(\xi) &= u_+(\mu) - \int_\xi^\infty u'(\zeta) d\zeta \\ &\geq u_+(\mu) - \frac{1}{n}\int_\xi^\infty \zeta u'(\zeta) d\zeta \geq \frac{1}{2}\underline{u}\,. \end{aligned} \tag{2.32}$$

Next, choose $\theta \in [n, n+1]$ such that $v'(\theta) = v(n+1) - v(n) \geq -(v_- - v_+)$. Integrating $(2.1)_2$ over (ξ, θ) with $0 < \xi \leq n \leq \theta \leq n+1$ and using (2.32), we obtain

$$\begin{aligned}
&\varepsilon k(u(\xi))v'(\xi) + \sigma(u(\xi)) \\
&\qquad = \sigma(u(\theta)) + \varepsilon k(u(\theta))v'(\theta) + \int_\xi^\theta \zeta v'(\zeta) d\zeta \\
&\qquad \geq \sigma(u(\theta)) - \varepsilon(v_- - v_+)k(u(\theta)) + (n+1)\Big(v(\theta) - v(\xi)\Big) \\
&\qquad \geq \min_{\frac{1}{2}\underline{u} \leq q \leq \bar{u}} \Big(\sigma(q) - \varepsilon(v_- - v_+)k(q)\Big) - (n+1)(v_- - v_+) =: -B
\end{aligned} \tag{2.33}$$

with B a (possibly negative) constant independent of μ and ε. Since $v' < 0$ on $(0, \infty)$, (2.33) in conjunction with $(H)_L$ imply

$$u(0) \geq \sigma^{-1}(-B) > -\infty\,, \tag{2.34}$$

and $(2.25)_L$ follows from (2.31), (2.34). ∎

The restrictions on shapes of solutions together with the L^∞-estimates in Lemmas 2.3 and 2.4 imply that families $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of solutions to $(\mathcal{P}_\varepsilon)$ are of uniformly bounded variation.

Corollary 2.5. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of solutions to the boundary-value problem $(\mathcal{P}_\varepsilon)$ corresponding to fixed data $(u_\pm, v_\pm)$. If $\sigma(u)$ satisfies (H) and $(H)_S$ or $(H)_L$, there exist constants C and δ depending on the data but independent of ε such that*

$$|u_\varepsilon(\xi)| \leq C\,, \quad |v_\varepsilon(\xi)| \leq C\,, \quad \text{on } (-\infty, \infty) \tag*{$(2.35)_S$}$$

for shearing motions,

$$0 < \delta \leq u_\varepsilon(\xi) \leq C\,, \quad |v_\varepsilon(\xi)| \leq C\,, \quad \text{on } (-\infty, \infty) \tag*{$(2.35)_L$}$$

for longitudinal motions, and

$$TV_{(-\infty,\infty)}\, u_\varepsilon \leq C\,, \quad TV_{(-\infty,\infty)}\, v_\varepsilon \leq C \tag{2.36}$$

in both cases.

3. CONNECTING TRAJECTORIES II - EXISTENCE

In this section we study the map $\mathcal{F} : [0,1] \times \bar{\Omega} \to X$ that carries $(\mu, U) \in [0,1] \times \bar{\Omega}$ to the first component u_μ of the solution (u_μ, v_μ) to (2.20), (2.21). Ω is defined by $(2.19)_L$ for longitudinal motions or $(2.19)_S$ for shearing motions; the constants δ and M in the definition of Ω are chosen as the minima and maxima, respectively, of the bounds in Lemmas 2.3 and 2.4. Our objectives are: (a) to show $\mathcal{F}$ is well defined, and (b) to establish existence of solutions for the equation $u - \mathcal{F}(\mu, u) = 0,\ \mu \in [0,1]$.

Let $T : \bar{\Omega} \to X,\ S : \bar{\Omega} \to X$ be the maps that carry $U \in \bar{\Omega}$ to the solution $(u, v) = (T(U), S(U))$ of the boundary value problem (2.20) subject to boundary data

$$u(-\infty) = 0,\ u(+\infty) = u_+ - u_- \ ; \quad v(-\infty) = 0,\ v(+\infty) = v_+ - v_- \ .$$

Then $(u_- + \mu T(U), v_- + \mu S(U))$ satisfy (2.20), (2.21) and thus we may express the map $\mathcal{F}$ in the form

$$\mathcal{F}(\mu, U) = u_- + \mu T(U) \ .$$

3.1. The Linearized Problem

Properties of T, S derive from analysis of the linearized problem

$$\left.\begin{aligned} -\xi u' - v' &= 0 \\ -\xi v' - a(\xi) u' &= \varepsilon (k(\xi) v')' \end{aligned}\right\} \quad -\infty < \xi < \infty \tag{3.1}$$

subject to data

$$\begin{aligned} u(-\infty) &= 0 \ , \quad u(+\infty) = u_+ - u_- \\ v(-\infty) &= 0 \ , \quad v(+\infty) = v_+ - v_- \end{aligned} \tag{3.2}$$

and with $a(\xi),\ k(\xi)$ continuous functions satisfying on $(-\infty, \infty)$

$$\begin{aligned} 0 < a_0 \le a(\xi) \le A_0 \\ 0 < k_0 \le k(\xi) \le K_0 \ . \end{aligned} \tag{3.3}$$

The solution (u, v) of (3.1), (3.2) can be calculated explicitly. Observe that v' satisfies the differential equation

$$\varepsilon (k(\xi) v')' + \frac{\xi^2 - a(\xi)}{\xi} v' = 0 \ , \quad \xi \neq 0$$

and thus

$$v'(\xi)=\begin{cases} c_+\dfrac{1}{k(\xi)}\exp\Big\{-\dfrac{1}{\varepsilon}\displaystyle\int_1^\xi \frac{s^2-a(s)}{sk(s)}ds\Big\}=: c_+\, I_+(\xi), & \xi>0\\ c_-\dfrac{1}{k(\xi)}\exp\Big\{-\dfrac{1}{\varepsilon}\displaystyle\int_{-1}^\xi \frac{s^2-a(s)}{sk(s)}ds\Big\}=: c_-\, I_-(\xi), & \xi<0\end{cases} \tag{3.4}$$

$$u'(\xi)=-\frac{1}{\xi}v'(\xi)=\begin{cases}-c_+\dfrac{1}{\xi}I_+(\xi)\,, & \xi>0\\ -c_-\dfrac{1}{\xi}I_-(\xi)\,, & \xi<0\end{cases} \tag{3.5}$$

with c_+, c_- arbitrary constants.

Lemma 3.1. *I_+, I_- have the following behavior. There exist constants α, β, γ, δ and C_ε depending on a_0, A_0, k_0, K_0, with C_ε also depending on ε, such that*

$$\frac{1}{C_\varepsilon}|\xi|^{\frac{\gamma}{\varepsilon}}\le I_\pm(\xi)\le C_\varepsilon|\xi|^{\frac{\alpha}{\varepsilon}}\,,\quad 0<|\xi|\le 1\,, \tag{3.6}$$

$$\frac{1}{C_\varepsilon}e^{-\frac{\delta}{\varepsilon}\xi^2}\le I_\pm(\xi)\le C_\varepsilon e^{-\frac{\beta}{\varepsilon}\xi^2}\,,\quad |\xi|\ge 1\,. \tag{3.7}$$

Proof. By virtue of (3.3), for $0<\xi\le 1$

$$\int_\xi^1\frac{s^2-a(s)}{sk(s)}ds=\int_\xi^1\frac{s}{k(s)}ds-\int_\xi^1\frac{a(s)}{sk(s)}ds \le \frac{1}{2k_0}+\frac{a_0}{K_0}\ell n\xi\,,$$

while for $\xi\ge 1$

$$-\int_1^\xi\frac{s^2-a(s)}{sk(s)}ds=-\int_1^\xi\frac{s}{k(s)}ds+\int_1^\xi\frac{a(s)}{sk(s)}ds \le -\frac{1}{2K_0}(\xi^2-1)+\frac{A_0}{k_0}\ell n\xi\le -\beta\xi^2+M$$

for some $\beta<\frac{1}{2K_0}$ and $M>0$. These bounds together with (3.4) show one part of (3.6), (3.7). The rest follows by similar arguments. ∎

