SEMiclassical WKB PROBLEM FOR THE NON-Self-ADJOINT DIRAC OPERATOR WITH A DECAYING POTENTIAL

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Abstract. In this paper we examine the semiclassical behavior of the scattering data of a non-self-adjoint Dirac operator with a fairly smooth but not necessarily analytic potential decaying at infinity. In particular, using ideas and methods going back to Langer and Olver, we provide the complete rigorous uniform semiclassical analysis of the scattering coefficients and the Bohr-Sommerfeld condition for the location of the eigenvalues. Our analysis is motivated by the potential applications to the focusing cubic NLS equation, in view of the well-known fact discovered by Zakharov and Shabat that the spectral analysis of the Dirac operator is the basis of the solution of the NLS equation via inverse scattering theory. This paper complements and extends a previous work of Fujiié and the second author, which considered a more restricted problem for a strictly analytic potential.

1. Introduction

Consider the initial value problem (IVP) of the one-dimensional focusing nonlinear Schrödinger equation (focusing NLS) for the complex field \( u(x,t) \), i.e.

\[
\begin{aligned}
    i \hbar \partial_t u + \frac{\hbar^2}{2} \partial_x^2 u + |u|^2 u & = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R} \\
    u(x,0) & = A(x), \quad x \in \mathbb{R}
\end{aligned}
\]  

(1.1)

for a real valued function \( A \) and a fixed positive \( \hbar \).

Zakharov and Shabat [22] have proved back in 1972 that the focusing NLS equation is integrable via the Inverse Scattering Transform (IST). A crucial step of the method is the analysis of the following Zakharov-Shabat eigenvalue (EV) (or Dirac) problem

\[
\begin{bmatrix}
    \hbar \partial_x & \hbar \partial_x \\
    \frac{\partial_x}{\partial x} & -\frac{\partial_x}{\partial x}
\end{bmatrix}
\begin{bmatrix}
    v_1(x;\hbar,\lambda) \\
    v_2(x;\hbar,\lambda)
\end{bmatrix}

= 
\begin{bmatrix}
    -i\lambda & A(x) \\
    -A(x) & i\lambda
\end{bmatrix}
\begin{bmatrix}
    v_1(x;\hbar,\lambda) \\
    v_2(x;\hbar,\lambda)
\end{bmatrix}
\]  

(1.2)

where \( \lambda \in \mathbb{C} \) is a “spectral” parameter; here prime denotes differentiation with respect to \( x \).

Now let us suppose that \( \hbar \) is small compared to the \( x, t \) we are interested in. The question raised is then: what is the behavior of solutions of the IVP (1.1) as \( \hbar \downarrow 0 \)? The rigorous analysis of this problem was initiated in [9]. Because of Zakharov-Shabat, the first step in the study of this IVP in the semiclassical limit \( \hbar \downarrow 0 \) has to be the asymptotic spectral analysis of the scattering problem (1.2) as \( \hbar \downarrow 0 \), keeping the function \( A \) fixed.

The EV problem (1.2) cannot be written as an EV problem for a self-adjoint operator. What we study here is a semiclassical WKB problem (or LG problem) for the corresponding non-self-adjoint Dirac operator with potential \( A \).
The question of the semiclassical approximation of the scattering data has a deep significance in view of the instability of the NLS problem which appears in many levels. In fact even away from the semiclassical regime, the focusing NLS is the main model for the so-called “modulational instability” ([4]), although for positive fixed \( \hbar \) the initial value problem is well-posed.

Semiclassically the instabilities become more pronounced. One way to see this is related to the underlying ellipticity of the formal semiclassical limit. To be more specific, consider the well-known Madelung transformation

\[
\begin{align*}
\rho &= |u|^2 \\
\mu &= \hbar \Im (u^* u_x)
\end{align*}
\]

where \( u^* \) denotes the complex conjugate of \( u \). Then the IVP (1.1) becomes

\[
\begin{align*}
\rho_t + \mu_x &= 0 \\
\mu_t + \left( \frac{\mu^2}{\rho} + \frac{\rho}{2} \right)_x &= \hbar^2 \partial_x [\rho (\log \rho)_{xx}]
\end{align*}
\]

with initial data \( \rho(x,0) = |u|^2(x,0) = A^2(x) \) and \( \mu(x,0) = 0 \).

The formal limit as \( \hbar \downarrow 0 \) is

\[
\begin{align*}
\rho_t + \mu_x &= 0 \\
\mu_t + \left( \frac{\mu^2}{\rho} + \frac{\rho}{2} \right)_x &= 0
\end{align*}
\]

with initial data \( \rho(x,0) = |u|^2(x,0) = A^2(x) \) and \( \mu(x,0) = 0 \). This is an IVP for an elliptic system of equations and so one expects that small perturbations of the initial data (independent of \( \hbar \)) can lead to large changes in the solution, at any given time.

Instabilities appear also at the spectral analysis of the related non-self-adjoint Dirac operator, as well as the related equilibrium measure problem [10], the related Whitham equations [9], the possibility of the appearance of rogue waves [3], and even in the numerical studies of the problem [15].

The semiclassical approximation of the scattering data, results in small changes of the initial data that depend on \( \hbar \). It is a priori unclear whether they can have a significant effect in the semiclassical asymptotics of the solution of the IVP (1.1) as \( \hbar \downarrow 0 \). Our ultimate aim is to provide a proof that they do not.

Our work complements the paper [6] of S. Fujiié and the second author where the potential is considered to be a real analytic bell-shaped function and in which the so-called exact WKB method (cf. [5], [19], [8] and [7]) is employed. In this work, we instead suppose that the bell-shaped potential function \( A \) has only some prescribed smoothness which we specify explicitly in §2. Our methods are necessarily different since the exact WKB method requires analyticity. Our ideas are rather influenced by the papers of D. R. Yafaev [20], [21] where an analogous problem is treated for the self-adjoint Schrödinger operator, which in turn rely on works of F. W. J. Olver [16], [17]. More precisely, Yafaev uses results from [17] while we rely heavily on the earlier work [16].

The present paper is arranged as follows. In section §2 we state all the necessary assumptions on the potential function \( A \) so that Olver’s work can be applied in our case. In section §3 we introduce a simple transformation that maps the Dirac...
problem to an equivalent Schrödinger problem. Section §4 shows how the Liouville transformation changes our Schrödinger equation into one containing a “junk term” which is a continuous function on the Liouville plane as will be demonstrated in section §5. The idea is to control this “negligible term” in section §6 so that we obtain approximate solutions expressed with the help of Parabolic Cylinder Functions (PCFs) in the Liouville variable \( \zeta \geq 0 \).

In section §7 we illustrate the previously mentioned results for the special case where the potential function is \( x \mapsto \frac{1}{1+x^2} \). Then in §8 we find the asymptotic behavior of the approximants introduced in §6. In §9 we present some connection formulas that relate the approximate solutions for \( \zeta \geq 0 \) to the ones for \( \zeta \leq 0 \). The significance of this connection becomes clear in §10 where we find Bohr-Sommerfeld quantization conditions for the EVs of our problem.

Next in §11 we study the EVs that lie somewhat close to zero and we extend the above results as we approach zero at distances of order \( \hbar b \) for \( b > 0 \). Finally, the last section §12 is concerned with the amplitudes of the transmission and reflection coefficients.

For the sake of the reader, as the approximate solutions to our problems involve Airy and Parabolic Cylinder Functions, we present all the necessary results concerning these functions in sections A and B of the appendix, where the reader can also find a section on the Variational Operator (section C) used in our proofs and a section (section D) on a theorem concerning integral equations which is the primary tool in the proof of the main Theorem 6.1.

Notationwise, a bar over a letter (or number) does not denote complex conjugation. For complex conjugation we have reserved the superscript “∗”; i.e. \( \bar{z} \) is the complex conjugate of \( z \). Also, the letter \( C \) denotes generically a positive constant (appearing in estimates). Finally, the notation \( f^2(x) \) denotes the square of the value of the function \( f \) at \( x \). Hence, the symbols \( f^2(x) \) and \( f(x)^2 \) are used interchangeably and are not to be confused with the composition \( f \circ f \) of \( f \) with itself.

2. Assumptions on the Potential

In this section we state precise assumptions on the potential function \( A \) which are sufficient to ensure that all the techniques and methods developed in the forthcoming sections can go through easily. In short, we consider “bell-shaped” functions \( A : \mathbb{R} \to \mathbb{R} \) with some smoothness.

To be more precise, we assume that \( A \) satisfies

- \( A(x) > 0 \) for \( x \in \mathbb{R} \)
- \( A(-x) = A(x) \) for \( x \in \mathbb{R} \)
- there exists \( \tau > 0 \) so that \( A(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \) as \( x \to \pm\infty \)
- \( A \) is in \( C^4(\mathbb{R}) \) and of class \( C^5 \) in a neighborhood of 0
- \( xA'(x) < 0 \) for \( x \in \mathbb{R} \setminus \{0\} \)
- \( A''(0) < 0 \); we set \( 0 < A(0) =: A_{\text{max}} \)

Now let \( \mu \in (0, A_{\text{max}}] \subset \mathbb{R} \). Observe that

- for \( \mu \in (0, A_{\text{max}}) \) the equation \( A(x) = \mu \) has exactly two solutions \( x_{\pm} \) which of course depend on \( \mu \) and by the symmetry of \( A \) satisfy \( x_+ = -x_- \).

They are called turning points (or transition points). Furthermore, \( A(x) > \mu \) for \( x \in (x_-, x_+) \) and \( A(x) < \mu \) for \( x \in (-\infty, x_-) \cup (x_+, +\infty) \).
• when $\mu = A_{\text{max}}$ the two turning points coalesce into one double at $x = 0$.

We believe that neither the evenness assumption, nor the single local maximum assumption are strictly necessary. If evenness is not imposed, we have what Klaus & Shaw call a “single lobe” potential. Essentially our discussion in this article goes through mostly unaltered, since the results of Klaus & Shaw mentioned below are still valid. One thing that changes is that the norming constants are no more real, but we still have uniform estimates for them, see Remark 10.4. If the single local maximum assumption is dropped we no more expect to have imaginary eigenvalues. But we do expect to have eigenvalues accumulate along curves as in Section §11 and Bohr-Sommerfeld conditions to appear. A forthcoming paper will hopefully show how to handle more general cases.

Figure 1. A bell-shaped function.

3. FROM DIRAC TO SCHRODINGER

As stated in the introduction, in this paper we examine an eigenvalue problem for a Dirac operator. Specifically, we study the eigenvalue problem

$$D_{\hbar}u = \lambda u$$

(3.1)

where $D_{\hbar}$ is the Dirac operator

$$D_{\hbar} := \begin{bmatrix} i\hbar \partial_x & -iA \\ -iA & -i\hbar \partial_x \end{bmatrix}$$

(3.2)

with $0 < \hbar \ll 1$ a small parameter (Planck), $A$ as in §2, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\lambda \in \mathbb{C}$ as usual plays the role of the spectral parameter.

