Nonlinear analysis techniques for shear band formation at high strain-rates

Athanasios E. Tzavaras

Department of Mathematics, University of Wisconsin, Madison, WI 57306.

One of the most striking manifestations of instability in solid mechanics is the localization of shear strain into narrow bands during high speed, plastic deformations of metals. According to one theory, the formation of shear bands is attributed to effective strain-softening response, which results at high strain rates as the net outcome of the influence of thermal softening on the, normally, strain-hardening response of metals. Our objective is to review some of the insight obtained by applying nonlinear analysis techniques on simple models of nonlinear partial differential equations simulating this scenario for instability. First, we take up a simple system, intended as a paradigm, that describes isothermal shear deformations of a material exhibiting strain softening and strain-rate sensitivity. As it turns out, for moderate amounts of strain softening strain-rate sensitivity exerts a dissipative effect and stabilizes the motion. However, once a threshold is exceeded, the response becomes unstable and shear strain localization occurs. Next, we present extensions of these results to situations where explicit thermal effects are taken into account.

INTRODUCTION

Shear bands are regions of intensely concentrated shear strain that are observed during the plastic deformation of many materials and often precede rupture. Their occurrence is typically associated with strain softening type response, past a critical strain, of the measured average shear stress versus the measured averaged shear strain. The diversity of materials and deformation conditions associated with observations of shear bands has lead to various theories and corresponding models for explaining their formation. Some type of strain-softening mechanism lies in the core of most theories, however, the origin of softening is often context dependent, hinging on the material under consideration and the geometric and loading circumstances.

In this article, we focus on a theory suggested for explaining the formation of shear bands during high speed, plastic deformations of metals. It was recognized by Zener and Hollomon [34] that, at high speed processes, the effect of the deformation speed is twofold: First, an increase in the deformation speed changes the deformation conditions from isothermal to nearly adiabatic. Second, strain rate has an effect *per se* and needs to be included in the constitutive modeling.

Under isothermal conditions, metals in general strain harden and exhibit a stable response. As the deformation speed increases, the heat produced by the plastic work triggers thermal effects. In particular, thermal-softening properties of metals may outweigh the tendency of the material to harden, so that the combined outcome results to (effective) softening. A destabilizing feedback mechanism is then induced, operating according to the following scenario (Clifton, Duffy, Hartley and Shawki [11]): Nonuniformities in the strain rate result in nonuniform heating. Since the material is softer at the hotter spots and harder at the colder spots, if heat diffusion is too weak to equalize the temperatures, the initial nonuniformities in the strain rate are, in turn, amplified. This mechanism tends to localize the total deformation into narrow regions. On the other hand, there is opposition to this process by "viscous effects" induced by strain-rate sensitivity. The two effects are competing and which one prevails depends on the relative weights of thermal softening, strain hardening and strain-rate sensitivity, as well as the loading circumstances.

Experimental, numerical and linearized analysis studies indicate that, at least when the degree of thermal softening is large, the competition results to instability in the form of strain localization and formation of shear bands (*e.g.* Clifton, Duffy, Hartley and Shawki [11], Drew and Flaherty [17], Wright and Batra [31], Burns [3], Anand, Kim and Shawki [1], Wright and Walter [32]). There is an extensive literature on the subject and the reader is referred to the article by Shawki and Clifton [23] for an excellent survey. In the present article we review recent analytical results on simple constitutive models, where the interplay of various contributing factors on shear strain localization is assessed.

DESCRIPTION OF THE MODEL

Typically, shear bands appear and propagate as one-dimensional structures (up to interaction times). Focusing on their development and evolution, we consider a one-dimensional, simple shearing deformation. In a Cartesian coordinate system, consider an infinite plate of unit thickness located between the planes $x = 0$ and $x = 1$. For a simple shearing motion, the only nonvanishing velocity component corresponds to the shearing direction and all the field variables depend only on the x-coordinate. Neglecting the normal stresses, the thermomechanical process is described by the list of variables:

$$\begin{aligned} \text{velocity}: &\ v(x,t) \\ \text{shear strain}: &\ u(x,t) \\ \text{temperature}: &\ \theta(x,t) \\ \text{heat flux}: &\ q(x,t) \\ \text{shear stress}: &\ \sigma(x,t)\,. \end{aligned}$$

They are connected through the balance of linear momentum

$$v_t = \sigma_x\,, \tag{1}$$

kinematic compatibility

$$u_t = v_x\,, \tag{2}$$

and the balance of energy equations

$$\theta_t = q_x + \sigma u_t\,. \tag{3}$$

Above, the reference density and specific heat have been taken constants (set equal to 1). The following set of constitutive assumptions are used throughout:

$$q = 0\,, \tag{4}$$

$$\sigma = f(\theta, u, u_t)\,, \tag{5}$$

where f is a smooth function with $f(\theta, u, 0) = 0$ and $f_p(\theta, u, p) > 0$, for $p \neq 0$. A discussion of the constitutive assumptions follows, but note that, in terms of classification, the resulting system (1-5) belongs to the framework of one-dimensional thermoviscoelasticity and is compatible with the requirements imposed by the Clausius-Duhem inequality.

It is instructive to present a derivation of $(4-5)$, starting from a constitutive theory appropriate for thermal elastic-viscoplastic materials (see Shawki and Clifton [23] for a detailed derivation starting from a three-dimensional model; also Needleman [22]). In this context, it is assumed that the shear strain u be decomposed, additively, into elastic and plastic components: $u = u^e + u^p$. The elastic component u^e satisfies linear elasticity with shear modulus G, that is $u^e = \frac{1}{G}\sigma$. The evolution of the plastic component is dictated by a plastic flow rule:

$$u^p_t = g(\theta, u^p, \sigma) \quad \text{or} \quad \sigma = f(\theta, u^p, u^p_t)\,, \tag{6}$$

where g is an increasing function in the variable σ, and $f(\theta, u, \cdot)$ is the inverse function of $g(\theta, u, \cdot)$. In summary,

$$\begin{aligned} \frac{1}{G}\sigma + u^p &= u \\ \frac{1}{G}\sigma_t + g(\theta, u^p, \sigma) &= v_x\,. \end{aligned} \tag{7}$$

For the heat flux either a Fourier law, $q = k\theta_x$, is used, or it is asserted that the process is adiabatic, *i.e.* $q = 0$. Imposing adiabatic conditions projects the belief that, at high strain rates, heat diffusion operates at a slower time scale than the one required for the development of a shear band. It appears a plausible assumption for the shear band initiation process, but not necessarily for the evolution of a developed band, due to the high temperature differences involved. Equations (1), (3) and (7), together with a Fourier law for the heat flux, are regarded as modeling the essential features of shear strain localization in thermal elastic-viscoplastic materials. They lead to the model $(1-5)$, by neglecting the elastic effects ($G = \infty$ and $u = u^p$) and working under adiabatic conditions.

Viewing (5) as a plastic flow rule suggests the following terminology: The material exhibits thermal softening at the state variables (θ, u, p) where $f_\theta(\theta, u, p) < 0$, strain hardening where $f_u(\theta, u, p) > 0$, and strain softening where $f_u(\theta, u, p) < 0$. The amounts of the slopes of f in the directions θ, u and p measure the degree of thermal softening, strain hardening (or softening) and strain-rate sensitivity, respectively. The difficulty of performing high strain-rate experiments causes uncertainty as to the specific form of the constitutive relations. As a result various models have been used in the literature. Examples of specific, experimentally fitted, constitutive models can be found in Shawki and Clifton [23]. The following are used for the purpose of illustration in this article:
Empirical power law with parameters $\nu < 0$, k and $n > 0$

$$\sigma = \theta^\nu u^k u_t^n\,. \tag{8}$$

Exponential type of law

$$\sigma = e^{-\beta\theta} u_t^n\,. \tag{9}$$

The parameters ν, k and n in the power law (8) serve as measures of the degree of thermal softening, strain hardening (or softening) and strain-rate sensitivity. The exponential law (9) does not exhibit any strain hardening and the parameters β and n measure the degree of thermal softening and strain-rate sensitivity, respectively.