It follows from (3.6), (3.7) that u', v' are integrable on $(-\infty,\infty)$ and thus (u,v) can be calculated by the formulas

$$u(\xi)=\begin{cases}(u_+-u_-)+c_+\displaystyle\int_\xi^\infty\frac{I_+(\zeta)}{\zeta}d\zeta\quad, & \xi>0\\ c_-\displaystyle\int_{-\infty}^\xi\frac{I_-(\zeta)}{-\zeta}d\zeta\quad, & \xi<0\end{cases} \tag{3.8}$$

$$v(\xi) = \begin{cases} (v_+ - v_-) - c_+ \displaystyle\int_\xi^\infty I_+(\zeta) d\zeta \quad , \quad \xi > 0 \\ c_- \displaystyle\int_{-\infty}^{\xi} I_-(\zeta) d\zeta \quad , \quad \xi < 0 \, . \end{cases} \tag{3.9}$$

The constants c_+, c_- are evaluated by requiring continuity of (u, v) across $\xi = 0$. That implies

$$\begin{aligned} c_+ \int_0^\infty \frac{I_+(\zeta)}{\zeta} d\zeta - c_- \int_{-\infty}^0 \frac{I_-(\zeta)}{-\zeta} d\zeta &= -(u_+ - u_-) \\ c_+ \int_0^\infty I_+(\zeta) d\zeta + c_- \int_{-\infty}^0 I_-(\zeta) d\zeta &= v_+ - v_- \, . \end{aligned} \tag{3.10}$$

As a consequence of (3.6), (3.7) the determinant Δ of the linear system (3.10) is bounded from below by a positive constant and thus (3.10) admits a unique solution (c_+, c_-).

We turn now to estimate the solution (u, v) defined by (3.8), (3.9). Let K_ε stand for a generic constant depending on a_0, A_0, k_0, K_0, ε and $u_\pm$, $v_\pm$. Then (3.6), (3.7), (3.8) and (3.9) imply

$$|c_+| + |c_-| \le K_\varepsilon \tag{3.11}$$

$$|u(\xi)| + |v(\xi)| \le K_\varepsilon \quad , \quad \xi \in (-\infty, \infty) \tag{3.12}$$

$$|u'(\xi)| + |v'(\xi)| \le K_\varepsilon e^{-\frac{\beta}{\varepsilon}\xi^2} \, , \; |\xi| \ge 1 \tag{3.13}$$

$$|v'(\xi)| \le K_\varepsilon \, , \; |u'(\xi)| \le K_\varepsilon |\xi|^{\frac{\alpha}{\varepsilon} - 1} \, , \; 0 < |\xi| \le 1 \, . \tag{3.14}$$

Therefore

$$\|v'\|_{L^\infty} \le K_\varepsilon \, , \; \|u'\|_{L^{p_\varepsilon}} \le K_\varepsilon$$

with $p_\varepsilon = \infty$ if $\varepsilon \le \alpha = \frac{a_0}{K_0}$ and $p_\varepsilon < 1/(1 - \frac{\alpha}{\varepsilon})$ if $\alpha < \varepsilon$. We conclude that v is Lipschitz, and

$$|u(\zeta) - u(\theta)| \le \|u'\|_{L^{p_\varepsilon}} |\zeta - \theta|^{(1 - \frac{1}{p_\varepsilon})} \tag{3.15}$$

u is Hölder continuous with exponent $1 - \frac{1}{p_\varepsilon} < \frac{\alpha}{\varepsilon}$ if $\alpha < \varepsilon$ and exponent 1 (Lipschitz) if $\varepsilon \le \alpha$. Note that the regularity improves as ε decreases.

3.2. Existence of Solutions for $(\mathcal{P}_\varepsilon)$

We state the main theorem of this section establishing existence of solutions for $(\mathcal{P}_\varepsilon)$:

Theorem 3.2. *Let $\varepsilon > 0$ be fixed and assume that $\sigma(u)$ satisfies (H) and $(H)_S$ in the case of shearing motions, or (H) and $(H)_L$ in the case of longitudinal motions. Then there exists a solution $(u(\xi), v(\xi))$ of the boundary value problem $(\mathcal{P}_\varepsilon)$ on $(-\infty, \infty)$. For longitudinal motions the solution satisfies $u(\xi) > 0$ on $(-\infty, \infty)$.*

Proof. Let $T,\ S : \bar{\Omega} \to X$ be the maps carrying $U \in \bar{\Omega}$ to the solution $(T(U), S(U))$ of (2.20), (3.2). T, S are properly defined by (3.8), (3.9) and (3.10), where $I_\pm$ are given by (3.4) with $k(\xi) = k(U(\xi))$, $a(\xi) = a(U(\xi))$. Define the map $\mathcal{F} : [0,1] \times \bar{\Omega} \to X$ by $\mathcal{F}(\mu, u) = u_- + \mu T(U)$. If u is a solution of $u = u_- + T(u)$ in Ω and we set $v = S(u)$ then (u, v) is a solution of $(\mathcal{P}_\varepsilon)$ with $u \in \Omega$. We will apply the Leray-Schauder degree theory (Rabinowitz [R, Ch V]) to solve

$$u - \mu T(u) = u_- \ , \quad \mu \in [0,1] \ . \tag{3.16}$$

First we show $T : \bar{\Omega} \to X$ is compact. This follows from (i), (ii) below.

(i) $T(\bar{\Omega})$ *is precompact in* X.

Consider a sequence $\{U_n\} \subset \bar{\Omega}$ and let $u_n = T(U_n)$. Estimates (3.12), (3.15) and (3.14) imply that the sequence of functions $\{u_n\}$ are uniformly bounded and uniformly equicontinuous on $[-L, L]$ for any $L \geq 1$. Also,

$$\begin{aligned} |u_n(\xi)| &\leq C_\varepsilon e^{-\frac{\beta}{\varepsilon}\xi^2} \quad , \quad \xi \leq -1 \\ |u_n(\xi) - (u_+ - u_-)| &\leq C_\varepsilon e^{-\frac{\beta}{\varepsilon}\xi^2} \quad , \quad \xi \geq 1 \ , \end{aligned} \tag{3.17}$$

with C_ε independent of n. It follows from the Ascoli-Arzela theorem and a diagonal argument that there is a subsequence $\{u_{n_k}\}$ and a continuous function u such that $u_{n_k} \to u$ uniformly on compact subsets of the reals. But then (3.17) implies that $u_{n_k} \to u$ in X.

(ii) $T : \bar{\Omega} \to X$ *is continuous.*

Let $\{U_n\} \subset \bar{\Omega}$ be a sequence such that $U_n \to U$ in X and set $u_n = T(U_n)$. Then u_n, U_n are related by

$$\begin{aligned} u_n(\xi) &= (u_+ - u_-) + c_+^n \int_\xi^\infty \frac{I_+^n(\zeta)}{\zeta} d\zeta \\ I_+^n(\xi) &= \frac{1}{k(U_n(\xi))} \exp\Big\{ -\frac{1}{\varepsilon} \int_1^\xi \frac{s^2 - a(U_n(s))}{sk(U_n(s))} ds \Big\} \end{aligned} \tag{3.18}$$

for $\xi > 0$ and corresponding formulas for $\xi < 0$; c_+^n, c_-^n solve (3.10). By part (i) there is a subsequence $\{u_{n_k}\}$ and $v \in X$ such that $u_{n_k} \to v$ in X. Using (3.6), (3.7) we pass to the limit in (3.10), (3.18) along the subsequence $\{U_{n_k}\}$ and obtain $v = T(U)$. Since any limiting point of $\{u_n\}$ in X is of the form $v = T(U)$ we deduce $T(U_n) \to T(U)$ in X.

The map $\mu T : [0,1] \times \bar{\Omega} \to X$ is compact, thus the Leray-Schauder degree of $I - \mu T$ is well defined. By Lemmas 2.3, 2.4 and the definition of Ω any solution u of (3.16) lies in the interior of Ω. Therefore

$$d(I - \mu T, \Omega, u_-) = d(I, \Omega, u_-) = 1 \ , \quad \mu \in [0,1] \ ,$$

and (3.16) admits at least one solution for each $\mu \in [0,1]$. ∎

4. SOLUTION OF THE RIEMANN PROBLEM

For the remainder of the article we study the $\varepsilon \to 0$ limit of solutions to $(\mathcal{P}_\varepsilon)$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0}$ be a family of such solutions, corresponding to fixed boundary data $(u_\pm, v_\pm)$, and having the property

$$(u_\varepsilon, v_\varepsilon) \ \text{ satisfy the uniform bounds } \ (2.35),\ (2.36) \ \text{ for any } \ \varepsilon > 0 \ . \tag{A}$$

Helly's selection principle implies there exists a subsequence of the original family, denoted again by $\{(u_\varepsilon, v_\varepsilon)\}$ with $\varepsilon \to 0$, and a pair of functions (u, v) of bounded variation such that

$$u_\varepsilon(\xi) \to u(\xi) \ , \quad v_\varepsilon(\xi) \to v(\xi), \qquad \text{pointwise on } \ (-\infty, \infty) \ . \tag{4.1}$$

Our objectives are to show (u, v) is a solution of the Riemann problem and to investigate its structure.

