ON THE ANALYTICITY OF THE SPECTRAL DENSITY FOR SEMICLASSICAL NLS

Spyridon Kamvissis

Department of Applied Mathematics, University of Crete, Greece

ABSTRACT

In [KMM], we analyze the semiclassical behavior of solutions to the focusing, completely integrable nonlinear Schrödinger equation, under the assumption of real analytic initial data (among others). We provide global semiclassical asymptotics under the so-called "finite gap" assumption. In a subsequent paper [KR] we have justified the "finite gap" assumption, again under several assumptions, the main assumption being that the limiting spectral density of the eigenvalues of the associated Dirac operator has an analytic extension in the upper half-plane. In this article, we show that this constraint is unnecessary. In fact, analyticity of the neccessary quantities in the analysis can be recovered via the solution of a scalar Riemann-Hilbert problem.

ACKNOWLEDGEMENT. The research leading to this article was supported by the Max Planck Society of Germany, the European Science Foundation (MISGAM program) and the European Union's Seventh Framework Programme (FP7-REGPOT-2009-1) under grant agreement n 245749. An earlier version of this note has appeared as Max Planck Institute preprint MPIM2002-43 in 2002.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

In [KMM], the authors have analyzed the semiclassical behavior of the solution $u(x,t)$ to the initial value problem for the focusing NLS equation:

$$ \begin{aligned} ihu_t + \frac{h^2}{2}u_{xx} + |u|^2u = 0, \\ u(x,0) = A(x), \end{aligned} \tag{1} $$

where h is small, and $A(x)$ is a real valued function, sufficiently decaying at infinity so that the theory of inverse scattering applies. Furthermore $A(x)$ was assumed to be "bell-shaped", i.e. even, with a single maximum A at $x = 0$, and not too flat at $x = 0$, i.e. $A"(0) < 0$. Finally, and most crucially, the "eigenvalue density" of the associated Lax operator (a Dirac operator) with data $A(x)$, namely $\int_{\mathbb{R}} Re(\frac{\eta}{(A(x)^2+\eta^2)^{1/2}})dx$, was assumed to have an analytic extension in the upper half-plane.

Under these assumptions, several results have been proved. Most importantly,

1. The existence of strong limits for the densities $\rho = |\psi|^2$ and $\mu = hIm(\psi^*\psi_x)$, has been proved, as $h \to 0$, for short times.

2. Asymptotics for the solution u have been provided as $h \to 0$, for any x, t, as long as the so-called "finite gap ansatz" is valid. The existence of weak limits for ρ, μ, as $h \to 0$, for any x, t, follows.

3. In [KR] the validity of the finite gap ansatz has been justified (under a certain simplifying assumption which can actually be dropped; see [K]).

In this article, we show that the assumption of analyticity is not necessary.

THEOREM. The results of [KMM] and [KR] are valid for the semiclassical analysis of (1), under the assumption that $A(x)$ is a C^2 bell-shaped function decaying rapidly enough at infinity (so that an inverse scattering theory is possible). The first result has to be qualified; it may not hold at points of non-analyticity of the initial data.

SKETCH OF PROOF:

It is essential for the proofs in [KMM] that the "density of eigenvalues" $\rho^0(\eta)$ (see (3.2) of [KMM]), derived by WKB theory and a priori defined in the straight line interval connecting 0 to iA, be analytically extensible to the closed upper half-plane $\mathbb{H}$. The main issue is whether the function

$$R^0(\eta) = \int_{x_-(\eta)}^{x_+(\eta)} (A(x)^2 + \eta^2)^{1/2} dx, \tag{2}$$

where the turning points are defined by

$$\begin{aligned} &A(x_\pm(\eta)) = -i\eta,\ 0 < -i\eta < A, \\ &-A < x_-(\eta) < 0 < x_+(\eta) < A, \end{aligned} \tag{3}$$

admits an analytic extension. We note here that we choose the branch of the square root that is positive for $x_- < x < x_+$.

We will show that even if R^0 does not admit an analytic extension in $\mathbb{H}$, the analysis of Chapter 5 in [KMM] can be amended via the solution of a scalar Riemann-Hilbert problem.

Indeed, consider the following scalar additive Riemann-Hilbert problem, with jump on the linear segment $\Sigma = [-iA, iA]$. Let p be a function analytic in $\mathbb{C} \setminus [-iA, iA]$, such that

$$p_+(\eta) + p_-(\eta) = \rho_0(\eta) = \frac{dR^0}{d\eta},\ \eta \in (-iA, iA),$$
$$lim_{\eta \to \infty} p(\eta) = 0.$$

Here $R^0(\eta)$ is extended to the lower half of Σ by the relation $R^0(\eta^*) = R^0(\eta)$. The "+" side is to the left of Σ and the "-" side is to the right of Σ.

