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EQUATIONS AUX DERIVEES PARTIELLES. Systèmes avec relaxation contractifs et système de particules avec interaction pour les lois de conservation scalaires,

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Résumé. Nous considérons une classe de systèmes hyperboliques semilinéaires avec relaxation, qui sont contractifs en norme L^1 et qui possédent des régions invariantes. Nous démontrons que, quand le paramètre de relaxation ε tend vers zéro, les solutions du système convergent vers une solution faible d'une loi de conservation, qui satisfait aux conditions d'entropie de Kruzhkov. Dans le cas de la dimension un d'espace, nous proposons des systèmes des particules avec interaction qui, à l' échelle mésoscopique, convergent vers les systèmes avec relaxation; leur dynamique macroscopique est décrite par les solutions entropiques d'une loi de conservation.

PARTIAL DIFFERENTIAL EQUATIONS. Contractive relaxation systems and interacting particles for scalar conservation laws.

Abstract. We consider a class of semilinear hyperbolic systems with relaxation that are contractive in the L^1-norm and admit invariant regions. We show that, as the relaxation parameter ε goes to zero, their solutions converge to a weak solution of the scalar multidimensional conservation law that satisfies the Kruzhkov entropy conditions. In the case of one space dimension, we propose certain interacting particle systems, whose mesoscopic limit is the systems with relaxation and their macroscopic dynamics is described by entropy solutions of a scalar conservation law.

Version Abrégée en Français.

Le problème de Cauchy pour la loi de conservation scalaire et multidimensionelle

$$(1)\qquad \begin{cases} \partial_t u + \sum_{i=1}^n \partial_{x_i} F^i(u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \end{cases}$$

a une solution faible unique qui satisfait les conditions d' entropie de Kruzhkov [7] :

$$(2)\qquad \partial_t |u-k| + \sum_{i=1}^n \partial_{x_i} \left[(F^i(u) - F^i(k))\text{sgn } (u-k) \right] \le 0, \quad \text{dans } \mathcal{D}', \text{ pour } k \in \mathbb{R}.$$

Nous introduisons un système hyperbolique avec relaxation dont la solution converge vers la solution de (1)-(2), quand le paramètre de relaxation ε tend vers 0:

$$(3)\qquad \begin{cases} \partial_t w + \sum_{j=1}^n \omega_j V_j \partial_{x_j} w = \frac{1}{\epsilon} \sum_{j=1}^n (g^j(w) - z^j) \\ \partial_t z^i - V_i \partial_{x_i} z^i = \frac{1}{\epsilon}(g^i(w) - z^i)\,, & i = 1, 2, ..., n, \end{cases}$$

où les fonctions $g^i(w)$ sont décroissantes et $g^i(0) = 0$, $i = 1, ..., n$. Les solutions de système (3) vérifient la loi de conservation

$$(4)\qquad \partial_t (w - \sum_{i=1}^n z^i) + \sum_{i=1}^n \partial_{x_i} V_i(\omega_i w + z^i) = 0\,.$$

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Quand $\varepsilon \to 0$, on s'attend à ce que les solutions de (3) convergent vers les solutions du problème relaxé (4) avec $z^i = g^i(w)$. Ce problème peut être mis sous la forme d'une loi de conservation scalaire si, étant donné $(u, F^1(u), ..., F^n(u))$, on peut trouver des fonctions décroissantes $g^i(w)$ qui satisfont aux relations

$$w - \sum_{i=1}^{n} g^i(w) = u\,, \qquad \omega_i w + g^i(w) = \frac{1}{V_i} F^i(u)\,. \tag{5}$$

Il est toujours possible de faire cette construction si on choisit les paramètres $\omega_i > 0$ et les vitesses $|V_i|$ suffisament grands.

On démontre dans [5] que le système (3) est contractif en norme L^1 et possède des régions invariantes. Quand le paramètre ε de relaxation tend vers 0, et si on choisit les paramètres $\omega_i > 0$ avec $\sum_i \omega_i < 1$ et $|V_i|$ assez grands, les solutions de (3) convergent dans L^1 vers une solution de (1)-(2) avec $u = w - \sum_i z^i$.