Concerning Hypothesis (A) we remark: First, Corollary 2.5 guarantees (A) for families corresponding to any data provided $\sigma(u)$ satisfies (H) and $(H)_L$ or $(H)_S$. Although (A) is not expected to be in general valid when $(H)_S$ or $(H)_L$ are violated, Lemmas 2.3 and 2.4 suggest (A) still holds under relaxed conditions on $\sigma(u)$ provided the data $(u_\pm, v_\pm)$ are appropriately restricted. To account for such cases, we work under the framework of (A) for functions $\sigma(u)$ that satisfy (H). We refer to Dafermos and DiPerna [DD], Keyfitz and

Kranzer [KKr] for studies of $\varepsilon \to 0$ limits in the absence of uniform bounds; see Antman [A] for a discussion of the problem of infinite compression in a different context.

Along families $\{(u_\varepsilon, v_\varepsilon)\}$ satisfying (A), the estimates (2.12), (2.13) also hold and the wave speeds $\lambda_1(u) = -\sqrt{\sigma_u(u)}$, $\lambda_2(u) = +\sqrt{\sigma_u(u)}$ of the hyperbolic system (E) are bounded and separated

$$\Lambda_- \le \lambda_1(u_\varepsilon(\xi)) \le \lambda_- < 0 < \lambda_+ \le \lambda_2(u_\varepsilon(\xi)) \le \Lambda_+ \tag{4.2}$$

by constants $\lambda_\pm = \pm\sqrt{a_0}$, $\Lambda_\pm = \pm\sqrt{A_0}$ independent of ε. First, we show that (u, v) is a solution of the Riemann problem.

Theorem 4.1. *Suppose $\sigma(u)$ satisfies (H) and let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of solutions of $(\mathcal{P}_\varepsilon)$ corresponding to data $(u_\pm, v_\pm)$ and satisfying (A). There exists a subsequence $\{(u_{\varepsilon_n}, v_{\varepsilon_n})\}$ with $\varepsilon_n \to 0$ and a bounded function (u, v) of bounded variation such that $u_{\varepsilon_n} \to u$, $v_{\varepsilon_n} \to v$ pointwise on the reals. The limit (u, v) satisfies* (1.2) *in the sense of measures and*

$$(u(\xi), v(\xi)) = \begin{cases} (u_-, v_-) & \xi < \Lambda_- \\ (u(0), v(0)) & \lambda_- < \xi < \lambda_+ \\ (u_+, v_+) & \xi > \Lambda_+ \, . \end{cases} \tag{4.3}$$

Proof. Let $\{(u_\varepsilon, v_\varepsilon)\}$ be a convergent subsequence as in (4.1). Integrating $(2.1)_2$ over $(0, \xi)$ and using $v_\varepsilon'(0) = 0$ yields

$$\varepsilon k(u_\varepsilon(\xi)) v_\varepsilon'(\xi) = \int_0^\xi v_\varepsilon(\zeta) d\zeta - \xi v_\varepsilon(\xi) + \sigma(u_\varepsilon(0)) - \sigma(u_\varepsilon(\xi))$$

and, in view of (A),

$$k(u_\varepsilon(\xi)) |v_\varepsilon'(\xi)| \le \frac{C}{\varepsilon}(|\xi| + 1) \, , \quad \xi \in (-\infty, \infty) \, . \tag{4.4}$$

Lemma 2.2, together with (4.4), implies the following bounds for $(u_\varepsilon', v_\varepsilon')$: Fix first any points $a < 0 < b$ with $[a, b] \subset (\lambda_-, \lambda_+)$. We see from (2.14), (2.16) and (4.4) that

$$|v_\varepsilon'(\xi)| \le \begin{cases} \dfrac{C}{\varepsilon}\Big(\dfrac{\xi}{b}\Big)^{\frac{d}{\varepsilon}} & 0 < \xi < b \\ \dfrac{C}{\varepsilon}\Big(\dfrac{\xi}{a}\Big)^{\frac{d}{\varepsilon}} & a < \xi < 0 \end{cases} \tag{4.5}$$

where d depends on the distances $|a-\lambda_-|$, $|b-\lambda_+|$ and degenerates as these distances go to zero. In a similar fashion for any $(-\infty,a]\subset(-\infty,\Lambda_-)$ and $[b,\infty)\subset(\Lambda_+,\infty)$

$$|v_\varepsilon'(\xi)|\le\begin{cases}\dfrac{C}{\varepsilon}\exp\Big\{-\dfrac{d}{\varepsilon}\Big((\dfrac{\xi}{a})^2-1\Big)\Big\}\,, & \xi<a\\ \dfrac{C}{\varepsilon}\exp\Big\{-\dfrac{d}{\varepsilon}\Big((\dfrac{\xi}{b})^2-1\Big)\Big\}\,, & \xi>b\end{cases} \tag{4.6}$$

where d now depends on $|a-\Lambda_-|$ or $|b-\Lambda_+|$ respectively. Using (4.5), (4.6) and (2.11), we deduce that $|v_\varepsilon'|$, $|u_\varepsilon'|\to 0$ uniformly on compact subsets of (λ_-,λ_+), $(-\infty,\Lambda_-)$ and (Λ_+,∞), and that the limiting (u,v) satisfies (4.3).

Observe next that if

$$\eta(u,v)=\frac{1}{2}v^2+\int_1^u\sigma(\tau)d\tau\ ,\quad q(u,v)=-v\sigma(u) \tag{4.7}$$

then (2.1) implies

$$-\xi\frac{d}{d\xi}\eta(u_\varepsilon,v_\varepsilon)+\frac{d}{d\xi}q(u_\varepsilon,v_\varepsilon)=\varepsilon v_\varepsilon(k(u_\varepsilon)v_\varepsilon')'\ . \tag{4.8}$$

Integrating (4.8) over any interval $[a,b]$

$$\begin{aligned}\int_a^b\eta(u_\varepsilon,v_\varepsilon)d\zeta+\varepsilon\int_a^b k(u_\varepsilon)(v_\varepsilon')^2d\zeta&\\ =\Big(\xi\eta(u_\varepsilon,v_\varepsilon)-q(u_\varepsilon,v_\varepsilon)+\varepsilon v_\varepsilon k(u_\varepsilon)v_\varepsilon'\Big)\Big|_a^b&\end{aligned} \tag{4.9}$$

and using (A) and (4.4), we obtain

$$\varepsilon\int_a^b k(u_\varepsilon)(v_\varepsilon')^2\,d\zeta\le C(|a|+|b|+1)\ . \tag{4.10}$$

Let now $\varphi,\ \psi\in C_c^\infty(\mathbb{R})$ be test functions and suppose that $\operatorname{supp}\varphi\subset[a,b]$. We integrate (2.1) against ψ,φ respectively and use (4.1) and the estimate

$$\begin{aligned}\varepsilon\int_{-\infty}^{\infty}k(u_\varepsilon)v_\varepsilon'\varphi'd\zeta&\le\varepsilon\Big(\int_a^b k(u_\varepsilon)(v_\varepsilon')^2d\zeta\Big)^{1/2}\Big(\int_a^b k(u_\varepsilon)\varphi'^2d\zeta\Big)^{1/2}\\&\le\varepsilon^{1/2}C(|a|+|b|+1)\Big(\int_a^b\varphi'^2d\zeta\Big)^{1/2}\end{aligned}$$

to conclude upon taking $\varepsilon\to 0$ that (u,v) satisfy

$$\begin{aligned}\int_{-\infty}^{\infty}u(\zeta\psi)'+v\psi'd\zeta&=0\\ \int_{-\infty}^{\infty}v(\zeta\varphi)'+\sigma(u)\varphi'd\zeta&=0\ .\end{aligned} \tag{4.11}$$

As (u, v) are functions of bounded variation (1.2) is satisfied in the sense of distributions and in the sense of measures. ∎

The function

$$\Big(u(\frac{x}{t}), v(\frac{x}{t})\Big) , \quad (x, t) \in (-\infty, \infty) \times (0, \infty) \tag{4.12}$$

is a weak solution of the Riemann problem for (E). Indeed, (1.1) follows from (4.3), and that (4.12) is a weak solution of (E) can be established by an argument similar to the one leading to (4.11). In fact, a solution of the form (4.12) is a weak solution of (E) in $(-\infty, \infty) \times (0, \infty)$ if and only if $(u(\xi), v(\xi))$ satisfies (4.11) (Dafermos [D_3]). Theorem 4.1 in conjunction with Corollary 2.5 lead to an existence theorem for the Riemann problem.