In (3.1), $\lambda$ is an eigenvalue if the equation has a non-trivial solution $u \in L^2(\mathbb{R}; \mathbb{R}^2)$; that is

$$0 < \int_{-\infty}^{+\infty} \left[ u_1^2(x) + u_2^2(x) \right] dx < +\infty.$$
The continuous spectrum of (3.1) with a potential $A$ satisfying the assumptions of §2 is the whole real line $\mathbb{R}$. On the other hand, the eigenvalues are simple, purely imaginary and symmetric with respect to the real axis. Their imaginary part lies in $[-A_{\max}, A_{\max}]$ (these spectral facts have been established in [11], [12] by M. Klaus and J. K. Shaw). This suggests writing $\lambda = i\mu$. Hence, (3.1) is written as

$$
\hbar \begin{bmatrix} u'_1(x; \hbar, \mu) \\ u'_2(x; \hbar, \mu) \end{bmatrix} = \begin{bmatrix} \mu & A(x) \\ -A(x) & -\mu \end{bmatrix} \begin{bmatrix} u_1(x; \hbar, \mu) \\ u_2(x; \hbar, \mu) \end{bmatrix}.
$$

(3.3)

Under the change of variables (cf. equation (4) in [14])

$$
y_{\pm} = \frac{u_2 \pm u_1}{\sqrt{A + \mu}}
$$

(3.4)

system (3.3) is equivalent to the following two independent eigenvalue equations

$$
y''_{\pm}(x; \hbar, \mu) = [\hbar^{-2}V_0(x; \mu) + F_{\pm}(x; \mu)]y_{\pm}(x; \hbar, \mu)
$$

(3.5)

in which the functions $V_0$ and $F_{\pm}$ are given by

$$
V_0(x; \mu) = \mu^2 - A^2(x)
$$

and

$$
F_{\pm}(x; \mu) = \frac{3}{4} \left[ \frac{A'(x)}{A(x) \mp \mu} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) \mp \mu}.
$$

We will only consider the “minus” case for the lower index (i.e. $y_-, F_-$ in (3.5)) because $F_-$ has no singularities and thus work with the equation

$$
d^2y
\over dx^2 = [\hbar^{-2}V(x; \mu) + F(x; \mu)]y
$$

(3.6)

where $V$ and $F$ satisfy

$$
V(x; \mu) = \mu^2 - A^2(x)
$$

(3.7)

and

$$
F(x; \mu) = \frac{3}{4} \left[ \frac{A'(x)}{A(x) + \mu} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + \mu}.
$$

(3.8)

Observe that the change of variables (3.4) with the “minus” choice does not alter the discrete spectrum; finding the discrete spectrum of (3.3) is equivalent to finding the values of $\mu \in (0, A_{\max})$ for which (3.5) has an $L^2(\mathbb{R})$ solution.

Let us choose any $A_0$ such that $0 < A_0 < A_{\max}$. In the last three equations, $\mu$ will play the role of a spectral parameter in $[A_0, A_{\max}] \subset \mathbb{R}$ where $x$ runs on the whole real line. The function $V(\cdot; \mu)$ is non-vanishing on $\mathbb{R}$ except for two distinct simple zeros (as in §2 these are called turning points or transition points) at $x = x_-$ and $x = x_+$ with $x_- < x_+$, or alternatively a single double zero at $x = 0$. Both $x_-, x_+$ are continuous functions of the parameter $\mu$ and tend to zero as $\mu \uparrow A_{\max}$.

We introduce a change of variables for the spectral parameter $\mu$ in order to rely on the results from §13. For this, we first define the function $B$ to be the restriction of $A$ on $[0, +\infty)$, i.e. $B = A \big|_{[0, +\infty)}$ and note that, by the assumptions on $A$, $B$ is invertible. We set

$$
a = \frac{x_+ - x_-}{2}.
$$

Since we have assumed that $A$ is even for simplicity, we have $a = x_+$ and

$$
A(x_+) = \mu \Leftrightarrow B(a) = \mu \Leftrightarrow a = B^{-1}(\mu).
$$
Since $B$ (and hence $B^{-1}$) is a decreasing function, we get

$$a \in B^{-1}\left([A_0, A_{\text{max}}]\right) = [0, B^{-1}(A_0)] = [0, a_0].$$

Thus, the zeros of $V(\cdot; a) = A^2(a) - A^2(\cdot)$ are located at $x_\pm = \pm a$. Furthermore, the critical value of $a$ is now zero and $a$ ranges over the compact interval $[0, a_0]$ (we should add that $a_0$ and consequently $A_0$ may depend on $\hbar$).

**Figure 2.** The relationship between parameters $\mu$ and $a$.

With this new parameter, equations (3.6), (3.7) and (3.8) are replaced by

$$\frac{d^2y}{dx^2} = [\hbar^{-2}f(x; a) + g(x; a)]y$$

where $f$ and $g$ satisfy

$$f(x; a) = A^2(a) - A^2(x)$$  

and

$$g(x; a) = 3 \left[ \frac{A'(x)}{A(x) + A(a)} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + A(a)}.$$  

Lastly, for convenience we introduce the notation

$$f(x; a) = (x^2 - a^2)p(x; a)$$

where

$$p(\pm a; a) = \mp \frac{A(a)A'(\pm a)}{a} > 0 \quad \text{for} \quad a \in (0, a_0]$$

and

$$p(0; 0) = -A_{\max}A''(0) > 0.$$
4. The Liouville transformation

In this section, we introduce new variables \( Y \) and \( \zeta \) according to the Liouville transform

\[
Y = \dot{x}^{-\frac{1}{2}} y
\]

where the dot signifies differentiation with respect to \( \zeta \). Equation (3.9) becomes

\[
\frac{d^2 Y}{d\zeta^2} = \left[ \hbar^{-2} \dot{x}^2 f(x; a) + \dot{x}^2 g(x; a) + \dot{x} \frac{1}{2} \frac{d^2}{d\zeta^2} (\dot{x}^{-\frac{1}{2}}) \right] Y. \tag{4.1}
\]

In our case, \( f(\cdot; a) \) is negative in \((-a, a)\) and positive in \((-\infty, -a) \cup (a, +\infty)\). Hence we prescribe

\[
\dot{x}^2 f(x; a) = \zeta^2 - \alpha^2, \tag{4.2}
\]

where \( \alpha \geq 0 \) is chosen in such a way that \( x = -a \) corresponds to \( \zeta = -\alpha \) and \( x = a \) to \( \zeta = \alpha \) accordingly. Indeed, after integration, (4.2) yields

\[
\int_{-a}^{x} [-f(t; a)]^{\frac{1}{2}} dt = \int_{-\alpha}^{\zeta} (\alpha^2 - \tau^2)^{\frac{1}{2}} d\tau \tag{4.3}
\]

provided that \(-a \leq x \leq a\). Notice that by taking these integration limits, \(-a\) corresponds to \(-\alpha\). For the remaining correspondence, we require

\[
\int_{-a}^{a} [-f(t; a)]^{\frac{1}{2}} dt = \int_{-\alpha}^{\alpha} (\alpha^2 - \tau^2)^{\frac{1}{2}} d\tau
\]

and hence

\[
\alpha^2 = \frac{2}{\pi} \int_{-a}^{a} [-f(t; a)]^{\frac{1}{2}} dt. \tag{4.4}
\]

For every value of \( \hbar \), relation (4.4) defines \( \alpha \) as a continuous and increasing function of \( a \) which vanishes as \( a \downarrow 0 \) and equals \( \alpha_0 \) when \( a = a_0 \). And so \( \alpha \in [0, \alpha_0] \).

Next, from (4.3) we find

\[
\int_{-a}^{x} [-f(t; a)]^{\frac{1}{2}} dt = \frac{1}{2} \alpha^2 \arccos \left( -\frac{\zeta}{\alpha} \right) + \frac{1}{2} \zeta (\alpha^2 - \zeta^2)^{\frac{1}{2}} \quad \text{for} \quad -a \leq x \leq a \tag{4.5}
\]

with the principal value choice for the inverse cosine taking values in \([0, \pi]\). For the remaining \( x \)-intervals, we integrate (4.2) to obtain

\[
\int_{x}^{a} [-f(t; a)]^{\frac{1}{2}} dt = -\frac{1}{2} \alpha^2 \text{arcosh} \left( -\frac{\zeta}{\alpha} \right) - \frac{1}{2} \zeta (\zeta^2 - \alpha^2)^{\frac{1}{2}} \quad \text{for} \quad x \leq -a \tag{4.6}
\]

and

\[
\int_{x}^{a} [-f(t; a)]^{\frac{1}{2}} dt = -\frac{1}{2} \alpha^2 \text{arcosh} \left( \frac{\zeta}{\alpha} \right) + \frac{1}{2} \zeta (\zeta^2 - \alpha^2)^{\frac{1}{2}} \quad \text{for} \quad x \geq a \tag{4.7}
\]

with \( \text{arcosh}(x) = \ln(x + \sqrt{x^2 - 1}) \) for \( x \geq 1 \).

Equations (4.5), (4.6) and (4.7) show that \( \zeta \) is a continuous and increasing function of \( x \) in \( \mathbb{R} \). Accordingly, this shows that there is a one-to-one correspondence between these two variables. Finally, we substitute (4.2) in (4.1) and obtain

\[
\frac{d^2 Y}{d\zeta^2} = \left[ \hbar^{-2} (\zeta^2 - \alpha^2) + \psi(\zeta; \alpha) \right] Y \tag{4.8}
\]

where

\[
\psi(\zeta; \alpha) = \dot{x}^2 g(x; a) + \dot{x} \frac{1}{2} \frac{d^2}{d\zeta^2} (\dot{x}^{-\frac{1}{2}}). \tag{4.9}
\]
or equivalently
\[
\psi(\zeta; \alpha) = \frac{1}{4} \frac{3 \zeta^2 + 2 \alpha^2}{(\zeta^2 - \alpha^2)^2} + \frac{1}{16} \frac{\zeta^2 - \alpha^2}{f(x; a)} \left\{ 4f(x; a)f''(x; a) - 5[f'(x; a)]^2 \right\} + (\zeta^2 - \alpha^2) \frac{g(x; a)}{f(x; a)} \tag{4.10}
\]
where prime denotes differentiation with respect to \(x\).

In the critical case in which the two (simple) turning points coalesce into one double point, the limiting form of the foregoing transformation is employed by setting \(a = 0\). Hence
\[
\int_{-\infty}^{0} \left[ f(t; 0) \right]^{1/2} dt = \frac{1}{2} \zeta^2 \quad \text{for} \quad x \leq 0 \tag{4.11}
\]
\[
\int_{0}^{\infty} \left[ f(t; 0) \right]^{1/2} dt = \frac{1}{2} \zeta^2 \quad \text{for} \quad x \geq 0 \tag{4.12}
\]
and equations (4.8), (4.9) and (4.10) apply with \(a = \alpha = 0\).

5. CONTINUITY OF \(\psi\)

In this section we prove that the function \(\psi(\zeta; \alpha)\) resulting from the Liouville transformation of \(\S 4\) is continuous in \(\alpha\) and \(\zeta\); a fact that will be used in \(\S 6\) to prove the existence and asymptotic behavior of approximate solutions of equation (4.8).

Our functions \(f, g\) and \(p\) defined by (3.10), (3.11) and (3.12) respectively satisfy the following properties

(i) \(p, \frac{\partial p}{\partial x}, \frac{\partial^2 p}{\partial x^2}\) and \(g\) are continuous functions of \(x\) and \(a\) (this means in \(x\) and \(a\) simultaneously and not separately) in the region \(\mathbb{R} \times [0, a_0]\)

(ii) \(p\) is positive throughout the same region

(iii) \(\frac{\partial^2 p}{\partial x^2}\) is bounded in a neighborhood of the point \((x, a) = (0, 0)\) in the same region and

(iv) \(f\) is a non-increasing function of \(a\) when \(x \in [-a, a]\) and \(a \in [0, a_0]\).

Indeed, (i) and (iii) follow from the fact that \(A\) is in \(C^4\) and of class \(C^5\) in some neighborhood of 0. For (ii) observe the sign of \(f\) and finally (iv) is a consequence of the monotonicity of \(A\) in \([0, +\infty)\).