Nous introduisons un système stochastique de particules avec interaction. Le modèle est posé sur un *réseau* $\mathbb{Z}$; $\eta(x, \sigma)$ représente le nombre de particules de "vitesse" $\sigma \in \{-1, 1\}$ au point $x \in \mathbb{Z}$. L' évolution en temps se déroule en deux étapes: "transport" et "collisions" des particules:

Transport: La particule occupant le site x et ayant la vitesse σ (notée (x, σ)) se déplace au point $(x + \sigma, \sigma)$ en conservant la même vitesse et en changeant de site d'une seule unité dans la direction σ avec le taux $V\gamma^{-1}\eta(x, \sigma)$, où on s'est donné $V, \gamma > 0$ et $\gamma << 1$.

Collisions: Les particules (x, σ) et $(x, -\sigma)$ entrent en collision et échangent leurs vitesses avec le taux $c(\eta, \sigma, x)$ satisfaisant les conditions suivantes: (i) $c(\eta, \sigma, x)$ est un polynôme en la variable $\eta(x, \sigma)$, (ii) $c(\eta, \sigma, x) \geq 0$ pour tout $\eta \in X$ avec égalité quand $\eta(x, \sigma) = 0$, et (iii) si $\mu_{w,z}$ est un produit de measures de Poisson sur X, tel que $E_{\mu_{w,z}} \eta(x, 1) = w$, $E_{\mu_{w,z}} \eta(x, -1) = -z$, $x \in \mathbb{Z}$, on a

$$c(\eta, -1, x) = \eta(x, -1), \quad E_{\mu_{w,z}} c(\eta, 1, x) = -g(w),$$

où g est un polynôme décroissant donné satisfaisant $g(0) = 0$.

Les deux mécanismes ci-dessus génèrent un processus Markovien des sauts $\eta_. = \{\eta_t(x, \sigma) \in \mathbb{N} : t \geq 0, (x, \sigma) \in \mathbb{Z} \times \{-1, 1\}\}$ dans l'espace des configurations $X = \mathbb{N}^{\mathbb{Z}\times\{-1,1\}}$. Nous obtenons des résultats sur le comportement du système de particules à l'échelle mésoscopique et macroscopique [6]. Après un changement de l'échelle spatiale, on obtient que la moyenne du processus Markovien $\eta_.$ tend vers la solution du système avec relaxation

$$\begin{cases} w_t + V w_x = g(w) - z \\ z_t - V z_x = g(w) - z\,. \end{cases}$$

Dans des échelles de temps plus longues, le système de particules se relaxe vers la solution entropique de la loi de conservation $(w - g(w))_t + (Vw + Vg(w))_x = 0$.

English Version.

1. Introduction. The Cauchy problem for the scalar multidimensional conservation law,

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} \partial_{x_i} F^i(u) = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \end{cases} \tag{1}$$

$$\partial_t |u - k| + \sum_{i=1}^{n} \partial_{x_i} \big[\, (F^i(u) - F^i(k))\mathrm{sgn}\ (u - k) \,\big] \leq 0, \text{ in } \mathcal{D}', \text{ for all } k \in \mathbb{R}, \tag{2}$$

has a unique solution that satisfies the Kruzhkov [7] entropy conditions (2), and its solution operator generates a semigroup of contraction in $L^1(\mathbb{R}^n)$, Crandall [3]. The first objective of this note is to present a class of relaxation systems whose dynamics, as the relaxation parameter $\varepsilon \to 0$, is described by entropy weak solutions of (1). We refer to Kurtz [8], Chen, Levermore and Liu [2], Jin and Xin [4] and Natalini [10] for studies on relaxation approximations of conservation laws.

The model introduced may be interpreted as discrete velocity kinetic equations – the relaxation parameter playing the role of the mean free path – and describes the macroscopic conservation law at a "mesoscopic" level. In that sense it is a discrete velocity analog of the kinetic formulation for (1), introduced by Perthame and Tadmor [12], Lions, Perthame and Tadmor [9]. The second objective is to return to the "microscopic" level and construct a stochastic interacting particle system, which: (a) in mesoscopic scales approximates the relaxation model and (b) in the fluid-dynamic limit converges to the entropy solution of the scalar conservation law. Related interacting particle systems were introduced by Caprino, De Masi, Presutti, and Pulvirenti [1] simulating the Carleman and Broadwell equations, and Perthame and Pulvirenti [11] for the kinetic equations in [12].