Theorem 4.2. *Assume that $\sigma(u)$ satisfies (H) and $(H)_L$ or $(H)_S$. Given any (u_-, v_-), (u_+, v_+) (with u_-, $u_+ > 0$ in the case of longitudinal motions), there exists a pair of functions of bounded variation $(u(\xi), v(\xi))$ defined on $(-\infty, \infty)$ (with $u(\xi) \geq \delta > 0$ in the case of longitudinal motions), such that $\Big(u(\frac{x}{t}), v(\frac{x}{t})\Big)$ is a weak solution of the Riemann problem for (E).*

Since (u, v) are of bounded variation, their domain can be decomposed into two disjoint subsets, $(-\infty, \infty) = \mathcal{C} \cup \mathcal{S}$, such that on $\mathcal{C}$ the function (u, v) is continuous while on $\mathcal{S}$ it undergoes jump discontinuities. $\mathcal{S} \subset [\Lambda_-, \lambda_-] \cup [\lambda_+, \Lambda_+]$ is at most countable and the right and left limits of (u, v) exist at any $\xi \in \mathcal{S}$. Moreover, u and v inherit the monotonicity properties of u_ε and v_ε listed in $C_1 - C_5$. We show next that at points of $\mathcal{S}$ the Rankine-Hugoniot conditions are satisfied.

Lemma 4.3. *At any $\xi \in \mathcal{S}$*

$$\begin{aligned} &-\xi\,[u(\xi+) - u(\xi-)] - [v(\xi+) - v(\xi-)] = 0 \\ &-\xi\,[v(\xi+) - v(\xi-)] - [\sigma(u(\xi+)) - \sigma(u(\xi-)] = 0 \,. \end{aligned} \tag{4.13}$$

Proof. This is a consequence of the fact that (u, v) of bounded variation satisfies (4.11). We outline a proof, which is in the spirit of self-similar viscous limits.

The shapes of $(u_\varepsilon, v_\varepsilon)$ together with the L^∞-bounds imply

$$\int_{-\infty}^{\infty} |u_\varepsilon'|\, d\zeta + \int_{-\infty}^{\infty} |v_\varepsilon'|\, d\zeta \leq C \,. \tag{4.14}$$

The solutions $(u_\varepsilon, v_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$ satisfy $(2.5-2.6)$. For $\xi \in \mathcal{S}$ fixed and any $\delta > 0$ these imply

$$\begin{aligned}
\int_\xi^{\xi+\delta} \theta u_\varepsilon(\theta) + v_\varepsilon(\theta)\ d\theta - \int_\xi^{\xi+\delta}\int_0^\theta u_\varepsilon(\zeta)\ d\zeta d\theta &= \delta v_\varepsilon(0) \\
\int_\xi^{\xi+\delta} \theta v_\varepsilon(\theta) + \sigma(u_\varepsilon(\theta))\ d\theta + \varepsilon \int_\xi^{\xi+\delta} k(u_\varepsilon(\theta)) v_\varepsilon'(\theta)\ d\theta & \\
- \int_\xi^{\xi+\delta}\int_0^\theta v_\varepsilon(\zeta)\ d\zeta d\theta &= \delta\sigma(u_\varepsilon(0))\ .
\end{aligned} \tag{4.15}$$

Take first $\varepsilon \to 0$, using (4.1), (A), (2.12) and (4.14), and then divide by δ and take $\delta \to 0+$ to obtain

$$\begin{aligned}
\xi u(\xi+) + v(\xi+) - \int_0^\xi u(\zeta) d\zeta &= v(0) \\
\xi v(\xi+) + \sigma(u(\xi+)) - \int_0^\xi v(\zeta) d\zeta &= \sigma(u(0))\ .
\end{aligned} \tag{4.16}$$

In a similar manner we establish

$$\begin{aligned}
\xi u(\xi-) + v(\xi-) - \int_0^\xi u(\zeta) d\zeta &= v(0) \\
\xi v(\xi-) + \sigma(u(\xi-)) - \int_0^\xi v(\zeta) d\zeta &= \sigma(u(0))\ ,
\end{aligned} \tag{4.17}$$

and (4.13) follows from a comparison of (4.16) and (4.17). ∎

5. STRUCTURE OF SOLUTIONS OF THE RIEMANN PROBLEM

Let $\{(u_\varepsilon, v_\varepsilon)\}$ be a sequence of solutions of $(\mathcal{P}_\varepsilon)$ as in Theorem 4.1 converging pointwise to the function (u, v) as $\varepsilon \to 0$. We study next the structure of (u, v).

First certain representation formulas, expressing the derivatives $(u_\varepsilon', v_\varepsilon')$ as averaging processes, are derived. Let

$$c_\varepsilon(\xi) = \frac{\xi^2 - \sigma_u(u_\varepsilon(\xi))}{\xi\ k(u_\varepsilon(\xi))} = \frac{1}{\xi\ k(u_\varepsilon(\xi))}\Big(\xi - \lambda_1(u_\varepsilon(\xi))\Big)\Big(\xi - \lambda_2(u_\varepsilon(\xi))\Big) \tag{5.1}$$

$$g_\varepsilon(\xi) = \begin{cases} \displaystyle\int_{\alpha_+}^\xi c_\varepsilon(\zeta) d\zeta\ , & \xi > 0 \\ \displaystyle\int_{\alpha_-}^\xi c_\varepsilon(\zeta) d\zeta\ , & \xi < 0 \end{cases} \tag{5.2}$$

where $\alpha_- < 0 < \alpha_+$ any fixed constants. Integrating (2.10) we obtain

$$v_+ - v_\varepsilon(0) = k\Big(u_\varepsilon(\alpha_+)\Big) v'_\varepsilon(\alpha_+) \int_0^\infty \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta$$
$$v_\varepsilon(0) - v_- = k\Big(u_\varepsilon(\alpha_-)\Big) v'_\varepsilon(\alpha_-) \int_{-\infty}^0 \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta \ .$$

In turn, using (2.10) and (2.11), we obtain a representation formula for v'_ε

$$v'_\varepsilon(\xi) = \begin{cases} (v_+ - v_\varepsilon(0)) \dfrac{\frac{1}{k(u_\varepsilon(\xi))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_0^\infty \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta} \ , & \xi > 0 \\ (v_\varepsilon(0) - v_-) \dfrac{\frac{1}{k(u_\varepsilon(\xi))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_{-\infty}^0 \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta} \ , & \xi < 0 \end{cases} \tag{5.3}$$

and a corresponding formula for u'_ε

$$u'_\varepsilon(\xi) = \begin{cases} -(v_+ - v_\varepsilon(0)) \dfrac{1}{\xi} \dfrac{\frac{1}{k(u_\varepsilon(\xi))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_0^\infty \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta} \ , & \xi > 0 \\ -(v_\varepsilon(0) - v_-) \dfrac{1}{\xi} \dfrac{\frac{1}{k(u_\varepsilon(\xi))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_{-\infty}^0 \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta} \ , & \xi < 0 \ . \end{cases} \tag{5.4}$$

Note that g_ε depends on the solution $(u_\varepsilon, v_\varepsilon)$.

It is instructive to use the correspondence between functions of bounded variation and finite Borel measures on the reals (Folland [F, Sec 3.5]). Let $(\lambda_\varepsilon, \nu_\varepsilon)$ be the measures associated with $(u'_\varepsilon, v'_\varepsilon)$ defined by

$$\begin{aligned} < \lambda_\varepsilon, \psi > &= \int \psi(\xi) u'_\varepsilon(\xi) d\xi \ , \\ < \nu_\varepsilon, \varphi > &= \int \varphi(\xi) v'_\varepsilon(\xi) d\xi \ , \end{aligned} \tag{5.5}$$

where $\psi, \ \varphi \in C_c(\mathbb{R})$ continuous functions with compact support. Then (2.36), (4.1) and (4.14) imply there exist finite (signed) Borel-Stieltjes measures (λ, ν) such that

$$\begin{aligned} \int \psi \ u'_\varepsilon d\xi \to \int \psi \ du =: < \lambda, \psi > \ , \quad \text{for} \ \ \psi \in C_c(\mathbb{R}) \ , \\ \int \varphi \ v'_\varepsilon d\xi \to \int \varphi \ dv =: < \nu, \varphi > \ , \quad \text{for} \ \ \varphi \in C_c(\mathbb{R}) \ , \end{aligned} \tag{5.6}$$

that is $\lambda_\varepsilon \rightharpoonup \lambda$, $\nu_\varepsilon \rightharpoonup \nu$ in the weak-$\star$ topology of measures (Folland [F , Sec 7.3]). Note that λ is generated by the right continuous function $\big(u(\xi+) - u_-\big)$, while ν is generated by $\big(v(\xi+) - v_-\big)$. It follows from (2.11), (5.3), (5.4), (4.5) and (4.6) that (λ, ν) enjoy the following properties :

(i) $\operatorname{supp} \lambda = \operatorname{supp} \nu \subset [\Lambda_+, \lambda_-] \cup [\lambda_+, \Lambda_-]$.