Isothermal vs. adiabatic deformations

To illustrate the effect of thermal softening on *spatially uniform* deformations, the isothermal and adiabatic cases are contrasted. Consider a theoretical experiment where the plate is subjected to steady shearing. Mathematically, that corresponds to prescribing the boundary velocities, say $v = 0$ at $x = 0$ and $v = 1$ at $x = 1$.
(i) In an *isothermal* deformation the temperature is kept constant, say θ_0, by appropriately removing the produced heat due to the plastic work. The "measured" stress-strain response in this idealized situation coincides with the $\Sigma - U$ graph of $\Sigma = f(\theta_0, U, 1)$. The slope of the graph is measured by $f_u(\theta_0, U, 1)$; hence, for a strain-hardening material Σ is monotonically increasing with U.
(ii) The situation in an *adiabatic* deformation is understood by studying a special class of solutions to $(1-5)$ describing uniform shearing. These are

$$\begin{aligned} \bar{v}(x,t) &= x \\ \bar{u}(x,t) &= U(t) := t + u_0 \\ \bar{\theta}(x,t) &= \Theta(t)\,, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \frac{d\Theta}{dt} &= f(\Theta, t+u_0, 1) \\ \Theta(0) &= \theta_0 \end{aligned} \tag{11}$$

and u_0, θ_0 are positive constants, standing for the initial strain and temperature. Accordingly, the resulting stress is given by

$$\Sigma(t) = f(\Theta(t), t + u_0, 1)\,. \tag{12}$$

The effective stress-strain curve coincides with the $\Sigma - t$ graph, and the material exhibits effective hardening in the increasing parts of the $\Sigma - t$ graph and effective softening in the decreasing parts. Thus, pending on the sign of the quantity $f_\theta f + f_u$, the combined effect of strain hardening and thermal softening can deliver net softening. For instance, consider the case of a strain-hardening ($k > 0$) power law (8). The uniform shearing solutions then read

$$\begin{aligned} U(t) &= t + u_0 \\ \Theta(t) &= \left[\theta_0^{1-\nu} + \frac{1-\nu}{k+1}\Big[(t+u_0)^{k+1} - u_0^{k+1}\Big]\right]^{\frac{1}{1-\nu}} \\ \Sigma(t) &= \Theta^\nu(t)(t+u_0)^k\,, \end{aligned} \tag{13}$$

and a simple computation yields

$$\frac{d\Sigma}{dt} = \Theta(t)^{2\nu-1}(t+u_0)^{2k}\left[\frac{\nu+k}{k+1} + \frac{k}{(t+u_0)^{k+1}}\left[\theta_0^{1-\nu} - \frac{1-\nu}{k+1}u_0^{k+1}\right]\right]. \tag{14}$$

For parameter values ranging in the region $\nu + k < 0$, $\Sigma(t)$ may initially increase but eventually decreases with t.

A hierarchy of models

It is generally maintained that strain softening has a destabilizing influence, tending to amplify small nonuniformities. At the analytical level, this belief is illustrated by the simple model

$$\begin{aligned} v_t &= \tau(u)_x \\ u_t &= v_x \, , \end{aligned} \tag{15}$$

with $\tau'(u) < 0$, describing isothermal motions of a strain-softening, inelastic material. The system (15) is elliptic in the t-direction, and the initial value problem is ill-posed. Nevertheless, it admits the class of uniform shearing solutions $\bar{v}(x,t) = x$, $\bar{u}(x,t) = t + u_0$.

Because of the inherent instability induced by strain softening, it has been postulated that higher order effects, such as strain-rate dependence, are triggered in the course of the motion and need to be accounted for in the constitutive modeling (*e.g.* Needleman [22], Wu and Freund [33]). Strain-rate dependence competes with strain softening, tending to diffuse nonuniformities in the strain-rate and/or the stress, and it may hinder or even altogether suppress instability [26, 28]. First, we present a quantitative analysis of this competition in the context of the isothermal model

$$\begin{aligned} v_t &= (\tau(u) v_x^n)_x \\ u_t &= v_x \, , \end{aligned} \tag{16}$$

with the function $\tau(u)$ satisfying

$$\tau(u) > 0 \quad \text{and} \quad \tau'(u) < 0 \, . \tag{H_0}$$

Note that (16) is a regularization of (15) incorporating rate-dependence effects. Different types of rate-dependent constitutive laws have also been employed in the study of shear bands (e.g. Wu and Freund [33]), as well as strain-gradient dependent constitutive laws (e.g. Coleman and Hodgdon [12], Zbib and Aifantis [35]).

Next, we proceed in the framework of the thermomechanical model $(1-5)$, and review investigations [27, 30] of the model

$$\begin{aligned} v_t &= (\mu(\theta,u) v_x^n)_x \\ u_t &= v_x \\ \theta_t &= \mu(\theta,u) v_x^{n+1} \end{aligned} \tag{17}$$

System (17) arises by substituting in $(1-4)$ the simplified version of the constitutive relation (5)

$$\sigma = \mu(\theta,u) u_t^n \, , \tag{18}$$

where the smooth function $\mu(\theta,u)$ satisfies

$$\mu(\theta,u) > 0 \, , \tag{H_1}$$

rendering $(17)_1$ parabolic, and

$$\mu_\theta(\theta, u) < 0\,, \tag{H_2}$$

determining a thermally softening material. Both strain hardening ($\mu_u(\theta, u) > 0$), strain softening ($\mu_u(\theta, u) < 0$), or combinations thereof are admitted as possibilities. Instead, it is postulated that

$$\theta\mu_\theta(\theta, u) + u\mu_u(\theta, u) \le 0\,. \tag{H_3}$$

This guarantees that, in the course of the motion, thermal softening prevails over strain hardening so that the material eventually exhibits net softening.

When $\mu(\theta, u)$ is independent of θ, the first and second equations in (17) decouple from the third and lead to (16). The system (16), when supplemented with Hypothesis (H_0), may be viewed as an isothermal model where thermal effects are *implicitly* taken into account. That is, the combined outcome of thermal softening and strain hardening, a coupling effected through the neglected energy equation, results to net softening, which is in turn reintroduced as an assumption by imposing (H_0).

Another interesting simplification occurs when $\mu(\theta, u)$ is independent of u, corresponding to materials that do not exhibit strain hardening. In this case the first and third equations in (17) decouple from the middle one and lead to the system

$$\begin{aligned} v_t &= (\mu(\theta)v_x^n)_x \\ \theta_t &= \mu(\theta)v_x^{n+1}\,. \end{aligned} \tag{19}$$

Note that (19) can also be interpreted as describing simple shearing of a non-Newtonian fluid with temperature dependent viscosity. Strain independent constitutive relations were the first to be analytically investigated [15, 5-7, 25, 8, 2], and their study motivated several of the ideas presented here.

Initial and boundary conditions

The governing system of partial differential equations is taken over $(x, t) \in [0, 1] \times \{t > 0\}$. It is supplemented with initial conditions

$$v(x, 0) = v_0(x)\,, \quad u(x, 0) = u_0(x)\,, \quad \theta(x, 0) = \theta_0(x)\,, \tag{20}$$

for $0 \le x \le 1$, and as a consequence

$$\sigma(x, 0) = \sigma_0(x) := \mu(\theta_0(x), u_0(x))v_{0x}^n(x)\,. \tag{21}$$

On occasion, the initial data are used to provide initial nonuniformities in one or more of the field variables in order to monitor their evolution.

Two distinct sets of boundary conditions are considered:

$$\sigma(0, t) = \sigma(1, t) = 1\,, \quad t > 0\,, \tag{22_S}$$

corresponding to prescribed tractions at the boundaries, and

$$v(0, t) = 0\,,\; v(1, t) = 1\,, \quad t > 0\,, \tag{22_V}$$

corresponding to steady shearing of the plate boundaries. The resulting initial-boundary value problems, consisting of (17) (or (16), or (19)), (21) and (22), are denoted by $(\mathcal{P})_S$ when $(22)_S$ are used, and by $(\mathcal{P})_V$ when $(22)_V$ are used.

Preliminary information

For the validity of the model, it is necessary that solutions comply with the sign restrictions $\sigma > 0$, $u_t > 0$, $u > 0$ and $\theta > 0$. Indeed, the constitutive law (16) arises as a simplification of (7) by neglecting the elastic effects; on the other hand, when u_t becomes zero the plastic flow stops and elastic effects become dominant. The last two restrictions are then a matter of a transformation of dependent variables.

An existence and continuation theory that covers the above models is carried out in Ref [28]. It requires smooth initial data, together with hypothesis (H_1), and yields a unique classical solution $(v(x,t), u(x,t), \theta(x,t))$ of $(\mathcal{P})_S$ (or $(\mathcal{P})_V$), satisfying $\sigma > 0$, $u_t > 0$, $u > 0$ and $\theta > 0$. The solution is defined on a maximal time interval of existence $[\,0, T^*)$, which in general could be infinite or finite. An existence theorem of weak solutions for (19) (with $n = 1$) is carried out by Charalambakis and Murat [8].