By Lemma I in Olver’s paper \(\text{[16]}\), the function \(\psi\) defined by (4.9) is continuous in the corresponding region of the \((\zeta, \alpha)\)-plane.

6. APPROXIMATE SOLUTIONS

In this section we exploit the arsenal assembled in the previous sections. We return to (4.8) and state a theorem concerning its approximate solutions. The only thing missing is a way to assess the error. For this, we introduce an error-control function.

First, we define\(^2\)
\[
\Omega(x) = 1 + |x|^\frac{1}{2}. \tag{6.1}
\]
\(^2\text{This is Olver’s balancing function. Actually we could choose any continuous function of the real variable } x \text{ which is positive (except possibly at } x = 0) \text{ and satisfies the asymptotics } \Omega(x) = \mathcal{O}(|x|^{1/2}) \text{ as } x \to \pm \infty.\)
Now for any \( b \leq 0 \) set
\[
I(b) := \sup_{x \in (0, +\infty)} \left\{ \frac{\Psi(x; b)}{\Gamma(\frac{1}{2} - b)} \right\}
\]
where \( M \) is a function defined in terms of parabolic cylinder functions in section B of the appendix. We note that the above supremum is finite for each value of \( b \). This fact is a consequence of (6.1) and the first relation in (B.9). Furthermore, because the relations (B.9) hold uniformly in compact intervals of the parameter \( b \), the function \( I \) is continuous.

This allows us to define the error-control function by
\[
H(\zeta; h, \alpha) = \int \frac{\psi(\zeta; \alpha)}{\Omega(\zeta \sqrt{2h^{-1}})} d\zeta
\]
in which the choice of integration constant is immaterial. We are now ready for the core theorem of this section.

**Theorem 6.1.** For each value of \( h \), equation (6.8) in the region \([0, +\infty) \times [0, \alpha_0]\) of the \((\zeta, \alpha)\)-plane has solutions \( Y_1 \) and \( Y_2 \) which are continuous, first and second partial \( \zeta \)-derivatives, and are given by
\[
Y_1(\zeta; h, \alpha) = U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2) + \epsilon_1(\zeta; h, \alpha)
\]
\[
Y_2(\zeta; h, \alpha) = U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2) + \epsilon_2(\zeta; h, \alpha)
\]
where \( U, U \) are the PCFs defined in Appendix B. Also,
\[
\frac{\epsilon_1(\zeta; h, \alpha)}{M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)} \leq \frac{1}{E(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)} \left\{ \exp \left[ \frac{1}{2} (\pi h)^{\frac{1}{2}} I(-\frac{1}{2}h^{-1} \alpha^2) \right] \right\} - 1
\]
and
\[
\frac{\epsilon_2(\zeta; h, \alpha)}{M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)} \leq \frac{1}{E(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)} \left\{ \exp \left[ \frac{1}{2} (\pi h)^{\frac{1}{2}} I(-\frac{1}{2}h^{-1} \alpha^2) \right] \right\} - 1
\]

**Proof.** By Theorem D.2 (cf. Theorem I in [16]), it suffices to prove two things. First that the function \( \psi \) is continuous in the region \([0, +\infty) \times [0, \alpha_0]\) and second that the integral (for the variational operator \( V \) see appendix C)
\[
V_{0, +\infty}(H) = \int_0^{+\infty} \frac{\psi(t; \alpha)}{\Omega(t \sqrt{2h^{-1}})} dt
\]
converges uniformly in \( \alpha \). The first assertion has already been proven in §4. For the second, we argue as follows.

We have \( f(x; \alpha) \sim A^2(\alpha) \) as \( x \to +\infty \) and \( \zeta^2 - \alpha^2 \sim \zeta^2 \) when \( \zeta \to +\infty \). Since by (4.7) \( x \to +\infty \) as \( \zeta \to +\infty \), (4.2) is translated as \( \hat{x} \sim \frac{1}{A(\alpha)} \) for \( \zeta \to +\infty \). Choosing \( x_0 \) to satisfy \( \zeta(x_0) = 0 \), we have \( \int_{x_0}^x \frac{\partial x}{\partial \alpha} d\alpha \) as \( \zeta \to +\infty \) or equivalently \( x \sim \frac{1}{l(\alpha)} \zeta^2 \) when \( \zeta \to +\infty \). Having in mind that \( A(x) = O(x^{-(n+1)}) \) as \( x \to +\infty \), these results combined with the assumptions of §4 (3.11) and (4.10) show...
that $|\psi(\zeta; \alpha)|/\zeta^{\frac{1}{2}}$ is integrable at $\zeta = +\infty$ uniformly with respect to $\alpha$. For $\Omega(x) = 1 + |x|^{\frac{1}{2}}$ the variation (6.8) is finite.

7. An Example

In this section we illustrate the theory developed so far to the special case of the potential $A(x) = \frac{1}{1 + x^2}$, $x \in \mathbb{R}$.

First, observe that this particular potential $A$ satisfies the assumptions of §2 indeed it is always positive, even, smooth, $\|A\|_{L^1(\mathbb{R})} = \pi$, it is increasing in $(-\infty, 0]$ and decreasing in $[0, +\infty)$, it has a maximum at $x = 0$, namely $A_{\text{max}} = A(0) = 1$, and if $\mu \in (0, 1)$ the equation $A(x) = \mu$ gives the two turning points $x_{\pm} = \pm \sqrt{\mu^{-1} - 1}$ while for $\mu = 1$ we get a double solution $x = 0$.

When $\mu \in [A_0, 1]$ for $A_0 > 0$ ($\mu = 1$ corresponding to the critical case), the parameter $a = x_+ = \sqrt{\mu^{-1} - 1}$ ranges over $[0, a_0]$ where $a_0 = \sqrt{A_0^{-1} - 1}$ (the criticality now being $a = 0$). The equation in question is

$$\frac{d^2y}{dx^2} = [h^{-2} f(x; a) + g(x; a)]y$$

(7.1)

where $f$ and $g$ satisfy

$$f(x; a) = A^2(a) - A^2(x) = \frac{(x^2 - a^2)(x^2 + a^2 + 2)}{[(1 + a^2)(1 + x^2)]^2}$$

(7.2)

and

$$g(x; a) = \frac{3}{4} \frac{A'(x)}{A(x) + A(a)} - \frac{1}{2} \frac{A''(x)}{A(x) + A(a)} = \frac{(1 + a^2)(-3x^4 - 2x^2 + a^2 + 2)}{[(1 + x^2)(x^2 + a^2 + 2)]^2}.$$  

(7.3)

The function $p$ that satisfies $f(x; a) = (x^2 - a^2)p(x; a)$ is

$$p(x; a) = \frac{x^2 + a^2 + 2}{[(1 + a^2)(1 + x^2)]^2}. 

(7.4)

For the non-critical case, applying the Liouville transform

$$Y = \dot{x}^{-\frac{1}{2}}y, \quad \dot{x}^2 f(x; a) = \zeta^2 - a^2,$$

where $\alpha \in (0, a_0]$ in which $a_0 > 0$ satisfies

$$\alpha_0^2 = \frac{2}{\pi} \int_{-\alpha_0}^{\alpha_0} [-f(t; a_0)]^{\frac{1}{2}} dt = \frac{4}{\pi(1 + a_0^4)} \int_0^{a_0} \sqrt{(a_0^2 - t^2)(t^2 + a_0^2 + 2)} dt,$$

(cf. (4.4)) we get

$$\frac{1}{1 + a^2} \int_{-a}^{a} \frac{\sqrt{(t^2 - a^2)(t^2 + a^2 + 2)}}{1 + t^2} dt = -\frac{1}{2} \alpha^2 \text{arccosh} \left( -\frac{\zeta}{a} \right) - \frac{1}{2} \zeta (\zeta^2 - a^2)^{\frac{1}{2}} \text{ for } x \leq -a$$

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\[11\] The potential $A$ is given by $A(x) = \sqrt{1 + x^2}$.
(cf. (4.6)) and
\[
\frac{1}{1 + a^2} \int_{-a}^{x} \frac{\sqrt{(a^2 - t^2)(t^2 + a^2 + 2)}}{1 + t^2} \, dt =
\frac{1}{2} a^2 \arccos \left( -\frac{\zeta}{\alpha} \right) + \frac{1}{2} \left( \alpha^2 - \zeta^2 \right)^{\frac{1}{2}} \text{ for } -a \leq x \leq a
\]
(cf. (4.5)), in which the inverse cosine takes its principal value (i.e. the value in \([0, \pi]\)) and
\[
\frac{1}{1 + a^2} \int_{a}^{x} \frac{\sqrt{(t^2 - a^2)(t^2 + a^2 + 2)}}{1 + t^2} \, dt =
-\frac{1}{2} a^2 \text{arccosh} \left( \frac{\zeta}{\alpha} \right) + \frac{1}{2} \left( \zeta^2 - \alpha^2 \right)^{\frac{1}{2}} \text{ for } x \geq a
\]
(cf. (4.7)). Additionally, equation (7.1) is transformed to
\[
\frac{d^2 Y}{d \zeta^2} = \left[ h^{-2} (\zeta^2 - \alpha^2) + \psi(\zeta; \alpha) \right] Y
\]
where
\[
\psi(\zeta; \alpha) = \frac{1}{4} \frac{3 \zeta^2 + 2 a^2}{(\zeta^2 - \alpha^2)^2} - (1 + a^2)^4 (\zeta^2 - \alpha^2)^2 \frac{5 x^6 + 9 x^4 + 3 x^2 + a^4 + 2 a^2}{(x^2 - a^2)(x^2 + a^2 + 2)^3}
+ (1 + a^2)^3 (\zeta^2 - \alpha^2)^2 \frac{-3 x^4 - 2 x^2 + a^2 + 2}{(x^2 - a^2)(x^2 + a^2 + 2)^3}.
\]

In the critical case (\(a = \alpha = 0\)) we have
\[
\int_{0}^{x} \frac{t \sqrt{2 + t^2}}{1 + t^2} \, dt = \frac{1}{2} \zeta^2 \text{ for } x \in \mathbb{R}
\]
(cf. (4.11) and (4.12)) and
\[
\psi(\zeta; 0) = \frac{3}{4} \frac{1}{\zeta^2} - \frac{\zeta^2}{x^4(x^2 + 2)^3} \left( 3 x^6 + 7 x^4 + 7 x^2 + 3 \right).
\]

We would like to note that all the integrals above can be reduced and subsequently evaluated using elliptic integral formulas and tables.

Using (7.4), easy but tedious calculations yield
\[
\frac{\partial p}{\partial x}(x; a) = \frac{2 x (-x^2 - 2 a^2 - 3)}{(1 + a^2)^2(1 + x^2)^3}
\]
\[
\frac{\partial^2 p}{\partial x^2}(x; a) = \frac{2(3 x^4 + 10 a^2 x^2 + 12 x^2 - 2 a^2 - 3)}{(1 + a^2)^2(1 + x^2)^4}
\]
\[
\frac{\partial^3 p}{\partial x^3}(x; a) = \frac{24 x (-x^4 - 5 a^2 x^2 - 5 x^2 + 3 a^2 + 4)}{(1 + a^2)^2(1 + x^2)^5}
\]
and from (7.2) we have
\[
\frac{\partial f}{\partial a}(x; a) = -\frac{4 a}{(1 + a^2)^3}.
\]

Hence, these last observations about \(f\) and \(p\) along with (7.3) clearly show that
(i) \(p, \frac{\partial p}{\partial x}, \frac{\partial^2 p}{\partial x^2}\) and \(g\) are continuous functions in the region \(\mathbb{R} \times [0, a_0]\)
(ii) \(p\) is positive in \(\mathbb{R} \times [0, a_0]\)
(iii) \(\frac{\partial^3 p}{\partial x^3}(0; 0) = 0\) and
is equal to

\[
\psi(\zeta; \alpha) = \int \frac{\psi(\zeta; \alpha)}{\Omega(\sqrt{2h-1})} d\zeta
\]

where \( \psi \) is given by (7.5) or (7.6) and \( \Omega \) satisfies \( \Omega(x) = O(|x|^{\frac{3}{2}}) \) as \( x \to \pm \infty \), has a variation \( V_{a,\infty}(H) \) that converges uniformly for \( \alpha \in [0, \alpha_0] \) as \( h \downarrow 0 \). Finally, we can obtain the two specific approximate solutions guaranteed by Theorem 6.1.