(ii) $\xi \notin \operatorname{supp} \lambda$ if and only if there is an open interval $I \ni \xi$ such that u is constant on I.

(iii) $\xi \notin \operatorname{supp} \nu$ if and only if there is an open interval $I \ni \xi$ such that v is constant on I.

(iv) $\mathcal{S} \subset \operatorname{supp} \lambda = \operatorname{supp} \nu$.

In the next proposition we establish an important property of the measures λ and ν, that incorporates admissibility restrictions induced by the self-similar viscosity. To this end, it is convenient to work with the quantity

$$\mu_\varepsilon(\xi) = \begin{cases} \dfrac{e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_0^\infty \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta}\,, & \xi > 0 \\ \dfrac{e^{-\frac{1}{\varepsilon} g_\varepsilon(\xi)}}{\int_{-\infty}^0 \frac{1}{k(u_\varepsilon(\zeta))} e^{-\frac{1}{\varepsilon} g_\varepsilon(\zeta)} d\zeta}\,, & \xi < 0\,. \end{cases} \tag{5.7}$$

Note that, since $\{\mu_\varepsilon\}$ are uniformly bounded in $L^1(\mathbb{R})$, along a further subsequence

$$< \mu_\varepsilon, \varphi >= \int \mu_\varepsilon(\xi)\varphi(\xi) d\xi \to < \mu, \varphi >, \quad \text{for } \varphi \in C_c(\mathbb{R})\,, \tag{5.8}$$

and that except for the trivial cases $v_+ = v(0)$ or $v_- = v(0)$ it is $\operatorname{supp} \mu = \operatorname{supp} \lambda = \operatorname{supp} \nu$. Along the convergent sequence $(u_\varepsilon, v_\varepsilon) \to (u, v)$, the functions c_ε, g_ε converge pointwise for $\xi \neq 0$

$$c_\varepsilon(\xi) \longrightarrow c(\xi) := \frac{1}{\xi\, k(u(\xi))} \Big(\xi - \lambda_1(u(\xi))\Big) \Big(\xi - \lambda_2(u(\xi))\Big) \tag{5.9}$$

$$g_\varepsilon(\xi) \longrightarrow g(\xi) := \begin{cases} \displaystyle\int_{\alpha_+}^{\xi} c(\zeta) d\zeta\,, & \xi > 0 \\ \displaystyle\int_{\alpha_-}^{\xi} c(\zeta) d\zeta\,, & \xi < 0\,. \end{cases} \tag{5.10}$$

We show that points in the support of μ are global minima for the function g.

Proposition 5.1. *Let $\xi \in \operatorname{supp} \mu$.*

(i) *If $\xi \in \operatorname{supp} \mu \cap (-\infty, 0)$ then $g(\zeta) \geq g(\xi)$ for $\zeta \in (-\infty, 0)$.*

(ii) *If $\xi \in \operatorname{supp} \mu \cap (0, \infty)$ then $g(\zeta) \geq g(\xi)$ for $\zeta \in (0, \infty)$.*

Proof. We show (ii); the proof of (i) is similar.

Consider the sequence $\{g_\varepsilon\}$ defined in (5.2). In view of (5.1) and (2.35) on any compact $[a,b] \subset (0,\infty)$ $\{g_\varepsilon\}$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies a subsequence $\{g_{\varepsilon'}\}$ converges uniformly. Since the whole sequence converges pointwise to g, we conclude

$$g_\varepsilon \to g \quad \text{uniformly on any} \quad [a,b] \subset (0,\infty)\,. \tag{5.11}$$

Suppose now that for some $\xi \in (0,\infty)$ the conclusion of (ii) is violated. Since g is continuous on $(0,\infty)$, there exists some $\alpha > 0$ such that the set

$$\mathcal{A} = \{\zeta \in (0,\infty) \ : \ g(\zeta) - g(\xi) < -\alpha < 0\} \tag{5.12}$$

has positive Lebesgue measure $m(\mathcal{A})$. We will show:

Claim. If $m(\mathcal{A}) > 0$ there exists an open interval $I \ni \xi$ such that $<\mu,\varphi> = 0$ for any $\varphi \in C_c(I)$.

The claim implies $\xi \notin \operatorname{supp}\mu$ and proves (ii).

It remains to establish the claim. In view of (5.1), (4.2) and (2.12) there are positive constants m_1, m_2 (independent of ε) such that

$$\begin{aligned} c_\varepsilon(s) &\le -\frac{m_1}{s} &&\text{for } \ 0 < s < \frac{\lambda_+}{2} \\ c_\varepsilon(s) &\ge m_2\,(s - \Lambda_+) &&\text{for } \ 2\Lambda_+ < s < 0\,. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\begin{aligned} c(s) &\le -\frac{m_1}{s} &&\text{for } \ 0 < s < \frac{\lambda_+}{2} \\ c(s) &\ge m_2\,(s - \Lambda_+) &&\text{for } \ 2\Lambda_+ < s < 0 \end{aligned}$$

and, hence, for $0 < \zeta < \frac{\lambda_+}{2}$

$$\int_{\frac{\lambda_+}{2}}^{\zeta} c(s)ds \ge -m_1\,\ell n\frac{2\zeta}{\lambda_+} \to +\infty, \quad \text{as} \quad \zeta \to 0+$$

while for $2\Lambda_+ < \zeta$

$$\int_{2\Lambda_+}^{\zeta} c(s)ds \ge \frac{m_2}{2}\left((\zeta - \Lambda_+)^2 - {\Lambda_+}^2\right) \to +\infty, \quad \text{as} \quad \zeta \to +\infty\,.$$

It follows that points ζ near the boundary of $(0,\infty)$ violate the inequality in (5.12) and thus $\mathcal{A}$ is contained in some compact interval $[c,d] \subset (0,\infty)$.

Fix now $\delta > 0$ such that

$$\xi' \in (\xi-\delta, \xi+\delta) \ \text{ implies } \ |g(\xi) - g(\xi')| < \frac{\alpha}{6} \, .$$

Since $\mathcal{A} \subset [c,d]$ and because of (5.11), we can choose $\varepsilon_0 > 0$ such that

$$\xi' \in (\xi-\delta, \xi+\delta),\ \varepsilon < \varepsilon_0,\ \zeta \in \mathcal{A} \quad \text{imply} \quad g^\varepsilon(\zeta) - g^\varepsilon(\xi') < -\frac{\alpha}{2} \, .$$

Now (5.7) and (2.12) yield for $\xi' \in I := (\xi-\delta, \xi+\delta)$

$$\begin{aligned} 0 < \mu_\varepsilon(\xi') &= \frac{1}{\int_0^\infty \frac{1}{k(u_\varepsilon(\zeta))} \exp\{-\frac{1}{\varepsilon}\big(g_\varepsilon(\zeta) - g_\varepsilon(\xi')\big)\}\, d\zeta} \\ &\leq \frac{1}{\frac{1}{K_0} \int_{\mathcal{A}} \exp\{-\frac{1}{\varepsilon}\big(g_\varepsilon(\zeta) - g_\varepsilon(\xi')\big)\}\, d\zeta} \\ &\leq \frac{K_0}{m(\mathcal{A})} e^{-\frac{\alpha}{2\varepsilon}} \, . \end{aligned} \tag{5.13}$$

Let $\varphi \in C_c(I)$. Then (5.8) and (5.13) imply

$$<\mu_\varepsilon, \varphi> = \int_{(\xi-\delta,\xi+\delta)} \mu_\varepsilon(\xi')\varphi(\xi')d\xi' \to 0 \, , \quad \text{as} \ \ \varepsilon \to 0 \, .$$

Hence, $<\mu, \varphi> = 0$ for $\varphi \in C_c(I)$, and the proof of the claim is complete. ▮

The minimization properties for g yield information on the structure of (u,v). In particular, a weak form of the Lax shock conditions is induced at points of discontinuity. Recall that $\lambda_1(u) < 0 < \lambda_2(u)$ are the two wave speeds of the hyperbolic system (E).