2. Contractive relaxation systems and the scalar multidimensional conservation law. Consider the semilinear hyperbolic system

$$
\begin{cases} \partial_t w + \sum_{j=1}^n \omega_j V_j \partial_{x_j} w = \frac{1}{\epsilon} \sum_{j=1}^n (g^j(w) - z^j) \\ \partial_t z^i - V_i \partial_{x_i} z^i = \frac{1}{\epsilon}(g^i(w) - z^i)\,, \qquad\qquad i = 1, 2, ..., n, \end{cases} \tag{3}
$$

in the variables (w, Z), with $Z = (z^1, ..., z^n)$. The convective velocities $a_0 = (\omega_1 V_1, ..., \omega_n V_n)$ and $a_i = (0, ..., V_i, ..., 0)$ are arbitrary for the moment, but restrictions will be placed when carrying the $\varepsilon \to 0$ limit. The functions $g^i(w)$ are strictly decreasing, $i = 1, ..., n$, and $g^i(0) = 0$ (so that $(0,0)$ is an equilibrium). Solutions of (3) satisfy the conservation law

$$
\partial_t (w - \sum_{i=1}^n z^i) + \sum_{i=1}^n \partial_{x_i} V_i (\omega_i w + z^i) = 0\,. \tag{4}
$$

The relaxation system (3) has the following properties [5]:

Theorem 1. *Let g^i be strictly decreasing. If $(w_0, Z_0) \in \left[L^1(\mathbb{R}^n)\right]^{n+1}$ take values in a region $\mathcal{R}_{a,b} := [a,b] \times \prod_{i=1}^n [g^i(b), g^i(a)]$, for some $a < 0 < b$, then there exists a unique globally defined weak solution (w, Z) of (3). Moreover, (3) satisfies the properties:*

(i) If (w, Z) and $(\hat{w}, \hat{Z})$ are two solutions then

$$
\int |w(x,t) - \hat{w}(x,t)| + \sum_{i=1}^n |z^i(x,t) - \hat{z}^i(x,t)|\, dx \le \int |w_0 - \hat{w}_0| + \sum_{i=1}^n |z_0^i - \hat{z}_0^i|\, dx\,. \tag{5}
$$

(ii) For any $\alpha < 0 < \beta$ the sets $\mathcal{R}_{\alpha,\beta} := [\alpha, \beta] \times \prod_{i=1}^n [g^i(\beta), g^i(\alpha)]$ are positively invariant.

(iii) If w_0, $z_0^i \in BV(\mathbb{R}^n)$ then $w(\cdot, t)$, $z^i(\cdot, t) \in BV(\mathbb{R}^n)$.

Next we study the relaxation limit $\varepsilon \to 0$. Formally, we expect that as $\varepsilon \to 0$ solutions of (3) will converge to solutions of (4) with $z^i = g^i(w)$, thus yielding the conservation law

$$\partial_t (w - \sum_{i=1}^{n} g^i(w)) + \sum_{i=1}^{n} \partial_{x_i} V_i(\omega_i w + g^i(w)) = 0\,. \tag{6}$$

The question arises whether the equation (1) can be mapped to the form (6). It turns out that, given the curve $(u, F^1(u), ..., F^n(u))$, if $\omega_i > 0$ and the speeds $|V_i|$ are selected sufficiently large, then there exist functions $g^i(w)$ *strictly decreasing* such that

$$w - \sum_{i=1}^{n} g^i(w) = u\,, \qquad \omega_i w + g^i(w) = \frac{1}{V_i} F^i(u)\,. \tag{7}$$

Let ω_i, V_i be fixed as above and the data $w_{0\varepsilon}, z^i_{0\varepsilon}$ be in a bounded set of $BV(\mathbb{R}^n)$. The following convergence theorem holds.