In the remainder we record information on the behavior of solutions, with the following broad objectives: (a) To assess in a quantitative fashion the interplay of thermal softening, strain hardening and strain-rate sensitivity and their effect on the response of shearing motions. (b) To examine the circumstances that lead to the developement or evolution of nonuniformities in the field variables.

ON THE COMPETITION OF STRAIN SOFTENING AND STRAIN-RATE SENSITIVITY

First, we take up the isothermal model (16), associated with the constitutive relation

$$\sigma = \tau(u)\, u_t^n \tag{23}$$

under hypothesis (H_0).The system (16) may be viewed for small n as a regularization of the elliptic system (15). On the other hand, (16) belongs to what is formally classified as hyperbolic-parabolic systems, what appears in discord with the view as a regularization of an elliptic system. To reconcile the two aspects, recall that (16) admits the uniform shearing solutions

$$\bar{v}(x,t) = x \quad \bar{u}(x,t) = t + u_0 \quad \bar{\sigma}(x,t) = \tau(t + u_0)\,, \tag{24}$$

and consider a perturbation of them

$$\begin{aligned} v(x,t) &= x + V(x,t) \\ u(x,t) &= (t + u_0) + U(x,t)\,. \end{aligned} \tag{25}$$

The linearized equations for (V, U) are easily computed:

$$\begin{aligned} V_t &= n\,\tau(t + u_0)\, V_{xx} + \tau'(t + u_0)\, U_x \\ U_t &= V_x \end{aligned} \tag{26}$$

Under hypothesis (H_0), when $n = 0$, (26) becomes an elliptic system.

Regarding the effect of the boundary conditions, we remark that, for a strain softening material, the amount of external work required to prescribe the tractions is greater than the amount required to prescribe the velocities. It is thus expected that the shearing deformation is more intense in the former case, causing favorable conditions for unstable response. In the course of shearing, under either kind of loading, the material is strained and the diffusion coefficient in the parabolic equation $(16)_1$ is decreasing. If the decrease is too rapid and/or spatially nonuniform, it is conceivable that the diffusion is unable to stabilize the motion, possibly resulting to spatially nonuniform structures.

Both of the initial-boundary value problems $(\mathcal{P})_S$ and $(\mathcal{P})_V$ for the quasilinear system (16) admit classical solutions $(v(x,t), u(x,t))$, defined on a maximal time interval $[0, T^*)$ and subject to the sign restrictions $\sigma > 0$, $u_t > 0$, $u > 0$ [28]. As an outcome of a continuation theorem, if T^* is finite, at the critical time the strain and strain rate blow up

$$\begin{aligned} \lim_{t \to T^*} \sup_{0 \le x \le 1} u(x,t) &= \infty \\ \limsup_{t \to T^*} \sup_{0 \le x \le 1} u_t(x,t) &= \infty \end{aligned} \tag{27}$$

in a way that the total stress remains bounded

$$0 < \sigma = \tau(u)\, u_t^n \le \max_{0 \le x \le 1} \sigma_0(x)\,. \tag{28}$$

The response predicted by (27) and (28) is compatible with expectations that the strain and strain-rate are very large in the interior of a shear band and suggests to look at shear band formation as a blow-up type of problem. This point of view appears instructive as a first attempt of analytical approach to the problem, but is probably too restrictive. It leads though to the question of characterizing whether solutions are globally defined or break down in finite time.

To this end, it is useful to write (23) in the form

$$\Phi(u(x,t)) = \Phi(u_0(x)) + \int_0^t \sigma^{\frac{1}{n}}(x,\tau)\, d\tau\,, \tag{29}$$

where

$$\Phi(u) := \int_1^u \tau(\xi)^{\frac{1}{n}}\, d\xi\,. \tag{30}$$

Then, blow up of solutions is related to the convergence of the integral

$$\Phi(\infty) = \int_1^\infty \tau(\xi)^{\frac{1}{n}}\, d\xi\,. \tag{31}$$

Indeed, in the case of the problem $(\mathcal{P})_S$, evaluating (31) at $x = 0$ or $x = 1$ leads to

$$\Phi(u(i,t)) = \Phi(u_0(i)) + t \quad , \quad i = 0, 1\,. \tag{32}$$

In turn, if $\Phi(\infty) < \infty$, (32) implies

$$u(i,t) \to \infty \quad \text{as} \quad t \to T_i\,, \tag{33}$$

with $T_i = \Phi(\infty) - \Phi(u_0(i)) < \infty$ for $i = 0, 1$, and the solution breaks down in finite time. It turns out that this criterion provides a complete characterization for blow-up of solutions in the case of $(\mathcal{P})_S$

$$T^* = \infty \quad \text{if and only if} \quad \Phi(\infty) = \infty\,. \tag{34}$$

However, in the case of the problem $(\mathcal{P})_V$,

$$\text{if} \quad \Phi(\infty) = \infty \quad \text{then} \quad T^* = \infty\,, \tag{35}$$

but the converse is an open problem.

The role of the integral (31) in localization problems is brought forth in the work of Molinari and Clifton [21]. They introduce the concept of "L_∞-localization", and use an approach developed by Hutchinson and Neale [19] for the uniaxial tension of a bar, to characterize L_∞-localization of the strain with the convergence of the integral (31).

In what follows, we record some information pertaining to the behavior of solutions. The two problems are treated separately, $(\mathcal{P})_S$ in this section and $(\mathcal{P})_V$ in the following. The reader is referred to [28, 29] for details of the derivations.

Prescribed tractions

The class of positive, decreasing constitutive functions $\tau(u)$ can be decomposed into two categories, depending on whether $\Phi(\infty)$ is finite or infinite. Roughly speaking, the dividing line consists of functions $\tau(u)$ that decay to zero like the power u^{-n}.

In case $\tau(u) \equiv \tau_0$ a constant, $\sigma(x,t)$ is a positive solution of

$$\left(\sigma^{\frac{1}{n}}\right)_t = \tau_0^{\frac{1}{n}} \sigma_{xx} \tag{36}$$

subject to the boundary conditions $(22)_S$, and $\sigma(x,t) \to 1$ uniformly in $x \in [0,1]$ as $t \to \infty$. We ask whether this behavior persists for positive and decreasing functions $\tau(u)$.

Two representative classes of functions $\tau(u)$ are considered: Class (H_c) consists of functions that decay to a positive constant $\tau(\infty)$ at a rate dominated by a power, i.e., for some $c > 0$ and $\alpha > 0$

$$\tau(u) > \tau(\infty) > 0 \quad , \quad 0 < -\tau'(u) \le \frac{c}{u^\alpha} \quad , \quad u > 0\,. \tag{H_c}$$

Class (H_p) consists of powers

$$\tau(u) = \frac{1}{u^m}\,. \tag{H_p}$$

According to (34), T^* is infinite when $m \le n$, but finite when $m > n$.

The system (16) can be recast into an equivalent formulation consisting of a reaction-diffusion equation (with variable diffusion) coupled to an ordinary differential equation

$$\left(\sigma^{\frac{1}{n}}\right)_t = \tau(u)^{\frac{1}{n}} \sigma_{xx} + \frac{1}{n} \frac{\tau'(u)}{\tau(u)^{1+\frac{1}{n}}} \left(\sigma^{\frac{1}{n}}\right)^2 \tag{37}$$

$$u_t = \frac{\sigma^{\frac{1}{n}}}{\tau(u)^{\frac{1}{n}}} \tag{38}$$

We ask the question: Which out of the reaction and diffusion terms of the parabolic equation (37) dominates in the course of the motion? The technical vehicle for answering this question is by classifying the steady super and sub-solutions of (37). The coefficients are viewed as unknown functions with the only available information that $u_t > 0$. For functions of class (H_c) the diffusion term always dominates, while for powers (H_p) the diffusion term dominates for parameter values $0 < m/n < 1/2$ and the reaction term dominates for $m/n \geq 1/2$.