8. Asymptotic Behavior of Solutions

In order to deduce the asymptotic behavior of the solutions \( Y_1(\zeta; h, \alpha), Y_2(\zeta; h, \alpha) \) when \( h \downarrow 0 \), we need to determine the asymptotic form of the error bounds (6.6), (6.7) examining closely \( l(-\frac{1}{2}h^{-1/2}) \) and \( V_{a,\infty}(H) \) as \( h \downarrow 0 \).

We start by investigating \( l(b) \) as in (6.2) for \( b \to -\infty \). Take \( \mu > 1 \) to be a large positive number and set \( b = -\frac{1}{2} \mu \) and \( x = \mu y \). Then by (B.8), (B.5) and (B.6) the quantity

\[
\frac{\psi(-\frac{1}{2} \mu^2)}{\Gamma(\frac{1}{2} + \frac{1}{2} \mu^2)}
\]

is equal to

\[
\sqrt{\frac{16\pi \mu^{-\frac{2}{3}} \eta}{y^2 - 1}} \left[ \begin{array}{ll}
\text{Ai}^2(\mu \frac{2}{3} \eta) + \text{Bi}^2(\mu \frac{2}{3} \eta) + E^2(\mu \frac{2}{3} \eta) M^2(\mu \frac{2}{3} \eta) O(\mu^{-2}) \end{array} \right], \quad 0 \leq y \leq \frac{\rho(-\frac{1}{2} \mu^2)}{\mu \sqrt{2}}
\]

where in each case, the estimate \( O(\mu^{-2}) \) is uniform with respect to \( y \). Using (B.7) we see that the changeover point from the one formula to the other satisfies

\[
\frac{\rho(-\frac{1}{2} \mu^2)}{\mu \sqrt{2}} = 1 + 2^{-\frac{2}{3}} c, \mu^{-\frac{2}{3}} + O(\mu^{-\frac{4}{3}}) \quad \text{as} \quad \mu \to +\infty
\]

and consequently \( \mu \frac{2}{3} \eta = c, + O(\mu^{-\frac{2}{3}}) \). But \( E \) is bounded in \([0, c, + O(\mu^{-\frac{2}{3}})]\). Hence we may write (8.1) as

\[
\frac{4 \sqrt{\pi}}{\mu \frac{2}{3}} \left( \frac{\eta}{y^2 - 1} \right)^{\frac{1}{2}} \left[ \begin{array}{ll}
\text{Ai}^2(\mu \frac{2}{3} \eta) + \text{Bi}^2(\mu \frac{2}{3} \eta) + M^2(\mu \frac{2}{3} \eta) O(\mu^{-2}) \end{array} \right], \quad 0 \leq y \leq \frac{\rho(-\frac{1}{2} \mu^2)}{\mu \sqrt{2}}
\]

where the \( O \)-terms are again uniform in \( y \).

Next, we employ the asymptotic approximations for the functions \( \text{Ai}, \text{Bi} \) and \( M \) (cf. section \( A \) in appendix) so that for \( y \geq 1 \) we obtain

\[
\frac{M(\mu y \sqrt{2}; -\frac{1}{2} \mu^2)}{\Gamma(\frac{1}{2} + \frac{1}{2} \mu^2)} \leq \frac{C}{\mu \frac{2}{3} \eta} \left( \frac{\eta}{y^2 - 1} \right)^{\frac{1}{2}} \frac{1}{1 + \mu \frac{2}{3} \eta^2}
\]

\( \ast \) The subsequent analysis follows the idea found in §6.2 of [10].

\( \dagger \) In this paragraph, \( \mu \) has nothing to do with the spectral parameter introduced in \( \S 3 \). It is just a large positive number as mentioned above.
where $C$ denotes a positive constant, used generically in what follows. By \((8.4)\) we have $\eta \sim (\frac{1}{2} \eta^2 y^2)$ as $y \to +\infty$, whence for $y \geq 0$ the estimate
\[
\left( \frac{\eta}{y^2 - 1} \right)^{\frac{1}{2}} \leq \frac{C}{1 + \eta^2}.
\] (8.3)

Also, from the continuity of $\Omega$ and its asymptotics \((6.1)\) we have
\[
\Omega(\mu y \sqrt{2}) \leq C(1 + \mu^\frac{3}{2} y^\frac{3}{2}) \leq C \mu^\frac{3}{2} (1 + \eta^\frac{3}{2}).
\] (8.4)

Finally, combining \((8.2)\), \((8.3)\) and \((8.4)\) we get
\[
\Omega(\mu y \sqrt{2}) \frac{M(\mu y \sqrt{2}, -\frac{1}{2} \mu^2)^2}{\Gamma(\frac{1}{2} + \frac{1}{2} \mu^2)} \leq C \frac{1}{1 + \mu^\frac{3}{2} \eta^\frac{3}{2}} \leq C
\]
implying that
\[
l(-\frac{1}{2} h^{-1} \alpha^2) = O(1) \quad \text{as} \quad h \downarrow 0.
\] (8.5)

Next, we examine $\mathcal{V}_{0, +\infty}(H)$. In the proof of Theorem \((6.1)\) we showed that $|\psi(\zeta; \alpha)|/\zeta^\frac{3}{2}$ is integrable at $\zeta = +\infty$ uniformly with respect to $\alpha$ and prescribed $\Omega(x) = 1 + |x|^\frac{3}{2}$ yielding the finite variation \((6.8)\). Thus
\[
\mathcal{V}_{0, +\infty}(H) \leq C \int_0^1 \frac{dt}{1 + (t^{\sqrt{2}h^{-1}})^{\frac{3}{2}}} \quad (8.6)
\]
\[
+ \left( \frac{\hbar}{2} \right) \int_1^{+\infty} \frac{|\psi(t; \alpha)|}{t^3} dt = O(h^{1/6}) \quad \text{as} \quad h \downarrow 0.
\]

The last two relations applied to \((6.6)\) and \((6.7)\) supply us with the required results:\(^5\)
\[
\varepsilon_1(\zeta; h, \alpha) = \frac{M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2)}{M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2)} O(h^\frac{3}{2})
\] (8.7)
\[
\varepsilon_2(\zeta; h, \alpha) = E(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2) M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2) O(h^\frac{3}{2})
\]
\[
\frac{\partial \varepsilon_1}{\partial \zeta} (\zeta; h, \alpha) = \frac{N(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2)}{N(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2)} O(h^\frac{3}{2})
\]
\[
\frac{\partial \varepsilon_2}{\partial \zeta} (\zeta; h, \alpha) = E(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2) N(\zeta \sqrt{2h^{-1}}; -\frac{1}{2} h^{-1} \alpha^2) O(h^\frac{3}{2})
\]
as $h \downarrow 0$ uniformly for $\zeta \geq 0$ and $\alpha \in [0, \alpha_0]$.

Closing this section, a remark has to be made about the interesting situation in which $\alpha = 0$ (i.e. when equation \((4.8)\) has a double turning point at $\zeta = 0$). Once again as in the proof of Theorem \((6.1)\) $|\psi(\zeta; 0)|/\zeta^\frac{3}{2}$ is integrable at $\zeta = +\infty$. Furthermore $l(0)$ is independent of $\hbar$ and by taking $\Omega(x) = 1 + |x|^\frac{3}{2}$ we see that the estimates above remain unchanged.

\(^5\)Observe that since $|\psi(\zeta; \alpha)|$ is integrable at $\zeta = +\infty$ the same results could be achieved by demanding $\Omega(x) = 1$ for all $x$. We chose to present the general case since it is more broadly applicable.
9. Connection Formulae

The results obtained so far are somewhat inadequate because Theorem 6.1 defines the character of solutions of equation (4.8) only for non-negative values of $\zeta$. Indeed, we are incapable of constructing error bounds like those ones in (6.6) and (6.7) for negative $\zeta$, a drawback pertinent to the nature of parabolic cylinder functions (cf. Miller’s [13]).

Consider $Y_1$ for example. As $\hbar \downarrow 0$ in a continuous manner, the asymptotic behavior of its approximant $U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)$ at $\zeta = -\infty$, changes abruptly as $h^{-1} \alpha^2$ goes through odd positive integers (cf. appendix B and exceptional values). $Y_1$ on the other hand is not expected to exhibit the same change at exactly the same values of $h^{-1} \alpha^2$.

But we can determine the asymptotic behavior of $Y_1, Y_2$ for small $h > 0$ and $\zeta < 0$ by establishing appropriate connection formulae. Since $|\psi(\zeta; \alpha)|/|\zeta|$ is integrable at $\zeta = \pm \infty$ uniformly with respect to $\alpha$, we can replace $\zeta$ by $-\zeta$ and appeal to Theorem 6.1 to ensure two more solutions $Y_3, Y_4$ of equation (4.8) satisfying

$$Y_3(\zeta; h, \alpha) = U(-\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2) + \frac{M(-\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)}{E(-\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2)} O(h^\frac{3}{2})$$

(9.1)

$$Y_4(\zeta; h, \alpha) = \overline{U}(-\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2) + E(-\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2) \overline{M}(-\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1} \alpha^2) O(h^\frac{3}{2})$$

(9.2)

as $h \downarrow 0$ uniformly for $\zeta \leq 0$ and $\alpha \in [0, \alpha_0]$. We express $Y_1, Y_2$ in terms of $Y_3, Y_4$ and write

$$Y_1(\zeta; h, \alpha) = \sigma_1^1 Y_3(\zeta; h, \alpha) + \sigma_1^2 Y_4(\zeta; h, \alpha)$$

(9.3)

$$Y_2(\zeta; h, \alpha) = \sigma_2^1 Y_3(\zeta; h, \alpha) + \sigma_2^2 Y_4(\zeta; h, \alpha)$$

(9.4)

The connection will become clear once we find approximations for the coefficients $\sigma^i_j, i, j = 1, 2$ in the linear relations (9.3) and (9.4).