Theorem 2. *If* $\|u_{0\varepsilon} - u_0\|_{L^1} = o(1)$ *and* $\|g^i(w_{0\varepsilon}) - z^i_{0\varepsilon}\|_{L^1} = O(\varepsilon)$, *then*

$$u_\varepsilon = w_\varepsilon - \sum_{i=1}^{n} z^i_\varepsilon \to u \quad in \quad L^1(\mathbb{R}^n \times \mathbb{R}^+)\,, \tag{8}$$

and u *satisfies (1) and the Kruzhkov entropy conditions (2).*

Remarks : 1. The system (3) is equipped with a class of entropy inequalities,

$$\partial_t \Big[(w-\kappa)^+ + \sum_{i=1}^{n} (z^i - g^i(\kappa))^- \Big] + \sum_{i=1}^{n} \partial_{x_i} \Big[V_i \omega_i (w-\kappa)^+ - V_i (z^i - g^i(\kappa))^- \Big] \le 0\,, \qquad \text{for } \kappa \in \mathbb{R}\,,$$

which, when carried to the limit $\varepsilon \to 0$, yield the Kruzhkov entropy conditions.
2. The estimates of Theorem 1 carry over to discrete, first order, upwind schemes for the one space dimensional version of (3),leading to a convergence result for this case also.

3. Interacting Particle Systems with relaxation. We introduce an Interacting Particle System, which will converge at a mesoscopic scale to a relaxation system (3) such that the functions g^i are polynomials; at macroscopic scales, we show that the particle system relaxes to the solution of the corresponding conservation law (1), (2). We restrict to the one dimensional case, in which case (1) reduces to $u_t + F(u)_x = 0$.

The model is set on the lattice $\mathbb{Z}$; $\eta(x, \sigma)$ denotes the number of particles with "velocity" $\sigma \in \{-1, 1\}$ at each site $x \in \mathbb{Z}$. The updating of the model is done through two mechanisms, "streaming" and "collisions" of particles.
Streaming: A particle at x with velocity σ (denoted by (x, σ)) jumps to $(x + \sigma, \sigma)$, by keeping its velocity and moving by one lattice unit in the direction σ, at a rate $V\gamma^{-1}\eta(x, \sigma)$ with $V, \gamma > 0$ and $\gamma << 1$.
Collisions: Particles (x, σ) and $(x, -\sigma)$ collide and exchange velocities with rate $c(\eta, \sigma, x)$, satisfying the following conditions: (i) $c(\eta, \sigma, x)$ is a polynomial in $\eta(x, \sigma)$, (ii) $c(\eta, \sigma, x) \ge 0$

for all $\eta \in X$ with equality holding if $\eta(x,\sigma) = 0$, and (iii) if $\mu_{w,z}$ is a product Poisson measure on X such that $E_{\mu_{w,z}}\eta(x,1) = w$, $E_{\mu_{w,z}}\eta(x,-1) = -z$, $x \in \mathbb{Z}$, then

$$c(\eta,-1,x) = \eta(x,-1), \quad E_{\mu_{w,z}} c(\eta,1,x) = -g(w)\,,$$

for a given decreasing polynomial g with $g(0) = 0$.

These two mechanisms give rise to a jump Markov process $\eta_{.} = \{\eta_t(x,\sigma) \in \mathbb{N} : t \geq 0, (x,\sigma) \in \mathbb{Z} \times \{-1,1\}\}$ set on the configuration space $X = \mathbb{N}^{\mathbb{Z}\times\{-1,1\}}$ with generator $V\gamma^{-1}L_s + L_c$, where

$$L_s f(\eta) = \sum_{(x,\sigma)} \eta(x,\sigma)[f(\eta + \delta_{(x+\sigma,\sigma)} - \delta_{(x,\sigma)}) - f(\eta)],$$

$$L_c f(n) = \sum_{x,\sigma} c(\eta,\sigma,x)[f(\eta + \delta_{(x,-\sigma)} - \delta_{(x,\sigma)}) - f(\eta)],$$

$f \in C(X;\mathbb{R})$, f cylindrical, and $\delta_{(x,\sigma)}(y,\sigma') = 1$ if $(y,\sigma') = (x,\sigma)$ and equal to 0 otherwise. We refer to Spohn [13] for the construction of such jump processes.