Dominant diffusion

Combining this analysis with parabolic type energy estimates it turns out that, for functions $\tau(u)$ of class (H_c) and for powers (H_p) satisfying $0 < m/n < 1/2$, the shear stress $\sigma(x,t)$ is attracted to the constant state $\sigma \equiv 1$, while $u(x,t)$ behaves asymptotically as a function of time alone [28]: As $t \to \infty$

$$\sigma(x,t) = 1 + O(t^{-\beta}), \tag{39}$$

$$\int_{u_0(x)}^{u(x,t)} \tau(\xi)^{\frac{1}{n}} d\xi = t + O\left(\int_1^t s^{-\beta} ds\right) \tag{40}$$

uniformly on $[0,1]$, with $0 < \beta < 1$. Precisely, $\beta = \alpha > 0$ for (H_c) and $\beta = \frac{n-2m}{n-m}$ for (H_p). The response in this case is similar to the response of (36).

Weak diffusion

For powers with exponent $m/n > 1/2$ (including the parameter values where solutions blow up) the following holds [29]: If $S(x)$ is a positive function satisfying

$$\begin{aligned} u_0(x)^{2\frac{m}{n}-1} S(x)^{\frac{2}{n}} &\geq \frac{n}{m} S_{xx}(x) \\ S(x) &\geq \sigma_0(x), \end{aligned} \tag{41}$$

then the corresponding solution of $(\mathcal{P})_S$ satisfies

$$\sigma(x,t) \leq S(x). \tag{42}$$

In turn, (29) induces corresponding bounds for $u(x,t)$. The implications of some specific choices for $S(x)$ are examined below.

Consider first the one-parameter family of functions $S_\alpha(x) = 1 - \alpha x(1-x)$, with $0 < \alpha < 4$. Then (41) implies the restriction on the initial data

$$\sigma_0(x) \leq S_\alpha(x) \quad , \quad u_0(x) \geq u_\alpha := \left[\frac{n}{m} \frac{2\alpha}{\left(1-\frac{\alpha}{4}\right)^{\frac{2}{n}}}\right]^{\frac{n}{2m-n}}, \quad 0 \leq x \leq 1. \tag{43}$$

Note that $u_\alpha \to 0$ and $S_\alpha \to 1$ as $\alpha \to 0$. No matter how close to the state $\sigma \equiv 1$ the initial stress is, it satisfies thereafter

$$\sigma(x,t) \le S_\alpha(x) = 1 - \alpha x(1-x) \tag{44}$$

and can no longer approach the state $\sigma \equiv 1$. We conclude that in the region $m/n < 1/2$ diffusion is too weak to uniformize the stress, and initial nonuniformities, however small, persist in time.

Next, consider a positive, convex, and symmetric with respect to $x = \frac{1}{2}$ function $S_\gamma(x)$, that drops fast from $S_\gamma(0) = 1$ to approximately a constant level γ, stays there over almost the entire interval, and then increases fast to the value $S_\gamma(1) = 1$. An example of such a function is provided by

$$S_\gamma(x) = \gamma + (1-\gamma)[x^N + (1-x)^N]\,, \tag{45}$$

for N a large positive integer and $0 < \gamma < 1$. Let the initial stress $u_0(x)$ be symmetric with respect to $x = \frac{1}{2}$, with $u_0(x) \le u_0(0) = u_0(1)$. Compatibility with $(41)_1$ dictates that $u_0(x)$ be sufficiently large in a neighborhood of $x = 0$ and $x = 1$. It could be a sufficiently large constant. Take $S_\gamma(x)$ as the initial stress and $u_0(x)$ as the initial strain and let $(\sigma_\gamma(x,t), u_\gamma(x,t))$ be the corresponding solution of $(\mathcal{P})_S$.

Then $\sigma_\gamma(x,t)$ satisfies (42). Regarding the behavior of u_γ, we look at two separate regions:

(i) $1/2 < m/n < 1$: Here $T^* = \infty$. Also

$$u_\gamma(x,t) \le \left[u_0^{1-\frac{m}{n}}(x) + (1-\frac{m}{n}) S_\gamma^{\frac{1}{n}}(x)\, t \right]^{\frac{n}{n-m}} \tag{46}$$

with (46), in fact, an equality at $x = 0$ and $x = 1$. Since u_0 may be a constant function, an initial nonuniformity in the stress, no matter how small, may induce spatial nonuniformities in the strain $u(x,t)$ that grow in time.

(ii) $m/n > 1$: Here $T^* < \infty$. Also,

$$\begin{aligned} u_\gamma(x,t) &\le \left[u_0^{-(\frac{m}{n}-1)}(x) - (\frac{m}{n} - 1) S_\gamma^{\frac{1}{n}}(x)\, t \right]^{-\frac{n}{m-n}} \\ \partial_t u_\gamma(x,t) &\le S_\gamma^{\frac{1}{n}}(x) \left[u_0^{-(\frac{m}{n}-1)}(x) - (\frac{m}{n} - 1) S_\gamma^{\frac{1}{n}}(x)\, t \right]^{-\frac{m}{m-n}} \end{aligned} \tag{47}$$

with equalities at $x = 0$ and $x = 1$. By the choice of $S_\gamma(x)$ and $u_0(x)$, the right hand sides blow up for the first time at the boundary points $x = 0$ and $x = 1$, where (47) holds as an equality. In blowing up, they appear like two shear bands located at the boundaries.

The above considerations suggest for the case of prescribed traction loading: First, that it is possible a small nonuniformity in the stress to produce large strain gradients in either long time (i), or even shorter times (ii). Second, that this type of response is concurrent with an inability of the material to diffuse the applied stresses.

ON THE STABILITY OF THE UNIFORM SHEARING SOLUTIONS

In this Section, we take up the initial-boundary value problem $(\mathcal{P})_V$, for a power law (H_p), and investigate the stability of the uniform shearing solutions (24). First, we present a linearized stability analysis, and then nonlinear results on the behavior of solutions.

The linearized problem

The uniform shearing solutions are time-dependent, thus leading to linearized equations with variable coefficients. Investigations that account for the time dependence of the coefficients in the linearized equations are presented by Burns [4], Fressengeas and Molinari [18] and Shawki [24]. The last two works concern models that include explicit thermal effects through the energy equation. Fressengeas and Molinari devise a relative linear perturbation analysis that examines the decay (or growth) of the ratio of the absolute perturbation over the time-dependent uniform solution. Shawki proceeds by means of a quasistatic approximation for the linearized equations and connects the issue of stability with the growth in the kinetic energy of the perturbation (see article in this volume).

Here, we work within the simple context of (26) for a power law. In compensation, no further approximations are imposed and a complete picture emerges. It is convenient to work with the equations for the linearized displacement

$$Y_{tt} = n\,(t+u_0)^{-m}\,Y_{xxt} - m(t+u_0)^{-m-1}\,Y_{xx}\,, \tag{48}$$

with $Y(0,t) = Y(1,t) = 0$. These lead to (26) by setting

$$V = Y_t \qquad U = Y_x\,. \tag{49}$$

The general solution of (48) can be written as a linear combination of eigensolutions

$$Y(x,t) = \sum_{k=1}^{\infty} c_k(t)\,\sin k\pi x\,, \tag{50}$$

where c_k are Fourier coefficients

$$c_k(t) = 2\int_0^1 Y(x,t)\,\sin k\pi x\,dx\,. \tag{51}$$

The evolution of c_k is monitored by the differential equation

$$\mathcal{L}[\,c_k\,] := \ddot{c}_k + \frac{(k\pi)^2}{(t+u_0)^m}\big(n\dot{c}_k - \frac{m}{t+u_0}c_k\big) = 0\,. \tag{52}$$

For simplicity, suppose that $Y(x,0) = \delta\,Y_0(x)$ and $Y_t(x,0) = \delta\,V_0(x)$, where $\delta > 0$ is measuring the amplitude of the initial perturbation. Then

$$|c_k(0)| + |\dot{c}_k(0)| \le \delta_k\,, \qquad k = 1,\,2,\,\ldots\,. \tag{53}$$

where $\delta_k = O(\delta)$ for k fixed, and its dependence with respect to k is determined by the smoothness of Y_0 and V_0. We state certain facts, that are proved in the Appendix, concerning the response of solutions of (52)-(53) for $0 < m < 1$ and $n > 0$.

(i) *Case* $m/n < 1$.

Given any $\varepsilon > 0$, there is a constant C_ε independent of k such that

$$|c_k(t)| \leq C_\varepsilon \, \delta_k \, (t + u_0)^{\frac{m}{n}+\varepsilon} \,, \quad |\dot{c}_k(t)| \leq C_\varepsilon \, \delta_k \, (t + u_0)^{\frac{m}{n}+\varepsilon-1} \,. \tag{54}$$

The above estimate is sharp, in the sense that there are initial data compatible with (53) such that

$$c_k(t) \geq C \, \delta_k \, (t + u_0)^{\frac{m}{n}} \tag{55}$$

(ii) *Case* $m/n > 1$.