Corollary 5.2. *Let $\xi,\ \xi' \in \mathbb{R}$ with $\xi < \xi'$.*

(a) *If $\xi \in \mathcal{C} \cap \operatorname{supp}\mu$ then*

$$\begin{aligned} \xi &= \lambda_1(u(\xi)) \quad \textit{for} \quad \xi < 0 \, , \\ \xi &= \lambda_2(u(\xi)) \quad \textit{for} \quad \xi > 0 \, . \end{aligned} \tag{5.14}$$

(b) *If $\xi \in \mathcal{S}$ then (u,v) satisfies at ξ the jump conditions (4.13) and the inequalities*

$$\begin{aligned} \lambda_1(u(\xi+)) \leq \xi \leq \lambda_1(u(\xi-)) \quad \textit{for} \quad \xi < 0 \, , \\ \lambda_2(u(\xi+)) \leq \xi \leq \lambda_2(u(\xi-)) \quad \textit{for} \quad \xi > 0 \, . \end{aligned} \tag{5.15}$$

(c) *If* $\xi,\ \xi' \in \operatorname{supp}\mu \cap (-\infty, 0)$ *then* $\lambda_1(u(\xi+)) = \xi,\ \lambda_1(u(\xi'-)) = \xi'$ *and for* $\theta \in (\xi, \xi')$

$$\begin{aligned} \theta = \lambda_1(u(\theta)) \quad & \textit{if} \quad \theta \in \mathcal{C}\ , \\ \lambda_1(u(\theta+)) = \theta = \lambda_1(u(\theta-)) \quad & \textit{if} \quad \theta \in \mathcal{S}\ . \end{aligned} \tag{5.16}$$

If $\xi,\ \xi' \in \operatorname{supp}\mu \cap (0, \infty)$ *then* $\lambda_2(u(\xi+)) = \xi,\ \lambda_2(u(\xi'-)) = \xi'$ *and for* $\theta \in (\xi, \xi')$

$$\begin{aligned} \theta = \lambda_2(u(\theta)) \qquad & \text{if} \qquad \theta \in \mathcal{C}\ , \\ \lambda_2(u(\theta+)) = \theta = \lambda_2(u(\theta-)) \qquad & \text{if} \qquad \theta \in \mathcal{S}\ . \end{aligned} \tag{5.17}$$

Proof. We present the proof for $\xi \in (0, \infty)$. The function g in (5.10) is continuous and satisfies $g(\xi) \to +\infty$ as $\xi \to 0+$ or $\xi \to +\infty$. Since c is of bounded variation

$$\lim_{\zeta \to \xi\pm} \frac{g(\zeta) - g(\xi)}{\zeta - \xi} = \lim_{\zeta \to \xi\pm} \frac{1}{\zeta - \xi} \int_\xi^\zeta c(s)ds = c(\xi\pm)\ , \tag{5.18}$$

that is $\frac{dg}{d\xi}$ exists and is continuous at points of $\mathcal{C}$, while only the right and left derivatives exist at points of $\mathcal{S}$. Let $\xi \in \operatorname{supp}\mu \cap (0, \infty)$, then

$$g(\zeta) \geq g(\xi) \qquad \text{for} \qquad 0 < \zeta < \infty$$

and thus

$$c(\xi+) \geq 0 \quad , \quad c(\xi-) \leq 0\ .$$

In turn, (5.9) and the separation properties of the wave speeds imply for $\xi > 0$

$$\xi - \lambda_2(u(\xi+)) \geq 0 \quad , \quad \xi - \lambda_2(u(\xi-)) \leq 0\ , \tag{5.19}$$

which leads to (5.14) for $\xi \in \mathcal{C}$ and to (5.15) for $\xi \in \mathcal{S}$.

It remains to show (c). Let $\xi,\ \xi' \in \operatorname{supp}\mu \cap (0, \infty)$ with $\xi < \xi'$. Then $\xi,\ \xi'$ are both global minima for g with $g(\xi) = g(\xi')$. We claim:

$$g(\theta) = g(\xi) \ \text{ for any } \ \theta \in (\xi, \xi')\ . \tag{5.20}$$

If (5.20) is violated at some point, there exist $a,\ b$ with $\xi \leq a < b \leq \xi'$ such that

$$g(a) = g(b) = g(\xi) \quad , \quad g(\theta) > g(\xi) \quad \text{for} \quad a < \theta < b\ .$$

At the points $a,\ b$ we have

$$\lambda_2(u(a+)) \le a \le \lambda_2(u(a-))$$
$$\lambda_2(u(b+)) \le b \le \lambda_2(u(b-))\ .$$

On the other hand it follows from Proposition 5.1 that the function $(u(\xi), v(\xi))$ remains constant on the interval (a, b) and thus $\lambda_2(u(a+)) = \lambda_2(u(b-))$. The inequalities then imply $b \le a$ which contradicts $a < b$; hence (5.20) follows. ∎

In summary, the region where (u, v) is nonconstant consists of two disjoint closed intervals: $I_{\lambda_1} = [a_1, b_1] \subset (-\infty, 0)$ associated with the negative characteristic speed $\lambda_1(u)$ and $I_{\lambda_2} = [a_2, b_2] \subset (0, \infty)$ associated with the positive characteristic speed $\lambda_2(u)$. Each of I_{λ_1} or I_{λ_2} could be empty or consist of just a single point. The function (u, v) takes constant values on the complement of $I_{\lambda_1} \cup I_{\lambda_2}$ and has the properties listed in Corollary 5.2 at points of I_{λ_1} or I_{λ_2}.

Let $l_1(u, v),\ l_2(u, v)$ be the left eigenvectors and $r_1(u, v),\ r_2(u, v)$ the right eigenvectors associated with the eigenvalues $\lambda_1(u),\ \lambda_2(u)$, respectively, for the hyperbolic system (E). They are normalized to satisfy

$$l_i(u, v) \cdot r_j(u, v) = \delta_{ij}\ .$$

The behavior of the wave speeds described in Corollary 5.2 together with a proposition from [Tz, Proposition 4.6] give:

Proposition 5.3. *Suppose that $I_{\lambda_k} = [a_k, b_k]$ is a full interval, $a_k < b_k$, for $k = 1$ or 2.*

(i) *For each $\xi \in [a_k, b_k)$ such that $\nabla\lambda_k(u(\xi+)) \cdot r_k(u(\xi+), v(\xi+)) \neq 0$*

$$\begin{aligned}\lim_{h\to 0, h>0} \frac{1}{h}\left[\begin{pmatrix} u \\ v \end{pmatrix}(\xi + h-) - \begin{pmatrix} u \\ v \end{pmatrix}(\xi+)\right] \\ = \frac{1}{\nabla\lambda_k(u(\xi+)) \cdot r_k(u(\xi+), v(\xi+))}\ r_k(u(\xi+), v(\xi+))\ .\end{aligned} \tag{5.21}$$

(ii) *For each $\xi \in (a_k, b_k]$ such that $\nabla\lambda_k(u(\xi-)) \cdot r_k(u(\xi-), v(\xi-)) \neq 0$*

$$\begin{aligned}\lim_{h\to 0, h<0} \frac{1}{h}\left[\begin{pmatrix} u \\ v \end{pmatrix}(\xi + h+) - \begin{pmatrix} u \\ v \end{pmatrix}(\xi-)\right] \\ = \frac{1}{\nabla\lambda_k(u(\xi-)) \cdot r_k(u(\xi-), v(\xi-))}\ r_k(u(\xi-), v(\xi-))\ .\end{aligned} \tag{5.22}$$

As a consequence, (u, v) has right and left derivatives at any point ξ that is not an accumulation point of $\mathcal{S}$. If such a point $\xi \in \mathcal{C}$ then (u, v) is Lipshitz there, and if, in addition, it is an interior point of I_{λ_k} then (u, v) is differentiable there. Moreover, a complete description of the structure of the two wave fans is obtained. We distinguish the following cases:

(i) If I_{λ_k} consists of a single point then the solution is a shock wave satisfying the weak form of the Lax shock conditions.