Evaluating at $\zeta = 0$ both equations (9.3) and (9.4) and their derivatives, after algebraic manipulations we obtain

$$\sigma^i_j = (-1)^{j+1} \frac{W[Y_3(\zeta; h, \alpha), Y_{5-i}(\zeta; h, \alpha)](0)}{W[Y_3(\zeta; h, \alpha), Y_4(\zeta; h, \alpha)](0)}$$

for $i, j = 1, 2$. (9.5)

Similar thinking can be argued for $Y_2$ as well.
But using the results and properties of parabolic cylinder functions and their auxiliary functions from section 13 in the appendix, we find
\[
Y_1(0; \hbar, \alpha) = M(0)[\sin \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y_2(0; \hbar, \alpha) = M(0)[\cos \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y_3(0; \hbar, \alpha) = M(0)[\sin \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y_4(0; \hbar, \alpha) = M(0)[\cos \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y'_1(0; \hbar, \alpha) = -\sqrt{2\hbar^{-1}N(0)}[\cos \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y'_2(0; \hbar, \alpha) = \sqrt{2\hbar^{-1}N(0)}[\sin \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y'_3(0; \hbar, \alpha) = \sqrt{2\hbar^{-1}N(0)}[\cos \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})] \\
Y'_4(0; \hbar, \alpha) = -\sqrt{2\hbar^{-1}N(0)}[\sin \varphi + \mathcal{O}(\hbar^{\frac{5}{2}})]
\]
as \hbar \downarrow 0 where \( \varphi = (1 + \hbar^{-1}\alpha^2)^{\frac{\pi}{4}} \). Finally substituting these estimates in (9.5) we obtain
\[
\sigma_1^1 = \sin\left(\frac{1}{2} \pi \hbar^{-1} \alpha^2\right) + \mathcal{O}(\hbar^{\frac{5}{2}}) \tag{9.6}
\]
\[
\sigma_1^2 = \cos\left(\frac{1}{2} \pi \hbar^{-1} \alpha^2\right) + \mathcal{O}(\hbar^{\frac{5}{2}}) \tag{9.7}
\]
\[
\sigma_2^1 = \cos\left(\frac{1}{2} \pi \hbar^{-1} \alpha^2\right) + \mathcal{O}(\hbar^{\frac{5}{2}})
\]
\[
\sigma_2^2 = -\sin\left(\frac{1}{2} \pi \hbar^{-1} \alpha^2\right) + \mathcal{O}(\hbar^{\frac{5}{2}})
\]
as \hbar \downarrow 0 uniformly for \( \alpha \in [0, \alpha_0] \).

10. A Quantization Condition for Eigenvalues

In this section, we will derive information about the eigenvalues of (3.1) by assembling the results of the previous paragraphs. This process will be facilitated by the equivalent equation (4.8) where eigenvalues appear for those values of \( \alpha \) for which there exists a solution that is decaying at both \( \zeta = -\infty \) and \( \zeta = +\infty \) of the real line. In the end, this approach will help us establish a quantization condition for the eigenvalues which in turn will provide a conclusion about the number of eigenvalues found in a given interval.

We have the following theorem

**Theorem 10.1.** Suppose that \( \lambda = i\mu \) is an eigenvalue of the operator \( \mathcal{D}_\hbar \) (see (3.2)). Consider a such that \( \mu = A(\alpha) \) (clearly \( \alpha \) depends on \( \hbar \) as well). Then there exists a non-negative integer \( n \) (depending both on \( \hbar \) and \( \mu \)) for which the Bohr-Sommerfeld quantization condition is satisfied, i.e.
\[
\int_{-a}^{a} [A^2(x) - \mu^2]^{1/2} \, dx = \pi \left(n + \frac{1}{2}\right)\hbar + \mathcal{O}(\hbar^{\frac{5}{2}}) \quad \text{as} \quad \hbar \downarrow 0. \tag{10.1}
\]
Conversely, for every non-negative integer \( n \) such that \( \pi (n + \frac{1}{2})\hbar \in [0, \alpha_0^2] \) there exists a unique \( \mu_n(\hbar) = A[a_n(\hbar)] \) satisfying
\[
\left| \int_{-a_n(\hbar)}^{a_n(\hbar)} [A^2(x) - \mu_n(\hbar)^2]^{1/2} \, dx - \pi \left(n + \frac{1}{2}\right)\hbar \right| \leq C\hbar^{\frac{5}{2}}
\]
with a constant \( C \) depending neither on \( n \) nor on \( \hbar \).
Proof. For the first part of the theorem, we observe the following. By referring to the asymptotic form of $Y_1(\zeta; h, \alpha)$ as $\zeta \to +\infty$ and the asymptotics for $Y_3(\zeta; h, \alpha)$ and $Y_4(\zeta; h, \alpha)$ as $\zeta \to -\infty$ (see (6.4), (8.7), (9.1) and (9.2)), equation (9.3) implies that in the presence of an eigenvalue, the coefficient $\sigma^2$ has to be zero. Accordingly, by (9.7) we have
\[
\cos\left(\frac{1}{2} \pi \delta^{-1} \alpha^2\right) = \mathcal{O}(\delta^2) \quad \text{as } \delta \downarrow 0
\]
or equivalently, there is a non-negative integer such that
\[
\alpha^2 = (2n + 1) \delta + \mathcal{O}(\delta^2) \quad \text{as } \delta \downarrow 0. \tag{10.2}
\]
In view of (4.4), this is exactly what we wanted.

For the converse: let us first prove existence. Define the map $\Phi : [0, a_0] \to \mathbb{R}$ by
\[
\Phi(a) := \frac{\pi}{2} \alpha^2(a) \tag{10.3}
\]
(cf. (4.4) and/or the LHS of (10.1)). Fix a non-negative integer $n$ such that $\pi (n + 1/2) \delta$ belongs to a neighborhood of $\Phi(\tilde{a})$ where $\tilde{\alpha} = \alpha(\tilde{a})$ and $A(\tilde{a}) = \tilde{\mu}$. From (9.2) we know that the functions $Y_1$, $Y_3$ belong to $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ respectively. It is enough to show that $\sigma(h, \alpha) := \sigma^2(h, \alpha(\alpha))$ vanishes for some $a$ -which will be denoted by $a_n(h)$ and similarly $\alpha(a_n(h)) = \alpha_n(h)$- satisfying
\[
\left| \Phi(a_n(h)) - \pi \left( n + \frac{1}{2} \right) \delta \right| \leq C h^{\frac{2}{3}}.
\]
Using (4.4) and Leibniz’s rule we have
\[
\frac{\partial \Phi}{\partial a}(a) = -A(a)A'(a) \int_{-\alpha}^{\alpha} [A^2(t) - A^2(a)]^{-1/2} dt > 0
\]
This result tells us that $\Phi$ maps a neighborhood $(a_1, a_2)$ of $\tilde{a}$ in a one-to-one way onto the neighborhood $(\Phi(a_1), \Phi(a_2))$ of $\Phi(\tilde{a})$. Let $X = \Phi(a)$, $a \in [0, a_0]$, $\tilde{X} = \Phi(\tilde{a})$ and set
\[
\chi(h, X) := \sigma(h, \Phi^{-1}(X)) - \cos(h^{-1}X), \quad X \in \Phi([0, a_0])
\]
By definition of $\sigma$ and (9.7) we have $|\chi(h, X)| \leq C h^{\frac{2}{3}}$ for $X$ in a neighborhood of $\tilde{X}$ where once more the constant $C$ is independent of $h$ and $X$. With the above definitions, our equation now reads
\[
0 = \sigma(h, a) = \chi(h, X) + \cos(h^{-1}X)
\]
So this equation has to have a solution $X_n(h)$ satisfying the estimate:
\[
\left| X_n(h) - \pi \left( n + \frac{1}{2} \right) \delta \right| \leq C h^{\frac{2}{3}}.
\]
A change of variables $s = h^{-1}X$ transforms our problem to the equivalent assertion that equation
\[
\chi(h, hs) + \cos s = 0 \tag{10.4}
\]
has to have a solution with respect to $s$, namely $s_n(h)$, such that
\[
\left| s_n(h) - \pi \left( n + \frac{1}{2} \right) \right| \leq C h^{\frac{2}{3}}. \tag{10.5}
\]
\[7\]Here we follow Yafaev’s idea found in §4 of [20].
But this is true because
\[ \chi(h, hs) = O(h^{\frac{7}{3}}) \quad \text{as} \quad h \downarrow 0. \]

To complete the proof of the theorem, we need uniqueness as well. Once again fix
\[ n \in \mathbb{Z}. \]
We have just proved that for this \( n \), equation (10.4) has a solution obeying
\[ (10.5). \]
We shall employ reductio ad absurdum. Suppose, on the contrary, that
there are \( s_1, s_2 \) - with \( s_1 < s_2 \) - satisfying (10.5) so that the function
\[ g(s) := \chi(h, hs) + \cos s, \quad s \in [s_1, s_2] \]
is zero; \( g(s_1) = g(s_2) = 0. \) Furthermore, \( g \) is continuous in \([s_1, s_2]\) and differentiable
in \((s_1, s_2)\) with
\[ g'(s) = h \frac{\partial \chi}{\partial X}(h, hs) - \sin s, \quad s \in (s_1, s_2). \]

By Rolle’s theorem there is \( \tilde{s} \in (s_1, s_2) \) such that
\[ 0 = g' (\tilde{s}) \]
\[ = h \frac{\partial \chi}{\partial X}(h, h\tilde{s}) - \sin \tilde{s}. \]

Recapping, we have found
- \( \tilde{s} \in (s_1, s_2) \) which says that \( \tilde{s} \) satisfies (10.5) too; namely
\[ \tilde{s} = \pi \left( n + \frac{1}{2} \right) + O(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0 \quad (10.6) \]
- \( \tilde{s} \) is a root of the equation
\[ \sin s = h \frac{\partial \chi}{\partial X}(h, h\tilde{s}). \quad (10.7) \]

Using (10.6), the left-hand side of (10.7) is seen to be \((-1)^n\) as \( h \downarrow 0 \). Now, using
(9.7) observe that
\[ \frac{\partial \sigma}{\partial a}(h, a) = -h^{-1} \Phi'(a) \sin \left[ h^{-1} \Phi(a) \right] + O(1) \quad \text{as} \quad h \downarrow 0 \]
which eventually leads to
\[ \frac{\partial \chi}{\partial X}(h, X) = O(1) \quad \text{as} \quad h \downarrow 0. \]

Hence the right-hand side of (10.7) is \( O(h) \) as \( h \downarrow 0 \). A contradiction. Thus, there
is only one such eigenvalue. \( \square \)

**Remark 10.2.** A result like equation (10.1) can also be found in [20] for the
Schrödinger operator, with the slightly better asymptotic estimate of order \( h^2 \). Al-
though the result we provide here is only \( O(h^{\frac{5}{3}}) \), it has the additional advantage of
holding for the critical case of a double turning point as well.

The following corollary is a straightforward application of the Theorem 10.1
giving the number of eigenvalues of the Dirac operator \( \mathfrak{D}_h \) in a fixed (independent
of \( h \)) interval not containing 0, on the imaginary axis.

**Corollary 10.3.** Let an interval \( (\mu_1, \mu_2) \subset [A_0, A_{\text{max}}] \) belong to a neighborhood of
a point \( \mu \) and take \( A(a_j) = \mu_j \) for \( j = 1, 2 \). Then the total number \( N_h \) of eigenvalues
\( \lambda = i\mu \) of the Dirac operator \( \mathfrak{D}_h \) lying in the set \( \{ i\mu \mid \mu \in (\mu_1, \mu_2) \} \subset \mathbb{C} \) is equal to
\[ N_h = \pi^{-1} [\Phi(a_1) - \Phi(a_2)] h^{-1} + r(h) \quad (10.8) \]
where \( |r(h)| \leq 1 \) for sufficiently small \( h \).
\[ \mu \text{-space} \]

\[ \mu_1 \quad \mu \quad \mu_2 \]

\[ a \text{-space} \]

\[ a_1 \quad a \quad a_2 \]

\[ X \text{-space} \]

\[ \Phi(a_1) \quad \Phi(a_2) \quad \pi(n + \frac{1}{2})h \]

\[ \Phi \]

**Figure 3.** Counting eigenvalues using the Bohr-Sommerfeld condition.

**Proof.** Observe that

\[ N_h = \# \{ a \in [0, a_0] \mid A(a) = \mu, \ \mu \text{ is an eigenvalue of } (3.6) \} \]

By Theorem 10.1, there is only one \( a \)-eigenvalue in a neighborhood of length \( C\hbar^{\frac{5}{3}} \) of every point \( \Phi^{-1}(\pi(n + 1/2)h) \). For sufficiently small \( \hbar \) these neighborhoods are mutually disjoint. But this means that the number \( N_h \) is equal to the number of the points \( \pi(n + 1/2)h \) that lie in the interval \( (\Phi(a_2), \Phi(a_1)) \), i.e.

\[ N_h = \# \{ n \in \mathbb{Z} \mid \pi(n + 1/2)h \in (\Phi(a_2), \Phi(a_1)) \} \]

for sufficiently small \( \hbar \). And this number is exactly \( \pi^{-1}[\Phi(a_1) - \Phi(a_2)]h^{-1} + r(h) \) with \( |r(h)| \leq 1 \).