For any initial data subject to (53), there is a positive constant C independent of k such that

$$|c_k(t)| \leq C \, \delta_k \, (t + u_0)^{\frac{m}{n}} \,, \quad |\dot{c}_k(t)| \leq C \, \delta_k \, (t + u_0)^{\frac{m}{n}-1} \,. \tag{56}$$

Moreover, there are initial data, compatible with (55), such that for any $\varepsilon > 0$ there is a constant C_ε with the property

$$c_k(t) \geq C_\varepsilon \, \delta_k \, (t + u_0)^{\frac{m}{n}-\varepsilon} \,. \tag{57}$$

Since the constants in (54) and (56) are independent of k, Y and $V = Y_t$ inherit these bounds, with the same time rates as c_k and $\dot{c}_k$ respectively. Also $U = Y_x$ satisfies a bound with the same time rate as Y.

To summarize, when $m/n < 1$ the amplitude of perturbations of the uniform flow grows, at worse, slower than the uniform flow. By contrast, in the parameter region $m/n > 1$ perturbations can grow faster than the uniform shear flow, and the uniform shearing solutions are linearly unstable.

Nonlinear response for prescribed velocities

The linearized stability analysis suggests a change in the response of $(\mathcal{P})_V$ across the parameter values $m/n = 1$. This is confirmed by analysis of the behavior of solutions to the nonlinear problem [28, 29].

Strong strain-rate dependence

First, we present a nonlinear stability result for the parameter region $m/n < 1$. Motivated by the form of the uniform shearing solutions and the scaling invariance of the underlying equations, we introduce the transformations

$$\begin{aligned} v(x,t) &= V(x, s(t)) \\ u(x,t) &= (t+1) \, U(x, s(t)) \\ \sigma(x,t) &= (t+1)^{-m} \, \Sigma(x, s(t)) \end{aligned} \tag{58}$$

with $s(t) = \log(t+1)$. A straightforward computation shows that $(\Sigma(x,s), U(x,s))$ satisfy the system of reaction-diffusion equations

$$\begin{aligned} \Sigma_s &= n e^{(1-m)s} U^{-\frac{m}{n}} \Sigma^{1-\frac{1}{n}} \Sigma_{xx} - m \frac{\Sigma}{U^{1-\frac{m}{n}}} \left(\Sigma^{\frac{1}{n}} - U^{1-\frac{m}{n}} \right) \\ U_s &= U^{\frac{m}{n}} \left(\Sigma^{\frac{1}{n}} - U^{1-\frac{m}{n}} \right) . \end{aligned} \tag{59}$$

Consider the system of ordinary differential equations obtained by neglecting the diffusion term in (59). Their integral curves are the lines $\Sigma = const.\, U^{-m}$ sketched in Fig 1, and their vector field vanishes along the line $\Sigma = U^{n-m}$. The form of the last line changes drastically across the critical parameter value $m/n = 1$. If $0 < m/n < 1$, the theory of Chueh, Conley and Smoller [10] guarantees that (59) admits positively invariant rectangles of arbitrary size in the first quadrant centered around the line $\Sigma = U^{n-m}$ (see Fig 1). (This property is lost in the complementary region $m/n > 1$.)

Fig. 1: Invariant regions for (59) when $0 < \frac{m}{n} < 1$.

Given positive initial data, let U_-, U_+, Σ_- and Σ_+ be the defining coordinates of the smallest rectangle containing the curve $(\sigma_0(x), u_0(x))$, $0 \le x \le 1$. Then

$$\Sigma_- \le \Sigma(x,s) \le \Sigma_+ \quad , \quad U_- \le U(x,s) \le U_+ . \tag{60}$$

In turn, (58) and (60) yield the relations:

$$\begin{aligned} U_-(t+1) \le & u(x,t) \le U_+(t+1) \\ \Sigma_-(t+1)^{-m} \le & \sigma(x,t) \le \Sigma_+(t+1)^{-m} \\ \Sigma_-^{\frac{1}{n}} U_-^{\frac{m}{n}} \le & v_x(x,t) \le \Sigma_+^{\frac{1}{n}} U_+^{\frac{m}{n}} . \end{aligned} \tag{61}$$

Relations (61) provide explicit bounds on perturbations of the uniform shear flow. Using them together with parabolic-type energy estimates, it turns out that, for $m < \min\{n, 1\}$, every solution is asymptotically attracted to a uniform shear flow: As $t \to \infty$,

$$\begin{aligned} v_x(x,t) &= 1 + O(t^{\beta-1}) \\ u(x,t) &= t + O(t^{\beta}) \\ \sigma(x,t) &= t^{-m}\big(1 + O(t^{\beta-1})\big) \end{aligned} \tag{62}$$

uniformly on $[0,1]$, with $\beta = \max\{\frac{m}{n}, m\} < 1$. The role of the constraint $m < 1$ is explained in [28]. We remark that (62) does not exclude the presence of nonuniformities in the strain that propagate at slower time rates, and is thus compatible with the findings of the linearized stability analysis. However, as indicated by the bound $(61)_3$ on the strain rate, no catastrophic growth of strain is predicted in this parameter range.

Asymptotic behavior results such as (62) have also been established for the case of the constitutive relation $\sigma = \tau(u)u_t$, with $\tau(u)$ increasing up to a critical strain and decreasing (but not very fast) thereafter [26]. However, the explicit bounds (61) are available only for a power law. Such results are interpreted as indicating that eventually rate dependence exerts a stabilizing influence that washes out nonuniformities. However, at intermediate stages, there may be local amplification of nonuniformities in strain connected, for instance, with undergoing through a sudden drop of the curve $\tau(u) - u$.

Weak strain-rate dependence

In the region $m/n > 1$, the behavior of solutions to $(\mathcal{P})_V$ is quite subtle and is not fully understood at the present time. The situation in the case of prescribed tractions suggests that unstable response be expected, at least for large values of the ratio m/n. It follows by (2) and $(22)_V$ that the average strain grows like t

$$\int_0^1 u(x,t)dx = t + \int_0^1 u_0(x)\,. \tag{63}$$

On the other hand, the linearized analysis indicates that perturbations of the strain tend to grow faster, like $t^{\frac{m}{n}}$. It is conceivable that soon large nonuniformities in the strain develop. At that point the linearized analysis ceases to be valid.

We discuss now the effect of a large nonuniformity in the strain (see [29], also Bertsch, Peletier and Verduyn Lunel [2] for a similar result for (19)). The analysis applies to constitutive functions that include powers with $m/n > 1$ and $n < 1$. Consider the initial data

$$\begin{aligned} v_0(x) &= \;x \qquad\qquad 0 \le x \le 1 \\ u_0(x) &= \begin{cases} \bar u & 0 \le x \le y-\delta \quad , \quad y+\delta \le x \le 1 \\ \bar U(x) & y-\delta \le x \le y+\delta \end{cases} \end{aligned} \tag{64}$$

corresponding to a concentration of strain in an interval of length 2δ around a fixed point y, and a linear initial velocity profile. We are thinking of $\bar U(x)$ as a bell shaped function that attains its maximum at y, say $\bar U(y) =: U_M$, and satisfies $\bar U(y-\delta) = \bar U(y+\delta) = \bar u$ and $\bar U(x) > \bar u$ for $x \in (y-\delta\,, y+\delta)$ (Fig 2).

It turns out that for U_M sufficiently large:
(i) either T^* is finite, in which case (27) holds,
(ii) or, as $t \to \infty$,

$$v(x,t) = \begin{cases} O(t^{-1}) & x \in [0, y-\delta] \\ 1 + O(t^{-1}) & x \in [y+\delta, 1] \end{cases} \tag{65}$$

and $u(x,t)$ converges monotonically, for x outside the band $(y-\delta, y+\delta)$, to some bounded limiting function.

Fig. 2: The initial data $u_0(x)$, $v_0(x)$ in (64).

The large time response predicted in (65) looks like a fully developped shear band, where the parts of the material to the left and right of the band have unloaded and move independently. The result indicates a collapse of momentum transfer across the band.

EXPLICIT THERMAL EFFECTS

In this Section we present information regarding the behavior of solutions of $(\mathcal{P})_S$ and $(\mathcal{P})_V$ for the model (17). The model (17) is based on the constitutive relation (18), appropriate for a material exhibiting thermal softening, strain hardening and strain-rate sensitivity, and is studied in this generality in Refs [27] (for $n = 1$) and [30].