We next present results about the mesoscopic and macroscopic behavior of this particle system [6]. First, after space rescalings only, we obtain that the average of the Markov process $\eta_{.}$ at each lattice site converges to the solution of of the unscaled relaxation system

$$\begin{cases} w_t + V w_x = g(w) - z\,, \\ z_t - V z_x = g(w) - z\,. \end{cases} \tag{9}$$

At longer time scales, we show that the particle system relaxes to the entropy solution of the conservation law (1), associated with the relaxation limit of (9) through inversion of the algebraic system (7). In both cases we also obtain the "propagation of chaos" property. Before stating our results we introduce some notation; we define, for a given constant $\rho_1 > 0$ and $N \in \mathbb{N}$, the sets

$$Q_\gamma = \gamma^{-1}[-\gamma^{-\rho_1}, \gamma^{-\rho_1}] \cap \mathbb{Z} \quad \text{and} \quad \mathcal{M}_\gamma^N = \{\underline{x} = (x_1,\ldots,x_N) : x_1 \neq \cdots \neq x_N \text{ and } x_i \in Q_\gamma\}\,.$$

A. Mesoscopic limit. Let μ_0^γ be a Poisson product measure such that for $w_0, z_0 : \mathbb{R} \to \mathbb{R}$,

$$E_{\mu_0^\gamma}\eta(x,1) = w_0(\gamma x) \quad \text{and} \quad E_{\mu_0^\gamma}\eta(x,-1) = -z_0(\gamma x)\,.$$

Theorem 3 *Let w, z the solutions to (9). If the initial data $w_0, z_0 \in L^\infty(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R})$, then for any $T > 0$*

$$\lim_{\gamma\to 0} \sup_{t\in[0,T]} \sup_{\underline{x}\in\mathcal{M}_\gamma^N} |\, E_{\mu_0^\gamma} \prod_{i=1}^n \eta_t(x_i,\sigma_i) - \prod_{i=1}^n f(\gamma x_i, \sigma_i, t)| = 0\,,$$

where $f(r,1,t) = w(r,t)$, $f(r,-1,t) = -z(r,t)$.

We can prove a similar theorem under weaker conditions on the intial data but the convergence holds only in L^1 (see Theorem 4 below).

B. Macroscopic Limits. We assume that:

(i) μ_0^γ is a Poisson product measure such that

$$E_{\mu_0^\gamma}\eta(x,1) = w_{0\varepsilon}(\gamma\epsilon x)\,, \quad E_{\mu_0^\gamma}\eta(x,-1) = -z_{0\varepsilon}(\gamma\epsilon x)\,.$$

(ii) $w_{0\varepsilon}, z_{0\varepsilon} \in BV(\mathbb{R})$, $\|w_{0\varepsilon}\|_\infty, \|z_{0\varepsilon}\|_\infty < L$ for some constant $L > 0$.

(iii) $\|g(w_{0\varepsilon}) - z_{0\varepsilon}\|_{L^1(\mathbb{R})} = O(\epsilon)$ and $\|u_{0\varepsilon} - u_0\|_{L^1(\mathbb{R})} = o(1)$, where $u_{0\varepsilon} = w_{0\varepsilon} - z_{0\varepsilon}$, and $\epsilon = \epsilon(\gamma) = \gamma^\rho$.

Theorem 4 *Let u be the entropy solution of the conservation law associated to the system (9). Then there are constants $\rho_1 > \rho^*$, such that for all $\rho < \rho^*$, and any fixed N and $T > 0$,*

$$\lim_{\gamma \to 0} \sup_{t \in [0, T/\epsilon]} \sum_{\underline{x} \in \mathcal{M}_\gamma^N} |E_{\mu_0^\gamma} \prod_{i=1}^N (\eta_t(x_i, 1) + \eta_t(x_i, -1)) - \prod_{i=1}^N u(\gamma \epsilon x_i, \epsilon t)| = 0 \, .$$

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