**Remark 10.4.** (norming constants) When \( \mu(h) \) is an eigenvalue of equation (3.6), then by (10.2) and (9.6) we have

\[ \sigma_1 = (-1)^n + O(h^\frac{5}{3}) \quad \text{as} \quad h \downarrow 0 \]

where \( n \) is the same as in (10.2). Thus for the corresponding \( \alpha \)-eigenvalue, namely \( \alpha(h) \), we have

\[ Y_1(\zeta; h, \alpha(h)) = \left[ (-1)^n + O(h^\frac{5}{3}) \right] Y_3(\zeta; h, \alpha(h)) \quad \text{as} \quad h \downarrow 0. \]

In particular we see that the corresponding norming constant is \( (-1)^n + O(h^\frac{5}{3}) \) which agrees with the known fact that (because of the symmetry of the potential, see Chapter 3 of [9]) the corresponding norming constant is exactly \( (-1)^n \). But of course, our method is easily extensible to the non-symmetric case, which we will in fact consider in a sequel to this paper.

**Remark 10.5.** (Weyl’s formula) Using the definition (10.3), we can write \( \Phi \) in a different way. Indeed, we have

\[ \Phi(a) = \int_{-a}^{a} \left[ A^2(x) - \mu^2 \right]^{1/2} dx = \frac{1}{2} \int_{-a}^{a} 2 \left[ A^2(x) - \mu^2 \right]^{1/2} dx = \]

\[ = \frac{1}{2} \int \int_{A^2(x) - k^2 \geq \mu^2} dk dx \]
With the help of this last equality, the difference \( \Phi(a_1) - \Phi(a_2) \) in Eq. (10.8) can be equivalently written as:

\[
\Phi(a_1) - \Phi(a_2) = \frac{1}{2} \int \int_{A^2(x) - k^2 \geq \mu_1^2} dkdx - \frac{1}{2} \int \int_{A^2(x) - k^2 \geq \mu_2^2} dkdx
\]

\[
= \frac{1}{2} \int \int_{\mu_1^2 \leq A^2(x) - k^2 \leq \mu_2^2} dkdx
\]

\[
= \frac{1}{2} \cdot \text{Area}\left\{(x, k) \in \mathbb{R}^2 | \mu_1^2 \leq A^2(x) - k^2 \leq \mu_2^2\right\}
\]

which means that the asymptotic coefficient in Eq. (10.8) is the WKB analogue of Weyl’s formula with a strong estimate on the remainder.

11. Eigenvalues Near Zero

In this section we focus on the eigenvalues of our Dirac operator which are close to zero. We wish to determine the order of the number of eigenvalues that are near the origin. To do so, we primarily use Corollary [10.3](actually a slight variant which shall be stated explicitly).

To be more precise, in the notation of Corollary [10.3], we choose \( b > 0 \) and set \( \mu_1 = \hbar^b, \mu_2 = A_{\text{max}} \). Of course, the result of the aforementioned corollary holds for a fixed interval \((\mu_1, \mu_2)\) independent of \( \hbar \), while now we are letting the left end of this interval depend on \( \hbar \). It is not a priori clear whether the result will remain the same in this more delicate situation. We shall explain why this result is indeed still true.

The \( \mu \)-interval \([\hbar^b, A_{\text{max}}]\) corresponds to the \( a \)-interval \([0, a_0(\hbar)]\) where \( 0 < a_0(\hbar) = B(\hbar^b) \) and \( B \) as in Eq. (10.8), i.e. the inverse of \( A^{-1}_{[0, +\infty)} \). The last interval is transformed to the \( \alpha \)-interval \([0, a_0(\hbar)]\) where \( 0 < a_0(\hbar) < +\infty \); observe that as \( \hbar \downarrow 0 \) then \( B(\hbar^b) \uparrow +\infty \) and consequently \( \alpha \uparrow \left( \frac{3}{\hbar^b} \| A \|_{L^1(\mathbb{R})}\right)^2 \) by Eq. (14).

In this setting, \( a \) depends on \( \hbar \) and using Eq. (3.10), (3.11) and (3.12) we have

\[
f(x; a(\hbar)) = A^2(a(\hbar)) - A^2(x)\]

\[
g(x; a(\hbar)) = \frac{3}{4} \left[ \frac{A'(x)}{A(x) + A(a(\hbar))}\right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + A(a(\hbar))}
\]

and

\[
f(x; a(\hbar)) = [x^2 - a^2(\hbar)]p(x; a(\hbar)).
\]

It is easy to see that for each value of \( \hbar \) the functions \( f \) and \( g \) satisfy properties (i) through (iv) of §6. This implies -again with the help of Lemma 1 in Eq. (10.1) - that for each \( \hbar \) the function

\[
\psi(\zeta; \alpha(\hbar)) = \frac{1}{4} \left[ 3 \zeta^2 + 2 \alpha^2(\hbar) \right] \cdot \frac{1}{4} \left[ \frac{\zeta^2 - \alpha^2(\hbar)}{f'(x; a(\hbar))}\right]^2 + \frac{1}{16} \frac{\zeta^2 - \alpha^2(\hbar)}{f'(x; a(\hbar))}
\]

\[
\cdot \left\{ 4f(x; a(\hbar))f''(x; a(\hbar)) - 5[f'(x; a(\hbar))]^2 \right\} + (\zeta^2 - \alpha^2(\hbar)) \frac{g(x; a(\hbar))}{f(x; a(\hbar))}
\]

is continuous in the corresponding region of the \((\zeta, \alpha)\)-plane.

Next, a variation of Theorem [6.1](is applied to guarantee the existence of approximate functions in this case too. To make it precise, notice that
• for each value of $h$, the function $\psi(\zeta; \alpha(h))$ is continuous in the region $[0, +\infty) \times [0, \alpha_0(h)]$ of the $(\zeta, \alpha)$-plane and

• for each $h$ the variation

$$V_{0, +\infty}(H) = \int_0^{+\infty} \frac{|\psi(t; \alpha(h))|}{\Omega(t\sqrt{2h^{-1}})} dt$$

where $\Omega$ as in (6.1), converges uniformly with respect to $\alpha$.

The argument of the previous paragraph still applies because the function $\psi$ does not depend on $h$ in an arbitrary way but only through $\alpha$ which in turn lives on a closed interval depending on $h$.

Now Theorem D.2 comes into play and guarantees that everything remains unchanged; for each value of $h$, equation (4.8), i.e.

$$\frac{dFY}{d\zeta^2} = [h^{-2}(\zeta^2 - \alpha^2(h)) + \psi(\zeta; \alpha(h))] Y$$

has in the region $[0, +\infty) \times [0, \alpha_0(h)]$ of the $(\zeta, \alpha)$-plane solutions $Y_+$ and $\overline{Y}_+$ (being extensions in $\alpha$ of $Y_1$ and $Y_2$ respectively, cf. (6.4) (6.5)) which are continuous, have continuous first and second partial $\zeta$-derivatives, and are given by

$$Y_+(\zeta; h, \alpha) = U(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) + \varepsilon(\zeta; h, \alpha)$$

$$\overline{Y}_+(\zeta; h, \alpha) = \overline{U}(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) + \overline{\varepsilon}(\zeta; h, \alpha)$$

(cf. (6.4), (6.5)) where

$$\left|\varepsilon(\zeta; h, \alpha)\right| \leq \frac{M(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{\sqrt{2h^{-1}}\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}$$

and

$$\left|\overline{\varepsilon}(\zeta; h, \alpha)\right| \leq \frac{M(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{\sqrt{2h^{-1}}\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}$$

(whence $l$ and $V_{0, +\infty}(H)$ satisfy the same asymptotics as before (cf. (6.5), (8.6)) and consequently one obtains the same asymptotic behavior of solutions as in §8) namely

$$\varepsilon(\zeta; h, \alpha) = \frac{M(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)} O(h^{\frac{1}{2}})$$

$$\overline{\varepsilon}(\zeta; h, \alpha) = \frac{\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)} O(h^{\frac{1}{2}})$$

$$\frac{\partial \varepsilon}{\partial \zeta}(\zeta; h, \alpha) = \frac{N(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)M(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)} O(h^{\frac{1}{2}})$$

$$\frac{\partial \overline{\varepsilon}}{\partial \zeta}(\zeta; h, \alpha) = \frac{\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{\Omega(\zeta\sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)} O(h^{\frac{1}{2}})$$
as \( h \downarrow 0 \) uniformly for \( \zeta \geq 0 \) and \( \alpha \in [0, \alpha_0(h)] \).

Arguing as in [3], we obtain two more solutions of \( (11.1) \), namely \( Y_- \) and \( \overline{Y}_- \) (the equivalent of \( Y_3 \) and \( Y_4 \) correspondingly), satisfying

\[
Y_- (\zeta; h, \alpha) = U(-\zeta \sqrt{2h^{-1}; -\frac{1}{2} h^{-1} \alpha^2}) + \frac{M(-\zeta \sqrt{2h^{-1}; -\frac{1}{2} h^{-1} \alpha^2})}{E(-\zeta \sqrt{2h^{-1}; -\frac{1}{2} h^{-1} \alpha^2})} \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\overline{Y}_- (\zeta; h, \alpha) = \overline{U}(-\zeta \sqrt{2h^{-1}; -\frac{1}{2} h^{-1} \alpha^2}) + \frac{M(-\zeta \sqrt{2h^{-1}; -\frac{1}{2} h^{-1} \alpha^2})}{E(-\zeta \sqrt{2h^{-1}; -\frac{1}{2} h^{-1} \alpha^2})} \mathcal{O}(h^{\frac{3}{2}})
\]
as \( h \downarrow 0 \) uniformly for \( \zeta \leq 0 \) and \( \alpha \in [0, \alpha_0(h)] \).

Consequently we have the same connection formulae (all the results of [3] are not altered at all). Indeed, expressing \( Y_+, \overline{Y}_+ \) in terms of \( Y_-, \overline{Y}_- \) and writing

\[
Y_+ (\zeta; h, \alpha) = \tau_1 Y_- (\zeta; h, \alpha) + \tau_2 \overline{Y}_- (\zeta; h, \alpha)
\]

\[
\overline{Y}_+ (\zeta; h, \alpha) = \tau_1 \overline{Y}_- (\zeta; h, \alpha) + \tau_2 Y_- (\zeta; h, \alpha)
\]
(confer [9.3], [9.4]) in the same way we find that

\[
\tau_1 = \sin \left( \frac{1}{2} \pi h^{-1} \alpha^2 \right) + \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\tau_2 = \cos \left( \frac{1}{2} \pi h^{-1} \alpha^2 \right) + \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\overline{\tau}_2 = \cos \left( \frac{1}{2} \pi h^{-1} \alpha^2 \right) + \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\overline{\tau}_2 = - \sin \left( \frac{1}{2} \pi h^{-1} \alpha^2 \right) + \mathcal{O}(h^{\frac{3}{2}})
\]

(like [9.6], [9.7]) as \( h \downarrow 0 \) uniformly for \( \alpha \in [0, \alpha_0(h)] \).