The analysis of the uniform shearing solutions (10) indicates that, when the degree of thermal softening is large, the combined effect of thermal softening and strain hardening results to net softening. Capturing this behavior for arbitrary solutions is the main source of analytical difficulty in dealing with (17), and establishing a precise analogy among the models (17) and (16). One approach to accomplish that is outlined below. Differentiating $\sigma^{\frac{1}{n}}$ with respect to t one obtains

$$(\sigma^{\frac{1}{n}})_t = \mu^{\frac{1}{n}}(\theta, u)\, u_{tt} + \frac{1}{n}\mu^{\frac{1}{n}-1}(\theta, u)\,(\partial_t \mu(\theta, u))\, u_t\,. \tag{66}$$

The term $\partial_t \mu(\theta, u)$ is positive or negative depending on whether the material exhibits effective hardening or softening along the deformation. Use of (66) and (17), leads to an alternative formulation of the problem, consisting of

$$(\sigma^{\frac{1}{n}})_t = \mu^{\frac{1}{n}}(\theta, u)\,\sigma_{xx} + \frac{1}{n}\frac{1}{\mu^{1+\frac{1}{n}}(\theta, u)}\big(\mu_\theta(\theta, u)\sigma + \mu_u(\theta, u)\big)\,(\sigma^{\frac{1}{n}})^2\,, \tag{67}$$

$$\theta_t = \sigma u_t\,, \tag{68}$$

and

$$u_t = \frac{\sigma^{\frac{1}{n}}}{\mu^{\frac{1}{n}}(\theta, u)} > 0\,. \tag{69}$$

As in the case of (37-38), we ask which out of the reaction or diffusion terms in (67) dominates. The analysis proceeds by classifying steady super and sub-solutions of (67). The coefficients are viewed as unknown functions connected through (68), which controls the relative weight of temperature and strain (see [30] for details).

This approach yields a characterization of the behavior of the field variables at the critical time, when T^* is finite. Hypotheses $(H_1 - H_3)$ for $\mu(\theta, u)$ are used; at places the additional hypothesis

$$\int_1^\infty \mu^{\frac{1}{n}}(\beta, \xi)\,d\xi = \infty \qquad \text{for any } \beta > 0\,, \tag{H_4}$$

is employed. If T^* is finite, then as $t \to T^*$ the stress remains bounded by the quantity $\Sigma = \max\{\max_{0\le x\le 1}\sigma_0(x)\,,\, \max_{0\le x\le 1}(\theta_0(x)/u_0(x))\}$, the strain and strain rate behave as in (27), while, in case (H_4) also holds, the temperature satisfies

$$\lim_{t\to T^*}\sup_{0\le x\le 1}\theta(x,t) = \infty\,. \tag{70}$$

Global existence or blow-up of solutions to $(\mathcal{P})_S$ or $(\mathcal{P})_V$, is related to the convergence or divergence of the integral

$$\mathcal{M}(\alpha) := \int_1^\infty \mu^{\frac{1}{n}}(\alpha\xi, \xi)\,d\xi\,, \tag{71}$$

which measures how fast the constitutive function $\mu^{\frac{1}{n}}$ decays (cf (H_3)) when the temperature is proportional to the strain. As it turns out:

(i) For either of the problems $(\mathcal{P})_S$ or $(\mathcal{P})_V$, if $\mathcal{M}(\Sigma) = \infty$ then $T^* = \infty$.

(ii) For the problem $(\mathcal{P})_S$, if $\mathcal{M}(k) < \infty$ then $T^* < \infty$.

Here $k := \min\{1\,, \theta_0(0)/u_0(0)\,, \theta_0(1)/u_0(1)\}$. This result provides a characterization of blow up for stress boundary conditions, but only sufficient conditions for global existence in the case of velocity boundary conditions.

In general, the above criteria depend on the initial data. However, for certain popular models used in the study of shear bands, they are actually independent of the data.

(a) For a power law (8),

$$\mathcal{M}(\alpha) = \begin{cases} \infty & \text{for } \nu + k \geq -n \\ \alpha^{\nu} \frac{-n}{\nu+k+n} < \infty & \text{for } \nu + k < -n \end{cases} \tag{72}$$

Hypothesis (H_3) dictates the restriction $\nu + k \leq 0$, while hypothesis (H_4) is fulfilled whenever $k \geq -n$. We conclude that for a power law solutions of $(\mathcal{P})_S$ and $(\mathcal{P})_V$ exist for all times for parameter values $-n \leq \nu + k < 0$. Solutions of $(\mathcal{P})_S$ blow up in finite time for parameter values $\nu + k < -n$.

(b) For an exponential law (9) with β positive,

$$\mathcal{M}(\alpha) = \frac{n}{\alpha\beta} < \infty \tag{73}$$

and solutions of $(\mathcal{P})_S$ blow up in finite time for any values of the parameters.

Power law and prescribed tractions

In the case of a power law and for shearing deformations caused by prescribed tractions it is possible to obtain a better understanding of the behavior of solutions [30]. The parameter region $\nu + k < 0$ can be decomposed into three subregions across which the material response changes drastically. The situation generalizes the results for strain softening materials (23) of class (H_p).

(i) In the region $0 < (\nu + k)/n < 1/2$ the shear stress $\sigma(x, t)$ is attracted to the constant state $\sigma \equiv 1$, as $t \to \infty$, while $u(x, t)$ and $\theta(x, t)$ behave asymptotically as functions of time.

(ii) In the region $1/2 \leq (\nu + k)/n \leq 1$ the constant state $\sigma \equiv 1$ loses its stability and spatial nonuniformities in the strain can develop and persist in time.

(iii) Finally, in the region $(\nu + k)/n > 1$, $u(x, t)$ becomes infinite in finite time.

Most interesting, when $(\nu + k)/n > 1/2$, there are initial data for which the strain $u(x, t)$ develops nonuniformities around $x = 0$ and $x = 1$ and looks like two shear bands located at the boundaries. This response is concurrent with an inability for diffusion of the applied stresses.

Strain independent materials

At the present time, the analysis of nonlinear stability of the uniform shearing solutions for (17) is incomplete. The available results concern the special cases that either thermal softening or strain dependence are absent. For a strain independent power law (17) takes the form

$$\begin{aligned} v_t &= (\theta^{\nu} v_x^n)_x \\ \theta_t &= \theta^{\nu} v_x^{n+1}, \end{aligned} \tag{74}$$

with $\nu < 0$. From a viewpoint of analysis there are certain similarities between (74) and (16) $-$ (H_p). As a consequence, the response of solutions to (74) parallels the results outlined in the previous section for (16).

First, if $\nu > -n$, every solution is asymptotically attracted as $t \to \infty$ to a uniform shearing solution (Dafermos and Hsiao [15], Tzavaras [25]). Similarly to the case of a strain-softening, rate-dependent power law, a derived system admits invariant regions (Bertsch, Peletier and Verduyn-Lunel [2] for $n = 1$, Tzavaras [30] for $n \neq 1$). As a result, perturbations in the temperature and strain rate of the uniform flow are controlled by bounds (of the form (61)) depending on the data and evolving according to the time rates of the uniform flow, given in (13) for $k = 0$.

Complementing the above, Bertsh, Peletier and Verduyn-Lunel [2] consider the case of a Newtonian fluid ($n = 1$) and show that, for parameter values $\nu < -1$ and for an initial temperature concentrated in a narrow region, either the solution of (74) blows up or the velocity has an asymptotic profile of the type described in (65). They also establish the asymptotic behavior of solutions in the borderline case $\nu = -1$.

The effect of other types of loading in the stability of shear flows for temperatures dependent fluids is studied by Charalambakis [5-7]. He considers loading effected by mixed velocity-stress boundary shearing, inertial forces or periodic boundary shearing, and establishes sufficient criteria for stability of such flows.

The above works proceed under the constitutive assumption (4) of a non-conducting material. Experience accumulated through studies of initial value problems in thermomechanics (Dafermos [13, 14], Dafermos and Hsiao [16]) suggests that heat conduction exerts a dissipative effect hindering unstable response. Due to the high temperature differences across a shear band, heat conduction is expected to play a significant role in problems concerning propagation of shear bands, and may conceivably provide shear bands with internal structure.