(ii) If I_{λ_k} is a full interval of points in $\mathcal{C}$ the solution is a k-rarefaction wave (provided that $\nabla \lambda_k \cdot r_k \neq 0$ on I_{λ_k} which is anyway necessary for rarefactions).

(iii) In general I_{λ_k} consists of an alternating sequence of k-rarefactions and shocks such that each shock adjacent to a rarefaction from one side is a contact discontinuity on that side.

The emerging picture is that typical of strictly hyperbolic but not genuinely nonlinear systems. In the absence of genuine nonlinearity, the Lax shock conditions are not sufficient to single out the admissible shocks and have to be strengthened (Wendroff [W], Liu [L]). In our setup, additional restrictions result from the analysis in the following section.

6. SELF-SIMILAR VISCOUS LIMITS AND TRAVELING WAVES

In this section we discuss the relation between self-similar viscous limits and traveling waves in the context of the elasticity equations (E). Let ξ be a point of discontinuity of (u, v) and note that $(u(\xi-), v(\xi-))$, $(u(\xi+), v(\xi+))$ satisfy the Rankine-Hugoniot conditions (4.13). Consider a sequence of points $\{\xi_\varepsilon\}$ with the property $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$. Define functions

$$U_\varepsilon(\zeta) = u_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta), \qquad V_\varepsilon(\zeta) = v_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta), \qquad -\infty < \zeta < \infty\,; \tag{6.1}$$

this introduces a stretching of the independent variable centered around the point ξ_ε, a shifted version of ξ. The uniform estimates (2.35), (2.36) imply that $(U_\varepsilon, V_\varepsilon)$ are uniformly bounded and that

$$\begin{aligned} TV_\zeta\ U_\varepsilon(\cdot) &= TV_\zeta\ u_\varepsilon(\xi_\varepsilon + \varepsilon\ \cdot) = TV_\xi\ u_\varepsilon(\cdot) \le C\ , \\ TV_\zeta\ V_\varepsilon(\cdot) &= TV_\zeta\ v_\varepsilon(\xi_\varepsilon + \varepsilon\ \cdot) = TV_\xi\ v_\varepsilon(\cdot) \le C\ . \end{aligned} \tag{6.2}$$

Using Helly's theorem and a diagonal argument we establish the existence of a subsequence and a function (U, V) such that

$$u_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \to U(\zeta), \quad v_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) \to V(\zeta), \quad \text{pointwise for } -\infty < \zeta < \infty\,. \tag{6.3}$$

Proposition 6.1. *Let $\xi \in \mathcal{S}$ and suppose that $\{\xi_\varepsilon\}$ is a sequence of points with $\xi_\varepsilon \to \xi$. Then the function $\big(U(\zeta), V(\zeta)\big)$ defined in (6.3) is continuously differentiable and satisfies on $(-\infty, \infty)$ the traveling wave equations*

$$\begin{aligned} -\xi\,[U - u(\xi-)] - [V - v(\xi-)] &= 0 \\ -\xi\,[V - v(\xi-)] - [\sigma(U) - \sigma(u(\xi-))] &= k(U)\,\frac{dV}{d\zeta} \end{aligned} \tag{6.4}$$

with initial conditions

$$U(0) = \lim_{\varepsilon\to 0} u_\varepsilon(\xi_\varepsilon) \quad , \quad V(0) = \lim_{\varepsilon\to 0} v_\varepsilon(\xi_\varepsilon)\,. \tag{6.5}$$

The limits $\lim\limits_{\zeta\to\pm\infty} U(\zeta) =: U_\pm$, $\lim\limits_{\zeta\to\pm\infty} V(\zeta) =: V_\pm$ exist, are finite, and (U_-, V_-), (U_+, V_+) satisfy the algebraic equations

$$\begin{aligned} -\xi\,[U - u(\xi-)] - [V - v(\xi-)] &= 0 \\ -\xi\,[V - v(\xi-)] - [\sigma(U) - \sigma(u(\xi-))] &= 0 \end{aligned} \tag{6.6}$$

Proof. The function $(u_\varepsilon, v_\varepsilon)$ satisfies the weak form (2.5), (2.6). First evaluate (2.5) between the points $\xi_\varepsilon + \varepsilon\zeta$ and θ to obtain

$$(\xi_\varepsilon + \varepsilon\zeta)\,u_\varepsilon(\xi_\varepsilon + \varepsilon\zeta) + v_\varepsilon(\xi_\varepsilon + \varepsilon\zeta) - \theta\,u_\varepsilon(\theta) - v_\varepsilon(\theta) - \int_\theta^{\xi_\varepsilon+\varepsilon\zeta} u_\varepsilon(\tau)d\tau = 0\,. \tag{6.7}$$

In (6.7) let first $\varepsilon \to 0$ and then let consecutively $\theta \to \xi+$ and $\theta \to \xi-$. Using (6.3), (4.1) and (2.35), we arrive at

$$\xi\Big(U(\zeta) - u(\xi\pm)\Big) + \Big(V(\zeta) - v(\xi\pm)\Big) = 0\,. \tag{6.8}$$

Next, we evaluate (2.6) at the points $\xi_\varepsilon + \varepsilon\,\zeta$, θ and integrate the resulting equation in θ between ξ and $\xi + \delta$, for some δ, to arrive at

$$\begin{aligned} (\xi_\varepsilon + \varepsilon\,\zeta)v_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta) + \sigma(u_\varepsilon(\xi_\varepsilon + \varepsilon\,\zeta)) - \frac{1}{\delta}\int_\xi^{\xi+\delta} \theta v_\varepsilon(\theta) + \sigma(u_\varepsilon(\theta))\,d\theta \\ -\frac{1}{\delta}\int_\xi^{\xi+\delta}\int_\theta^{\xi_\varepsilon+\varepsilon\zeta} v_\varepsilon(\tau)\,d\tau d\theta = -\,\varepsilon k(u_\varepsilon(\xi_\varepsilon + \varepsilon\zeta))\,v_\varepsilon'(\xi_\varepsilon + \varepsilon\zeta) \\ + \varepsilon\frac{1}{\delta}\int_\xi^{\xi+\delta} k(u_\varepsilon(\theta))v_\varepsilon'(\theta)d\theta\,. \end{aligned}$$

After some algebraic manipulations and an integration in ζ we get

$$\begin{aligned}
&\int_0^\zeta \frac{(\xi_\varepsilon + \varepsilon s)\, v_\varepsilon(\xi_\varepsilon + \varepsilon s) + \sigma(u_\varepsilon(\xi_\varepsilon + \varepsilon s))}{k(u_\varepsilon(\xi_\varepsilon + \varepsilon s))} ds \\
&\quad - \Big(\frac{1}{\delta}\int_\xi^{\xi+\delta} \theta v_\varepsilon(\theta) + \sigma(u_\varepsilon(\theta))\, d\theta\Big) \Big(\int_0^\zeta \frac{ds}{k(u_\varepsilon(\xi_\varepsilon + \varepsilon s))}\Big) \\
&\quad - \frac{1}{\delta}\int_0^\zeta \frac{1}{k(u_\varepsilon(\xi_\varepsilon + \varepsilon s))} \int_\xi^{\xi+\delta} \int_\theta^{\xi_\varepsilon + \varepsilon s} v_\varepsilon(\tau)\, d\tau d\theta ds \\
&\quad = v_\varepsilon(\xi_\varepsilon + \varepsilon\zeta) - v_\varepsilon(\xi_\varepsilon) - \varepsilon\Big(\frac{1}{\delta}\int_\xi^{\xi+\delta} k(u_\varepsilon(\theta))\, v_\varepsilon'(\theta)\, d\theta\Big) \Big(\int_0^\zeta \frac{ds}{k(u_\varepsilon(\xi_\varepsilon + \varepsilon s))}\Big) .
\end{aligned}$$

Letting $\varepsilon \to 0$ and using (6.3), (4.1), (2.35), (4.14) and (2.12), we deduce

$$\begin{aligned}
&\int_0^\zeta \frac{\xi\, V(s) + \sigma(U(s))}{k(U(s))} ds - \Big(\frac{1}{\delta}\int_\xi^{\xi+\delta} \theta\, v(\theta) + \sigma(u(\theta)) d\theta\Big) \Big(\int_0^\zeta \frac{ds}{k(U(s))}\Big) \\
&\quad - \Big(\int_0^\zeta \frac{ds}{k(U(s))}\Big) \Big(\frac{1}{\delta}\int_\xi^{\xi+\delta} \int_\theta^\xi v(\tau) d\tau d\theta\Big) = -V(\zeta) + V(0) .
\end{aligned} \tag{6.9}$$