Eventually, this means that the results of [10] for the eigenvalues remain the same (eg. Theorem 10.1 but this time for \( \mu \in [h^0, \max A] \)). Hence, we arrive at the following theorem the proof of which -remembering the facts of this current paragraph that have just been pointed out- has been already provided in the previous paragraph (cf. Theorem 10.1).

**Theorem 11.1.** Suppose that \( \lambda(h) = i\mu(h) \) -where \( \mu(h) \in [h^0, \max A] \) (for an arbitrary \( h \)-independent positive constant \( b \)-) is an eigenvalue of the operator \( \mathfrak{D}_h \) (see [3.3]). Consider \( a(h) \) such that \( \mu(h) = A[a(h)] \). Then there exists a non-negative integer \( n \) for which

\[
\int_{-a(h)}^{a(h)} \left[ A^2(x) - \mu^2(h) \right]^{1/2} dx = \pi \left( n + \frac{1}{2} \right) h + \mathcal{O}(h^{\frac{3}{2}}) \quad \text{as} \quad h \downarrow 0.
\]

Conversely, for every non-negative integer \( n \) such that \( \pi (n + \frac{1}{2}) h \in [0, \alpha_0^2] \) there exists a unique \( \mu_n(h) = A[a_n(h)] \) satisfying

\[
\left| \int_{-a_n(h)}^{a_n(h)} \left[ A^2(x) - \mu_n(h)^2 \right]^{1/2} dx - \pi \left( n + \frac{1}{2} \right) h \right| \leq C h^{\frac{3}{2}}
\]
with a constant \( C \) depending neither on \( n \) nor on \( h \).

This theorem has some interesting corollaries. Specifically, Corollary 10.3 can be applied in this new case as well. Stating it explicitly, we have

**Corollary 11.2.** Take \( b \) to be an arbitrary \( h \)-independent positive constant and let \( A(a_*) = h^0 \) (clearly \( a_* \) depends on \( h \)). Then the total number \( N_h \) of eigenvalues
\( \lambda = i \mu \) of the Dirac operator \( \mathcal{D}_h \) lying in the set \( \{ i \mu \mid \mu \in (\hbar b, A_{\max}) \} \subset \mathbb{C} \) is equal to

\[
N_h = \pi^{-1} \Phi(a_n) \hbar^{-1} + r(h)
\]

where \( |r(h)| \leq 1 \) for sufficiently small \( \hbar \).

**Proof.** The arguments prior to the statement of the corollary allow us to use Corollary 10.3 with \( \mu_2 = A_{\max} \). In the notation of this same corollary, this leads to \( a_2 = 0 \) and consequently \( \Phi(a_2) = 0 \). And the desired result follows. \( \square \)

Now, let us define

\[
a_n^{WKB}(\hbar) := \Phi^{-1} \left[ \pi(n + 1/2) \hbar \right]
\]

(11.4) to be the approximant (provided by the WKB analysis) to an actual \( a \)-eigenvalue \( a_n(h) \). We have the following corollary

**Corollary 11.3.** Let \( 0 < b < \frac{5}{3} \) (independent of \( \hbar \)). Then for every non-negative integer \( n \) such that \( \pi(n + 1/2) \hbar \) belongs to \( (0, a_0^2) \) there exists a unique eigenvalue \( a_n(h) \) satisfying

\[
|a_n(h) - a_n^{WKB}(\hbar)| = O(\hbar^{5/3}) \quad \text{as} \quad \hbar \downarrow 0
\]

uniformly for \( a_n \) in \( (0, a_0(\hbar)) \), i.e. for \( \mu_n = A(a_n) \) in \( (\hbar^b, A_{\max}) \).

**Proof.** The analysis of this section along with Theorem 10.1 and formula (11.4) give rise in a straightforward way to the desired expression. \( \square \)

**Remark 11.4.** Having reached close enough to 0, at a distance \( \hbar^b \) with \( b > 1 \), it is possible to show that even in the remaining interval \( (0, i \hbar^b] \) the absolute difference \( |a_n(h) - a_n^{WKB}(\hbar)| \) is bounded by \( \hbar^b \). The argument relies on the fact that there exists a very accurate semiclassical estimate of the total number of eigenvalues due to Klaus & Shaw (see e.g. section 6 of [6]). Since neighboring Bohr-Sommerfeld approximations are at distance \( O(\hbar) \) from each other asymptotically, it follows that there is at most one such in the interval \( (0, i \hbar^b] \). Because of the previous corollary, and the established 1-1 correspondence in \( (\hbar^b, A_{\max}) \) it is clear that there is also at most one eigenvalue in the interval \( (0, i \hbar^b] \) and the absolute difference \( |a_n(h) - a_n^{WKB}(\hbar)| \) is indeed bounded by \( \hbar^b \).

12. Scattering Coefficients

In this section we will consider the scattering coefficients for our Dirac operator (3.2)\(^8\). As mentioned in \( \square \) the continuous spectrum of our Dirac operator is the whole real line. So now \( \lambda \in \mathbb{R} \) and under the change of variables

\[
y_\pm = \frac{u_2 \pm u_1}{\sqrt{A \pm i \lambda}}
\]

equation (3.1) -with the help of (3.2)- is transformed to the following two independent equations

\[
y_\pm''(x; \hbar, \lambda) = [\hbar^{-2}V_1(x; \lambda) + \tilde{F}_\pm(x; \lambda)]y_\pm(x; \hbar, \lambda)
\]

(12.1) in which the functions \( V_1 \) and \( \tilde{F}_\pm \) are given by

\[
V_1(x; \lambda) = -A^2(x) - \lambda^2
\]

\[
\tilde{F}_\pm(x; \lambda)
\]

---

\(^8\)In this part, we work using as guide ideas presented in section IV of [6].
and
\[ \tilde{F}_\pm(x; \lambda) = 3 \left[ \frac{A'(x)}{A(x) \pm i\lambda} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) \pm i\lambda}. \]

Again we only consider the “minus” lower index (i.e. \( y_-, \tilde{F}_- \) in [12.1]), so, from now on, we drop all the indices and work with the equation
\[ \frac{d^2 y}{dx^2} = [-\hbar^{-2} \hat{f}(x; \lambda) + \tilde{g}(x; \lambda)]y \]  
(12.2)
where \( \hat{f} \) and \( \tilde{g} \) satisfy
\[ \hat{f}(x; \lambda) = A^2(x) + \lambda^2 \]  
(12.3)
and
\[ \tilde{g}(x; \lambda) = 3 \left[ \frac{A'(x)}{A(x) - i\lambda} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) - i\lambda}. \]

In this case we define the error-control function to be
\[ \tilde{H}(x; \lambda) = \int \left[ \frac{1}{\hat{f}^2(x; \lambda)} \frac{\partial^2}{\partial x^2} \left( \frac{1}{\hat{f}^2(x; \lambda)} \right) - \frac{\tilde{g}(x; \lambda)}{\hat{f}^2(x; \lambda)} \right] dx \]  
(12.4)
where the integration constant is once again immaterial.

Observe that \( \text{12.3} \) implies \( \hat{f} > 0 \) in \( \mathbb{R} \). Consequently, equation \( \text{12.2} \) has no turning points. Furthermore notice that \( \tilde{g} \) is complex-valued. Also, \( \hat{f} \) is twice continuously differentiable (with respect to \( x \)), a fact that comes from the properties of \( \hat{A} \) found in \( \text{2} \) while \( \tilde{g} \) is continuous. These properties allow one (cf. Theorem 2.2 of §2.4 from chapter 6 of \( \text{17} \) along with the remarks from §5.1 of the same chapter) to state that for \( x \) in the (finite or infinite) interval \( (x_1, x_2) \subset \mathbb{R} \) and \( \kappa \) an arbitrary finite or infinite point in the closure of \( (x_1, x_2) \), equation \( \text{12.2} \) has twice continuously differentiable solutions \( w_\pm \) with
\[ w_\pm(x; \hbar) = \hat{f}^{-\frac{i}{2}}(x; \lambda) \exp \left\{ \pm \frac{i}{\hbar} \int \hat{f}^\frac{1}{2}(t; \lambda) dt \right\} (1 + \epsilon_\pm(x; \hbar)) \]
where
\[ |\epsilon_\pm(x; \hbar)|, \quad \hbar \hat{f}^{-\frac{i}{2}}(x; \hbar) \frac{\partial \epsilon_\pm}{\partial x}(x; \hbar) \leq \exp\{\hbar \mathcal{V}_{\kappa,x}(\tilde{H})\} - 1 \]  
(12.5)
provided that \( \mathcal{V}_{\kappa,x}(\tilde{H}) < +\infty \). As usual, the symbol \( \int \hat{f}^\frac{1}{2}(t; \lambda) dt \) denotes any primitive of \( \hat{f}^\frac{1}{2}(t; \lambda) \). The choice of the reference point \( \kappa \) leads to the following results satisfied by the initial conditions of the solutions
\begin{itemize}
  \item \( \epsilon_\pm(x; \hbar) \to 0 \) as \( x \to \kappa \) and
  \item \( \hbar \hat{f}^{-\frac{i}{2}}(x; \hbar) \frac{\partial \epsilon_\pm}{\partial x}(x; \hbar) \) as \( x \to \kappa \).
\end{itemize}

Notice that \( \tilde{H} \) in \( \text{12.4} \) is independent of \( \hbar \) whence the right-hand side of \( \text{12.5} \) is \( \mathcal{O}(\hbar) \) as \( \hbar \downarrow 0 \) and fixed \( x \). But \( \mathcal{V}_{x_1,x_2}(\tilde{H}) < +\infty \) which implies that this \( \mathcal{O} \) term is uniform with respect to \( x \) since \( \mathcal{V}_{\kappa,x}(\tilde{H}) \leq \mathcal{V}_{x_1,x_2}(\tilde{H}) \). Hence
\[ w_\pm(x; \hbar) \sim \hat{f}^{-\frac{i}{2}}(x; \lambda) \exp \left\{ \pm \frac{i}{\hbar} \int \hat{f}^\frac{1}{2}(t; \lambda) dt \right\} \quad \text{as} \quad \hbar \downarrow 0 \]
uniformly in \( (x_1, x_2) \).