Insight in that direction is offered by recent works (Chen, Douglas and Malek-Madani [9], Maddocks and Malek-Madani [20]). Their analysis applies to the system of equations consisting of (1), (3) and the constitutive relations

$$q = k\theta_x \quad , \quad \sigma = e^{-\beta\theta} v_x \tag{75}$$

under isothermal boundary conditions for the temperature, and various possibilities of loading, including prescribed tractions and prescribed velocities. The existence of steady state solutions is studied and the presence of bifurcating branches of solutions is established. In addition, the stability of these steady states is studied, using a direct linearized analysis and energy estimates [9], or a variational stability approach [20].

APPENDIX

We present here the technical steps that provide the stability analysis for the linearized problem (48). It is based on properties of the differential operator

$$\mathcal{L}[\,g\,] := \ddot{g} + \frac{(k\pi)^2}{(t+u_0)^m}\big(n\dot{g} - \frac{m}{t+u_0}g\big)$$

with $k = 1, 2, \ldots$. Note that if $m/n = 1$ then $\mathcal{L}[\,(t+u_0)\,] = 0$.

Lemma. Suppose that $0 < m < 1$ and $n > 0$.

For parameter values $m/n > 1$:

(i_1) If $f_+(t) = (t+u_0)^{\frac{m}{n}}$ then $\mathcal{L}[f_+] > 0$.

(i$_2$) Given any $\varepsilon > 0$, there is a positive constant K_ε depending on ε but not on k, such that the function

$$f_-(t) = (t+u_0)^{\frac{m}{n}-\varepsilon} + K_\varepsilon$$

satisfies $\mathcal{L}[f_-] < 0$, for $t \geq 0$.

For parameter values $m/n < 1$:

(ii$_1$) There is $0 < \alpha < 1$ such that the function $H_+(t) = (t+u_0)^\alpha$ satisfies $\mathcal{L}[H_+] > 0$, for $t \geq 0$.

(ii$_2$) Given any $\varepsilon > 0$, there is a positive T_ε such that the function

$$h_+(t) = (t+u_0)^{\frac{m}{n}+\varepsilon}$$

satisfies $\mathcal{L}[h_+] > 0$, for $t \geq T_\varepsilon$.

(ii$_3$) If $h_-(t) = (t+u_0)^{\frac{m}{n}}$ then $\mathcal{L}[h_-] < 0$.

Proof. First, observe that

$$\mathcal{L}[\,(t+u_0)^{\frac{m}{n}}\,] = \frac{m}{n}(\frac{m}{n}-1)(t+u_0)^{\frac{m}{n}-2}$$

is positive for $m/n > 1$ and negative for $m/n < 1$. This shows (i$_1$) and (ii$_3$).

Next we turn to (i$_2$). A straightforward but lengthy computation yields

$$\mathcal{L}[\,f_-\,] = (t+u_0)^{\frac{m}{n}-\varepsilon-2}\left[\lambda - \varepsilon n(k\pi)^2(t+u_0)^{1-m}\right] - K_\varepsilon\frac{m(k\pi)^2}{(t+u_0)^{m+1}}\,,$$

where $\lambda = (\frac{m}{n}-\varepsilon)(\frac{m}{n}-\varepsilon-1)$. If λ is negative the result follows. So, we consider the case $0 < \varepsilon < \frac{m}{n}-1$, for which λ is strictly positive. Let T_ε be such that $(T_\varepsilon+u_0)^{1-m} > \lambda/\varepsilon n\pi^2$. Then $\mathcal{L}[f_-] < 0$ for $t > T_\varepsilon$. For the remaining interval $[0, T_\varepsilon]$ we attain $\mathcal{L}[f_-] < 0$, by choosing the positive constant K_ε sufficiently large.

Finally we turn to (ii$_1$) and (ii$_2$). A computation yields

$$\mathcal{L}[\,(t+u_0)^\alpha\,] = (t+u_0)^{\alpha-2}\left[(n\alpha-m)\,(k\pi)^2\,(t+u_0)^{1-m} - \alpha(1-\alpha)\right].$$

Let first α be sufficiently close to 1, so that that $\alpha(1-\alpha) < \pi^2 {u_0}^{1-m}(n\alpha - m)$ and $m/n < \alpha < 1$. For this choice of α, the corresponding function $H_+(t)$ satisfies $\mathcal{L}[H_+] > 0$ for $t \geq 0$. This shows (ii$_1$). Set now $\alpha = m/n + \varepsilon$ and let T_ε be such that $(T_\varepsilon + u_0)^{1-m} > (\frac{m}{n}+\varepsilon)(1-\frac{m}{n}-\varepsilon)/\varepsilon n\pi^2$. Then for $t \geq T_\varepsilon$ the function $h_+(t)$ satisfies $\mathcal{L}[h_+] > 0$. This shows (ii$_2$) and completes the proof of the lemma.

The second ingredient of the linearized analysis is a comparison estimate for the operator $\mathcal{L}[g]$. Namely:

(iii) If $g(\tau) > 0$, $\dot{g}(\tau) > 0$ and $\mathcal{L}[g] > 0$ for $t \geq \tau$, then $g(t) > 0$ for $t \geq \tau$.

Indeed, if the conclusion is violated then g admits a positive, local maximum at some point $s > \tau$. At s it is $g(s) > 0$, $\dot{g}(s) = 0$, $\ddot{g}(s) \leq 0$ and thus $\mathcal{L}[g](s) < 0$, which contradicts the assumption.

Behavior of solutions to (52)

Fix k a positive integer and let $c_k(t)$ be a solution of (52) with initial data subject to (53). Our goal is to monitor the time evolution of $c_k(t)$ in terms of the size δ_k of the data. In the sequel the k-dependence of δ_k is suppressed, and C will stand for a generic positive constant that is independent of t, $\delta = \delta_k$ and the parameter k. Within the range $0 < m < 1$ and $n > 0$ two separate regions are considered.

First case: $m/n < 1$

Our first task is to show:

(iv) If c_k satisfies $c_k(0) < \delta$, $\dot{c}_k(0) < \delta$ and $\mathcal{L}[c_k] = 0$, then for any $\varepsilon > 0$ there is a constant C_ε such that

$$c_k(t) \leq C_\varepsilon \delta (t+u_0)^{\frac{m}{n}+\varepsilon}\,, \qquad \dot{c}_k(t) \leq C_\varepsilon \delta (t+u_0)^{\frac{m}{n}+\varepsilon-1}\,, \qquad k = 1,\,2,\,\ldots. \tag{$a.1$}$$

To this end, let

$$g(t) = C_1 \delta H_+(t) - c_k(t)\,,$$

where $H_+(t)$ is as in (ii$_1$), and the constant C_1 is chosen sufficiently large so that $g(0) > 0$ and $\dot{g}(0) > 0$. Since $\mathcal{L}[g] > 0$ for $t \geq 0$, (iii) yields

$$c_k(t) \leq C_1 \delta H_+(t) = C_1 \delta (t+u_0)^\alpha \qquad t \geq 0\,. \tag{$a.2$}$$

To improve this bound, observe that (52) implies

$$\begin{aligned} \ddot{c}_k + \frac{(k\pi)^2 n}{(t+u_0)^m}\dot{c}_k &= \frac{(k\pi)^2 m}{(t+u_0)^{m+1}} c_k \\ &\leq C_1 \frac{\delta (k\pi)^2 m}{(t+u_0)^m} \end{aligned} \tag{$a.3$}$$

Whenever $\dot{c}_k$ is larger than $C_1 \delta m/n$, $\dot{c}_k$ is decreasing. Hence,

$$\dot{c}_k(t) \leq \max\{\dot{c}_k(0)\,,\, C_1\delta m/n\} \leq C_2 \delta \qquad t \geq 0\,.$$

Consider now the comparison function

$$g(t) = C_3 \delta h_+(t) - c_k(t)\,.$$

In view of (ii$_2$), there is T_ε so that $\mathcal{L}[g] > 0$ for $t \geq T_\varepsilon$. Choose $C_{3,\varepsilon} = C_3(\varepsilon)$ such that $g(T_\varepsilon) > 0$ and $\dot{g}(T_\varepsilon) > 0$. This is possible because of the preliminary estimates that are at our disposal. Then (iii) implies

$$c_k(t) \leq C_{3,\varepsilon} \delta (t+u_0)^{\frac{m}{n}+\varepsilon} \qquad t \geq T_\varepsilon\,. \tag{$a.4$}$$

Combining (a.2) and (a.4) yields the first part of (a.1).