Note that if R^0 is entire, then we can choose $p = \rho^0 = 1/2\frac{dR^0}{d\eta}$. In general, our choice of initial data only ensures that ρ^0 is continuous.

Now, the analysis of Chapter 5 in [KMM] can be amended as follows. First, let's amend the definition of X in Chapter 3, which describes the interpolant of the norming constants. We simply set

$$X(\lambda) = i\pi(2K+1) \int_\lambda^{iA} (p_+(\eta) + p_-(\eta)) d\eta,$$

for λ in the linear segment $[0, iA]$. Then, the discussion of Chapter 5 in [KMM], in particular from relation (5.4) to (5.8), is amended by substitutitng $\overline{\rho}^{\sigma} = p - \rho$. More precisely, taking $\sigma = 1$,

$$\int_0^{iA} L_\eta^0(\lambda) p_-(\eta) d\eta = \int_{C_I} L_{\eta-}^C(\lambda) p(\eta) d\eta,$$

and similarly, by symmetry,

$$\int_{-iA}^0 L_\eta^0(\lambda) p_-(\eta^*)^* \, d\eta = \int_{C_I^*} L_{\eta-}^C(\lambda) p(\eta^*)^* \, d\eta.$$

(Recall here that $L_\eta^0(\lambda) = log(\lambda - \eta)$, with a cut along the imaginary axis from η to $-i\infty$. In the above integral we integrate over the "-" side, while in the integral just following we integrate over the "+" side.) Also

$$\int_0^{iA} L_\eta^0(\lambda) p_+(\eta) d\eta = \int_{C_F} L_{\eta-}^C(\lambda) p(\eta) d\eta,$$

and similarly, by symmetry,

$$\int_{-iA}^0 L_\eta^0(\lambda) p_+(\eta^*)^* d\eta = \int_{C_F^*} L_{\eta-}^C(\lambda) p(\eta^*)^* d\eta.$$

Next, note that $L_{\eta+}^C(\lambda) = L_{\eta-}^C(\lambda)$ for all $\eta \in C_I \cup C_I^*$ "below" $\lambda \in C_I$ and at the same time $L_{\eta+}^C(\lambda) = 2\pi i + L_{\eta-}^C(\lambda)$ for $\eta \in C_I$ "above" λ. This means that for $\lambda \in C$,

$$\int_C L_{\eta\pm}^C(\lambda) p(\eta) d\eta + \int_{C^*} L_{\eta\pm}^C(\lambda) p(\eta^*)^* d\eta =$$
$$\int_C \overline{L_\eta^C}(\lambda) p(\eta) d\eta + \int_{C^*} \overline{L_\eta^C}(\lambda) p(\eta^*)^* d\eta \pm \pi i/2 \int_{C_I} p(\eta) d\eta \pm \pi i/2 \int_{C_F} p(\eta) d\eta,$$

with $\overline{L_\eta^C}(\lambda) = \frac{L_{\eta+}^C(\lambda) + L_{\eta-}^C(\lambda)}{2}$. Assembling these results gives the expression

$$\tilde{\phi}(\lambda) = \int_C \overline{L_\eta^C}(\lambda) \overline{\rho}(\eta) d\eta + \int_{C^*} \overline{L_\eta^C}(\lambda) \overline{\rho}(\eta^*)^* d\eta$$
$$+ J(2i\lambda x + 2i\lambda^2 t) - (J(2K+1)+1) \left(\pm \pi i/2 \int_{C_I} p(\eta) d\eta \pm \pi i/2 \int_{C_F} p(\eta) d\eta\right),$$

valid for $\lambda \in C$, where we have introduced the complementary density for $\eta \in C$: $\overline{\rho}(\eta) := p(\eta) - \rho(\eta)$. Choosing K so that $J(2K+1)+1 = 0$, the last term vanishes and we simply have

$$\tilde{\phi}(\lambda) = \int_C \overline{L_\eta^{C,\sigma}}(\lambda) \overline{\rho}(\eta) d\eta + \int_{C^*} \overline{L_\eta^C}(\lambda) \overline{\rho}(\eta^*)^* d\eta + J(2i\lambda x + 2i\lambda^2 t). \quad (4)$$

Compare with (5.11) of [KMM]. Formula (4) is less awkward, since it does not depend on the a priori constraint that the contour C has to go through iA, a constraint that is eventually suspended anyway.

The rest of the proofs of [KMM] go through, with p substituting ρ^0. We omit the detailed discussion, but we *do* stress one major point on the variational problem of Chapter 8 of [KMM].