From (6.9), by letting consecutively $\delta \to 0+$ and $\delta \to 0-$, we obtain

$$\int_0^\zeta \frac{1}{k(U(s))} \Big(\xi\big(V(s) - v(\xi\pm)\big) + \sigma(U(s)) - \sigma(u(\xi\pm))\Big) ds = -V(\zeta) + V(0) . \tag{6.10}$$

It follows from (6.8), (6.10) that $(U(\zeta), V(\zeta))$ are continuously differentiable functions that satisfy the equations (6.4) and the initial conditions (6.5). Furthermore, the function $U(\zeta)$ satisfies the differential equation

$$\xi\, k(U)\, \frac{dU}{d\zeta} = [\,\sigma(U) - \sigma(u(\xi-))\,] - \xi^2\, [\,U - u(\xi-)\,] \tag{6.11}$$

Since U, V are of bounded variation on $\mathbb{R}$, the limits $\lim_{\zeta \to \pm\infty} \big(U(\zeta),\, V(\zeta)\big) =: (U_\pm,\, V_\pm)$ exist and are finite. It is then straightforward to show that U_+, U_- are equilibrium states of (6.11)

$$[\,\sigma(U_\pm) - \sigma(u(\xi-))\,] - \xi^2\, [\,U_\pm - u(\xi-)\,] = 0\,. \tag{6.12}$$

In turn, (6.8) implies that $(U_-,\, V_-)$, $(U_+,\, V_+)$ solve the algebraic equations (6.6). ∎

The function $(U,\, V)$, as well as the limiting values $U_\pm$, $V_\pm$, depends on the choice of the sequence ξ_ε. For certain choices of $\{\xi_\varepsilon\}$ it may happen that $(U_-,\, V_-) = (U_+,\, V_+)$ and the traveling wave disintegrates to a constant solution. However, whenever a choice of shifts produces a nonconstant solution $(U,\, V)$ this will be a heteroclinic orbit for (6.4) connecting two distinct equilibria $(U_-,\, V_-)$ and $(U_+,\, V_+)$ solving (6.6). In the latter case $U(\zeta)$ is a solution of (6.11) connecting U_- to U_+, what imposes restrictions on all intermediate states U. If, for instance, $U_- < U_+$ and $\xi > 0$ then

$$[\,\sigma(U) - \sigma(u(\xi-))\,] - \xi^2\,[\,U - u(\xi-)\,] > 0\,, \quad \text{for } U \in (U_-,\, U_+)\,. \tag{6.13}$$

We proceed to investigate the relation that $\{(u_\varepsilon, v_\varepsilon)\}$ bears to the corresponding limit (u, v) at a point $\xi \in \mathcal{S}$. Issues of interest are to understand what types of discontinuities are admissible and under what circumstances it is

$$(U_-,\, V_-) = \big(u(\xi-),\, v(\xi-)\big)\,, \quad (U_+,\, V_+) = \big(u(\xi+),\, v(\xi+)\big)\,. \tag{6.14}$$

We analyze a concrete case: $\xi > 0$, $u(\xi-) < u(\xi+)$ and, in accordance with the Rankine-Hugoniot conditions, $v(\xi-) > v(\xi+)$; other cases can be treated in a similar fashion.

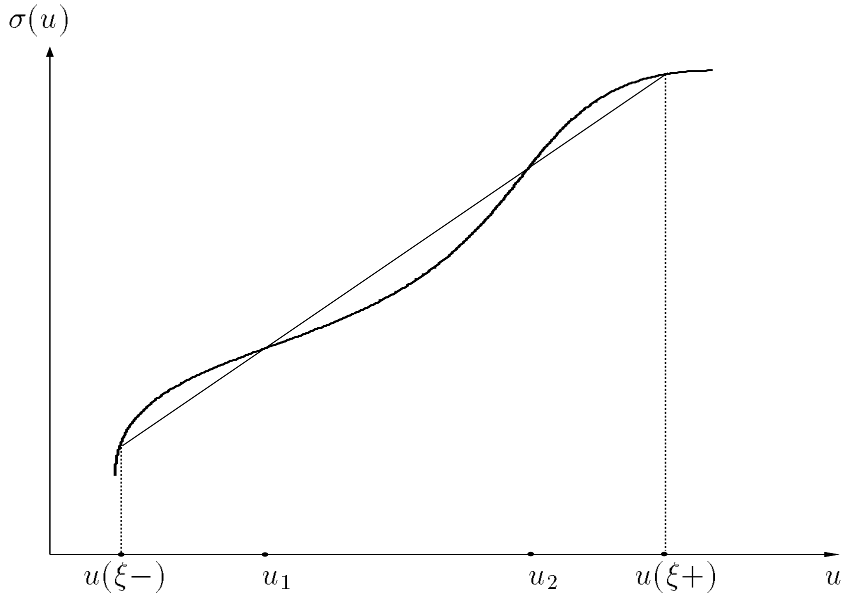

Figure 1

In the case under study (4.13) and (5.15) take the form

$$\xi^2 = \frac{\sigma(u(\xi+)) - \sigma(u(\xi-))}{u(\xi+) - u(\xi-)} \tag{6.15}$$

$$\sigma_u(u(\xi+)) \leq \xi^2 \leq \sigma_u(u(\xi-)) \tag{6.16}$$

They impose restrictions on the placement of the graph of $\sigma(u)$ relative to the chord connecting the points with coordinates $(u(\xi-), \sigma(u(\xi-)))$, $(u(\xi+), \sigma(u(\xi+)))$. As a case study consider a configuration where the graph cuts the chord at exactly two intermediate points u_1 and u_2 (see Fig. 1). Then the set of states (U, V) solving (6.6) with $u(\xi-) \leq U \leq u(\xi+)$ consists of $(u(\xi-), v(\xi-))$, (u_1, v_1), (u_2, v_2), $(u(\xi+), v(\xi+))$ with v_1, v_2 defined by

$$v_i = v(\xi-) - \xi\,(u_i - u(\xi-))\,, \quad i = 1,\, 2\,.$$

We will show that such a discontinuity, although compatible with (6.15) and the Lax conditions (6.16), can not appear as an $\varepsilon \to 0$ limit of solutions $(u_\varepsilon, v_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$.

For $\xi > 0$, $u(\xi-) < u(\xi+)$, the restrictions on the shapes of solutions dictate that u_ε is increasing and v_ε is decreasing on $(0, \infty)$. Fix states u_{01}, u_{02} and u_{03} in the intervals $((u(\xi-), u_1)$, (u_1, u_2) and $(u_2, u(\xi+))$, respectively, and define three sequences of points $\{\xi_{i,\varepsilon}\}$ such that

$$u(\xi_{i,\varepsilon}) = u_{0i}\,, \quad i = 1,\, 2,\, 3.$$

According to Proposition 6.1, the corresponding traveling waves U_i, defined by (6.3), satisfy (6.11) with initial data $U_i(0) = u_{0i}$. Therefore U_1 connects $u(\xi-)$ to u_1, U_2 connects u_1 to u_2, while U_3 connects u_2 to $u(\xi+)$. Note that $U_2(\zeta)$ violates (6.13) and thus such a traveling wave cannot appear, except in the case u_1 and u_2 coincide.

We conclude that for a shock at $\xi > 0$ to arise as an $\varepsilon \to 0$ limit of solutions $(u_\varepsilon, v_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$, it must satisfy

$$[\,\sigma(U) - \sigma(u(\xi-))] - \xi^2\,[U - u(\xi-)] \geq 0\,, \qquad U \in (u(\xi-),\, u(\xi+))\,, \tag{6.17}$$

that is the graph of $\sigma(u)$ lies above the chord joining the end states. If (6.17) holds as a strict inequality there is a choice of $\{\xi_\varepsilon\}$ producing a single shock profile connecting $u(\xi-)$ to $u(\xi+)$. If (6.17) holds as an equality at a finite number of points u_j, $j = 1, ..., m$, there will be a chain of traveling waves connecting $u(\xi-)$ to u_1, each of the points u_j to the next, and u_m to $u(\xi+)$. Finally, if $(6,17)$ holds as an equality on an interval, there is no shock profile associated with the part of the solution taking values on the interval, but there are shock profiles associated with the complementary part as before.

The same argument shows that when $\xi < 0$ the inequality in (6.17) is reversed, which geometrically means that the graph of $\sigma(u)$ must now lie below the chord. These conditions are sufficient for solving the Riemann problem by patching together elementary solutions, in the class of rerefaction waves, shocks and contact discontinuities [W, L].

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