To show the second part, note that (52) implies

$$\ddot{c}_k + \frac{(k\pi)^2 n}{(t+u_0)^m}\dot{c}_k \le C_{3,\varepsilon}\delta \frac{(k\pi)^2 m}{(t+u_0)^{m+1}}(t+u_0)^{\frac{m}{n}+\varepsilon}$$

Integrating the differential inequality and performing an integration by parts gives

$$\begin{aligned}
\dot{c}_k(t) &\le \dot{c}_k(0)\exp\{-C_4 k^2[(t+u_0)^{1-m} - {u_0}^{1-m}]\} \\
&\quad + C_{5,\varepsilon}\,\delta\, k^2 \int_0^t (s+u_0)^{\frac{m}{n}+\varepsilon-1-m} \exp\{C_4 k^2[(s+u_0)^{1-m} - (t+u_0)^{1-m}]\}\, ds \\
&\le \delta\, \exp\{-C_4[(t+u_0)^{1-m} - {u_0}^{1-m}]\} \\
&\quad + \frac{C_{5,\varepsilon}\,\delta}{(1-m)C_4}\Big[(t+u_0)^{\frac{m}{n}+\varepsilon-1} - {u_0}^{\frac{m}{n}+\varepsilon-1}\exp\{-C_4 k^2\,[(t+u_0)^{1-m} - {u_0}^{1-m}]\} \\
&\quad + (1-\frac{m}{n}-\varepsilon)\int_0^t (s+u_0)^{\frac{m}{n}+\varepsilon-2}\exp\{C_4 k^2[(s+u_0)^{1-m} - (t+u_0)^{1-m}]\}\, ds\Big]
\end{aligned}$$

The last integral is estimated, with the help of L'Hopital's rule, to arrive at

$$\dot{c}_k(t) \le C_{6,\varepsilon}\,\delta (t+u_0)^{\frac{m}{n}+\varepsilon-1}\,.$$

with $C_{6,\varepsilon}$ independent of k.

Now (54) follows by applying (iv) consecutively to c_k and $-c_k$. To show (55), fix the initial data to be $c_k(0) = \dot{c}_k(0) = \delta/2$ and test the comparison function

$$g(t) = c_k(t) - C_7\delta h_-(t)$$

Then $\mathcal{L}[g] > 0$ and, upon choosing C_7 appropriately small, we will have $g(0) > 0$ and $\dot{g}(0) > 0$. It follows that $g(t) \ge 0$, which yields (55).

Second case: $m/n > 1$

A similar in spirit argument as in the previous case, using the functions $f_+(t)$ and $f_-(t)$, provides (56) and (57).

ACKNOWLEDGEMENT

Research partially supported by the Army Research Office under Contract No. DAAL03-88-K-0185 and the National Science Foundation under grant DMS-8716132.

REFERENCES

1. Anand L, Kim KH, and Shawki TG, Onset of shear localization in viscoplastic solids, *J Mech Phys Solids* **35** (1987), 407-429.

2. Bertsch M, Peletier LA, and Verduyn Lunel SM, The effect of temperature dependent viscosity on shear flow of incompressible fluids, *SIAM J Math Anal* **22** (1991), 328-343.

3. Burns TJ, Approximate linear stability analysis of a model of adiabatic shear band formation, *Quart Appl Math* **43** (1985), 65-83.

4. Burns TJ, Similarity and bifurcation in unstable viscoplastic shear, *SIAM J Appl Math* **49** (1989), 314-329.

5. Charalambakis N, Adiabatic shearing flow caused by time dependent inertial force, *Quart Appl Math* **42** (1984), 275-280.

6. Charalambakis N, Adiabatic shearing of an incompressible fluid under periodic or steady boundary conditions, *J Thermal Stresses* **8** (1985), 425-434.

7. Charalambakis N, Stabilité asymptotique lorsque $t \to \infty$ en thermoviscoelasticité et thermoplasticité, in *Nonlinear partial differential equations and their applications: Collège de France Seminar Vol IX*, H Brezis and J-L Lions (eds), Pitman Research Notes in Mathematics No 181, Longman Scientific & Technical, Essex, England, 1988.

8. Charalambakis N and Murat F, Weak solutions to the initial-boundary value problem for the shearing of non-homogeneous thermoviscoplastic materials, *Proc Royal Soc Edinburgh* **113A** (1989), 257-265.

9. Chen HT, Douglas AS, and Malek-Madani R, An asymptotic stability condition for inhomogeneous simple shear, *Quart Appl Math* **47** (1989), 247-262.

10. Chueh K, Conley C, and Smoller J, Positively invariant regions for systems of nonlinear diffusion equations, *Indiana Univ Math J* **26** (1977), 372 - 411.

11. Clifton RJ, Duffy J, Hartley KA, and Shawki TG, On critical conditions for shear band formation at high strain rates, *Scripta Met* **18** (1984), 443-448.

12. Coleman BD and Hodgdon ML, On shear bands in ductile materials, *Arch Rational Mech Analysis* **90** (1985), 219-247.

13. Dafermos CM, Global smooth solutions to the initial boundary value problem of one-dimensional nonlinear thermoviscoelasticity, *SIAM J Math Analysis* **13** (1982), 397-408.

14. Dafermos CM, *Contemporary Issues in the Dynamic Behavior of Continuous Media*, LCDS Lecture Notes # 85-1, 1985.

15. Dafermos CM and Hsiao L, Adiabatic shearing of incompressible fluids with temperature dependent viscosity, *Quart Appl Math* **41** (1983), 45 - 58.

16. Dafermos CM and Hsiao L, Development of singularities in solutions of the equations of nonlinear thermoelasticity, *Quart Appl Math* **44** (1986), 463-474.

17. Drew DA and Flaherty JE, Adaptive finite element methods and the numerical solution of shear band problems, in *Phase Transformations and Material Instabilities in Solids*, M Gurtin (ed), Academic Press, New York, 1984.

18. Fressengeas C and Molinari A, Instability and localization of plastic flow in shear at high strain rates, *J Mech Phys Solids* **35** (1987), 185-211.

19. Hutchinson JW and Neale KW, Influence of strain-rate sensitivity on necking under uniaxial tension, *Acta Metallurgica* **25** (1977), 839-846.

20. Maddocks JH and Malek-Madani R, Steady-state shear-bands in thermo-plasticity, Part I: Vanishing yield stress (1991) (preprint).

21. Molinari A and Clifton RJ, Analytical characterization of shear localization in thermoviscoplastic materials, *J Appl Mech* **54** (1987), 806-812.

22. Needleman A, Material rate dependence and mesh sensitivity in localization problems, *Comp Methods in Appl Mech and Engineering* **67** (1988), 69 - 85.

23. Shawki TG and Clifton RJ, Shear band formation in thermal viscoplastic materials, *Mech Materials* **8** (1989), 13 - 43.

24. Shawki TG, An energy criterion for the onset of localization in thermal viscoplastic materials. Part I - Necessary and sufficient initiation conditions. Part II - Applications and implications, (1991) (preprint).

25. Tzavaras AE, Shearing of materials exhibiting thermal softening or temperature dependent viscosity, *Quart Appl Math* **44** (1986), 1 - 12.

26. Tzavaras AE, Plastic shearing of materials exhibiting strain hardening or strain softening, *Arch Rational Mech Analysis* **94** (1986), 39 - 58.

27. Tzavaras AE, Effect of thermal softening in shearing of strain-rate dependent materials, *Arch Rational Mech Analysis* **99** (1987), 349 - 374.

28. Tzavaras AE, Strain softening in viscoelasticity of the rate type, *J Integral Equations Appl* **3** (1991), 195-238.

29. Tzavaras AE, Shear strain localization in plastic deformations, in *Shock Induced Transitions and Phase Structures in General Media*, R Fosdick, E Dunn and M Slemrod (eds), IMA Volumes in Mathematics and its Applications, Springer Verlag, 1991 (to appear).

30. Tzavaras AE, On adiabatic shear bands. (in preparation)

31. Wright TW and Batra RC, The initiation and growth of adiabatic shear bands, *Int J Plasticity* **1** (1985), 205-212.

32. Wright TW and Walter JW, On stress collapse in adiabatic shear bands, *J Mech Phys Solids* **35** (1988), 701-720.

33. Wu FH and Freund LB, Deformation trapping due to thermoplastic instability in one-dimensional wave propagation, *J Mech Phys Solids* **32** (1984), 119-132.

34. Zener C, and Hollomon JH, Effect of strain rate upon plastic flow of steel, *J Appl Physics* **15** (1944), 22-32.

35. Zbib HM and Aifantis EC, On the structure and width of shear bands, *Scripta Met* **22** (1988), 703-708.