The contour C and the measure $\rho d\eta$ are characterized by a solution of a Green's variational problem of electrostatic kind. Indeed

$$E_\phi(\rho d\eta) = max_{C'} min_{\mu: supp(\mu)\in C} E_\phi(\mu),$$

where the contours C' are a priori supported in the upper half-plane minus the linear segment $[0, iA]$, and E_ϕ is the weighted energy of a measure with respect to the external field given by

$$\phi(z) = \int log \frac{|z-\eta^*|}{|z-\eta|} \rho^0(\eta) d\eta - Re(i\pi J \int_z^{iA} p(\eta) d\eta + 2iJ(zx + z^2 t)).$$

The harmonicity of ϕ is important to the structure of $C, supp(\rho)$. But again, even if ρ^0 is not analytically extended, it can be written as a sum of two terms that *are*.

One could write ϕ as

$$\phi(z) = \int log \frac{|z-\eta^*|}{|z-\eta|} (p_+ + p_-)(\eta) d\eta - Re(i\pi J \int_z^{iA} p(\eta) d\eta + 2iJ(zx + z^2 t)). \tag{5}$$

Again, this representation is perhaps more natural, since in setting the variational problem it is more appropriate to think of the "left" and "right" sides of the linear segment $[0, iA]$ as distinct.

CONCLUSION: The moral of the story is that if ρ^0 does not admit an entire extension, we can write it as the average of two functions p_-, p_+ that can be extended to the left and right of the segment $[0, iA]$ respectively, and proceed as before, with ρ^0 substituted by p.

CAVEAT: In [KR] we assume that the solution of the variational problem does not touch the spike $[0, iA]$ except possibly at a finite number of points. As shown in

[K], this obstacle can be overcome by setting the variational problem on an infinite sheeted Riemann surface $\mathbb{L}$. For this, we use the analyticity of ρ^0 even across the spike. Now, here we don't have that (in fact this is the whole point of this note). But a careful examination of [K] shows that what we actually need is analyticity across all but one liftings of the spike on $\mathbb{L}$. This we can get by simply setting our scalar Riemann-Hilbert problem on $\mathbb{L}$ and letting the jump be a single copy of the spike $[0, iA]$ in $\mathbb{L}$. The scalar Riemann-Hilbert problem on $\mathbb{L}$ can be explicitly solved by mapping conformally $\mathbb{L}$ to $\mathbb{C}$.

REMARKS: 1. Because of the analysis of [KMM] it is now believed that the behavior of the solutions of the focusing NLS equation, under any analytic initial data, is described semiclassically by algebro-geometric solutions of the same equation, corresponding to Riemann Surfaces of slowly modulated moduli. In this respect, focusing NLS is surprisingly not very different to defocusing NLS. However, the geometry of the phase space of the periodic focusing NLS is much richer, since it also contains complicated homoclinic and heteroclinic manifolds, apart from the invariant tori where algebro-geometric solutions live. For this reason, it has been conjectured that, at least for non-analytic data, the semiclassical focusing NLS should be qualitatively different from the defocusing NLS, and that modulated algebro-geometric solutions should only give part of the semiclassical picture.

The above result shows that this is not the case. We believe that the full geometry of the phase space of the periodic focusing NLS should only be involved in a study of small perturbations of the semiclassical problem. A numerical study of the perturbed problem has been initiated in [BKB].

2. If the initial data $A(x)$ are not analytic, then no solution of the Euler system that appears as a formal limit of (1) is guaranteed. So, no strong limit can in general exist. However, the finite gap ansatz may well hold, albeit with a higher number of "gaps" than one would attain with analyticity.

3. A study of the semiclassical problem for focusing NLS under steplike initial data can be found in [K1].

BIBLIOGRAPHY

[BKB] J.Bergamin, S.Kamvissis, T.Bountis, A Numerical Analysis of the Perturbed Semiclassical Nonlinear Schrödinger Equation, Physics Letters A, v.304, n.3-4, 2002, pp.85-94.

[KMM] S.Kamvissis, K.McLaughlin, P.Miller, Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation, nlin.SI/0012034, 2000; Annals of Mathematics Studies, v.154, Princeton University Press, 2003.

[KR] S. Kamvissis, E. A. Rakhmanov, Existence and Regularity for an Energy Maximization Problem in Two Dimensions, Journal of Mathematical Physics, v.46, n.8, 2005.

[K] S.Kamvissis, From Stationary Phase to Steepest Descent, arXiv:math-ph/0701033; Contemporary Mathematics, v.458, AMS 2008, pp.145-162.

[K1] S.Kamvissis, Semiclassical Focusing NLS with Barrier Data, arXiv:math-ph/0309026.