Next we define the \textit{Jost solutions}. Equation \( \text{12.2} \) can be put in the form
\[ -\frac{d^2 y}{dx^2} + [-\hbar^{-2} A^2(x) + \tilde{g}(x; \lambda)] y = \left( \frac{\lambda}{\hbar} \right)^2 y. \]

\[ \text{Since} \; \tilde{g} \; \text{is not real, we cannot expect these solutions to be complex conjugates.} \]
A Schrödinger equation with momentum \( \frac{1}{\hbar} \), energy \((\frac{\lambda}{\hbar})^2\) and a complex potential. The Jost solutions are defined as the components of the bases \( \{ J^I_-, J^I_+ \} \) and \( \{ J^I'_-, J^I'_+ \} \) of the two-dimensional linear space of solutions of equation \( (\ref{e1.2}) \), which satisfy the asymptotic conditions

\[
J^I_\pm(x; \lambda) \sim \exp \left\{ \pm i \frac{\lambda}{\hbar} x \right\} \quad \text{as} \quad x \to -\infty
\]

\[
J^I'_\pm(x; \lambda) \sim \exp \left\{ \pm i \frac{\lambda}{\hbar} x \right\} \quad \text{as} \quad x \to +\infty.
\]

From scattering theory, we know that the reflection \( R(h, \lambda) \) and transmission \( T(h, \lambda) \) coefficients for the waves incident on the potential from the right, can be expressed in terms of wronskians of the Jost solutions. More precisely, we have

\[
R(h, \lambda) = \frac{\mathcal{W}[J^I_-, J^I'_-]}{\mathcal{W}[J^I_+, J^I'_+]},
\]

\[
T(h, \lambda) = \frac{\mathcal{W}[J^I_-, J^I'_-]}{\mathcal{W}[J^I_+, J^I'_+]}.
\]  

(12.6)

(12.7)

The next step is to construct the Jost solutions as WKB solutions. For this, we define the following four WKB solutions

\[
w^I_\pm(x; h) = \tilde{f}^{-\frac{i}{\hbar}}(x; \lambda) \exp \left\{ \pm \frac{i}{\hbar} \left( \lambda x + \int_{-\infty}^{x} [\tilde{f}^2(t; \lambda) - \lambda] dt \right) \right\} (1 + \epsilon^I_\pm(x; h))
\]

\[
\bar{w}^I_\pm(x; h) = \tilde{f}^{-\frac{i}{\hbar}}(x; \lambda) \exp \left\{ \pm \frac{i}{\hbar} \left( \lambda x + \int_{-\infty}^{x} [\tilde{f}^2(t; \lambda) - \lambda] dt \right) \right\} (1 - \epsilon^I_\pm(x; h))
\]

which we are going to modify slightly in a while. If we take the limits as \( x \to \pm \infty \) of the above, we instantly notice the following relations between \( w^I_\pm, \bar{w}^I_\pm \) and the Jost solutions \( J^I_\pm, J^I'_\pm \); we have

\[
J^I_\pm = \lambda^\frac{1}{2} \bar{w}^I_\pm
\]

\[
J^I'_\pm = \lambda^\frac{1}{2} w^I_\pm.
\]

Let now \( w^I_\pm, \bar{w}^I_\pm \) be four WKB solutions satisfying

\[
w^I_\pm(x; h) = \tilde{f}^{-\frac{i}{\hbar}}(x; \lambda) \exp \left\{ \pm \frac{i}{\hbar} \int_0^x [\tilde{f}^2(t; \lambda) - \lambda] dt \right\} (1 + \epsilon^I_\pm(x; h))
\]

\[
w^I'_\pm(x; h) = \tilde{f}^{-\frac{i}{\hbar}}(x; \lambda) \exp \left\{ \pm \frac{i}{\hbar} \int_0^x [\tilde{f}^2(t; \lambda) - \lambda] dt \right\} (1 - \epsilon^I_\pm(x; h)).
\]

(12.8)

(12.9)

Once again, the connection between \( w^I_\pm, \bar{w}^I_\pm \) and \( w^I'_\pm, \bar{w}^I'_\pm \) is evident. It is

\[
\bar{w}^I_\pm = \exp \left\{ \pm \frac{i}{\hbar} \int_{-\infty}^{x} [\tilde{f}^2(t; \lambda) - \lambda] dt \right\} w^I_\pm
\]

\[
\bar{w}^I'_\pm = \exp \left\{ \mp \frac{i}{\hbar} \int_{-\infty}^{x} [\tilde{f}^2(t; \lambda) - \lambda] dt \right\} w^I'_\pm.
\]

Subsequently, for the Jost solutions we have

\[
J^I_\pm = \lambda^\frac{1}{2} \exp \left\{ \pm \frac{i}{\hbar} \int_{-\infty}^{x} [\tilde{f}^2(t; \lambda) - \lambda] dt \right\} w^I_\pm
\]

\[
J^I'_\pm = \lambda^\frac{1}{2} \exp \left\{ \mp \frac{i}{\hbar} \int_{-\infty}^{x} [\tilde{f}^2(t; \lambda) - \lambda] dt \right\} w^I'_\pm.
\]

(12.10)

(12.11)
Remember from \(\S 4\) that the properties of \(A\) show that the function \(t \mapsto \tilde{f}^\perp(t; \lambda) - \lambda\) is in \(L^1(\mathbb{R})\). Furthermore, we have
\[
\int_{-\infty}^{0} [\tilde{f}^\perp(t; \lambda) - \lambda]dt = \int_{0}^{+\infty} [\tilde{f}^\perp(t; \lambda) - \lambda]dt = \frac{1}{2} \|\tilde{f}^\perp(\cdot; \lambda) - \lambda\|_{L^1(\mathbb{R})}
\]
and we define
\[
\sigma(\lambda) := \|\tilde{f}^\perp(\cdot; \lambda) - \lambda\|_{L^1(\mathbb{R})}
\]
Substituting (12.10), (12.11), (12.12) and (12.13) in (12.6), (12.7) and using \(\mathcal{W}[J^+_r, J^-_r] = -2i\frac{\lambda}{\hbar}\), we have
\[
R(h, \lambda) = e^{i \frac{\sigma(\lambda)}{\hbar}} \frac{\mathcal{W}[w^-_r, w^-_r]}{\mathcal{W}[w^+_r, w^-_r]}
\]
\[
T(h, \lambda) = -\frac{2i}{\hbar} e^{i \frac{\sigma(\lambda)}{\hbar}} \frac{1}{\mathcal{W}[w^+_r, w^-_r]}
\]
Now, using (12.8), (12.9) we find
- \(\mathcal{W}[w^-_r, w^-_r] = \mathcal{O}(\hbar^{-3b})\) as \(\hbar \downarrow 0\) and
- \(\mathcal{W}[w^+_r, w^-_r] = -\frac{2i}{\hbar}[1 + \mathcal{O}(\hbar^{1-3b})]\) as \(\hbar \downarrow 0\)
and \(\lambda \in [\hbar^b, +\infty)\). Substituting these last results in (12.14), (12.15) we finally obtain that
\[
R(h, \lambda) = \frac{i\hbar}{2} e^{i \frac{\sigma(\lambda)}{\hbar}} \mathcal{O}(\hbar^{-3b}) \quad \text{as} \quad \hbar \downarrow 0
\]
\[
T(h, \lambda) = e^{i \frac{\sigma(\lambda)}{\hbar}} [1 + \mathcal{O}(\hbar^{1-3b})] \quad \text{as} \quad \hbar \downarrow 0
\]
So, we have proved the following

**Theorem 12.1.** Let \(A\) satisfy the assumptions of \(\S 4\) take \(0 < b < \frac{1}{3}\) and define \(\sigma\) by (12.13). Then the reflection coefficient and the transmission coefficient of equation (12.3) as defined by (12.6) and (12.7) respectively, satisfy
\[
R(h, \lambda) = \frac{i\hbar}{2} e^{i \frac{\sigma(\lambda)}{\hbar}} \mathcal{O}(\hbar^{-3b}) \quad \text{as} \quad \hbar \downarrow 0
\]
\[
T(h, \lambda) = e^{i \frac{\sigma(\lambda)}{\hbar}} [1 + \mathcal{O}(\hbar^{1-3b})] \quad \text{as} \quad \hbar \downarrow 0
\]
uniformly for \(\lambda\) in any closed interval of \([\hbar^b, +\infty)\).

**Remark 12.2.** We check that
\[
|R(h, \lambda)|^2 + |T(h, \lambda)|^2 = 1 + \mathcal{O}(\hbar^{1-3b}) \quad \text{as} \quad \hbar \downarrow 0
\]
as of course it should be the case.

**Remark 12.3.** The result (12.16) above only guarantees asymptotics of order \(\mathcal{O}(\hbar^{1-\epsilon})\) for small positive \(\epsilon\), as \(\hbar \downarrow 0\), while the results provided by Theorems 2.1 and 2.4 in [6], where the potential is real-analytic, actually imply exponential decay as \(\hbar \downarrow 0\). Still this is good enough for the applications to the theory of focusing NLS.

**Remark 12.4.** The results in the last three paragraphs are stronger than those of [6] in the sense that they cover analytic bell-shaped potentials as well as non-analytic potentials with a certain smoothness. The reflection coefficient is weaker but this does not affect the results and proofs pertaining to the applications to the semiclassical limit of the NLS equation. On the other hand, the more important
Bohr-Sommerfeld estimate is stronger. We refer to [6] for the actual statements of the precise results and the detailed proofs which we can now immediately apply to our more general potentials (and are actually a bit simpler because of the stronger Bohr-Sommerfeld estimate).

APPENDIX A. AIRY FUNCTIONS

In this section, some basic properties of Airy functions are presented. For further reading one may consult [17].

Consider the Airy equation
\[- \frac{d^2 w}{dt^2}(t) + t \cdot w(t) = 0, \quad t \in \mathbb{R}\]

We denote by \(Ai\) and \(Bi\) its two linearly independent solutions having the asymptotics
\[Ai(t) = 2^{-1} \pi^{-1/2} t^{-1/4} \exp(-2t^{3/2}/3)(1 + O(t^{-3/2})) \quad \text{as} \quad t \to +\infty \quad (A.1)\]
and
\[Bi(t) = -\pi^{-1/2} |t|^{-1/4} \sin(2|t|^{3/2}/3 - \pi/4) + O(|t|^{-7/4}) \quad \text{as} \quad t \to -\infty \quad (A.2)\]

Figure 4. The Airy functions \(Ai, Bi\) on the real line.
Their behavior on the opposite side of the real line is known to be
\[
Ai(t) = \pi^{-1/2}|t|^{-1/4} \sin(2|t|^{3/2}/3 + \frac{\pi}{4}) + O(|t|^{-7/4}) \quad \text{as} \quad t \to -\infty \quad (A.3)
\]
and
\[
Bi(t) \leq C(1 + t)^{-1/4} \exp(2t^{3/2}/3), \quad t \geq 0
\]
where \(C\) is a positive constant. Observe that as \(t \to -\infty\), \(Ai\) and \(Bi\) only differ by a phase shift. Also \(Ai(t), Bi(t) > 0\) for all \(t \geq 0\). Note that all asymptotic relations \(A.1, A.2\) and \(A.3\) can be differentiated in \(t\); for example
\[
Ai'(t) = -\pi^{-1/2}|t|^{1/4} \cos(2|t|^{3/2}/3 + \frac{\pi}{4}) + O(|t|^{-5/4}) \quad \text{as} \quad t \to -\infty
\]
and
\[
Ai'(t) = -2^{-1/2}\pi^{-1/2}t^{1/4} \exp(-2t^{3/2}/3)(1 + O(t^{-3/2})) \quad \text{as} \quad t \to +\infty.
\]
Another property says that
\[
|Ai(t)| \leq C(1 + |t|)^{-1/4}, \quad t \in \mathbb{R}
\]
where \(C\) is a positive constant. The wronskian of \(Ai, Bi\), namely \(W[Ai, Bi]\), satisfies
\[
W[Ai, Bi](t) := Ai(t)Bi'(t) - Ai'(t)Bi(t) = \pi^{-1}, \quad t \in \mathbb{R}.
\]
In order to have a convenient way of assessing the magnitudes of \(Ai\) and \(Bi\) we introduce a modulus function \(M\), a phase function \(\vartheta\) and a weight function \(E\) related by
\[
E(x)Ai(x) = M(x)\sin \vartheta(x), \quad \frac{1}{E(x)}Bi(x) = M(x)\cos \vartheta(x), \quad x \in \mathbb{R}.
\]
Actually, we choose \(E\) as follows: denote by \(c_*\) the negative root with the smallest absolute value of the equation \(Ai(x) = Bi(x)\) (numerical calculations show that \(c_* = -0.36605\) correct up to five decimal places); then define
\[
E(x) = \begin{cases} 
1, & x \leq c_* \\
\frac{Bi(x)}{Ai(x)}^{1/2}, & x > c_*, 
\end{cases}
\]
With this choice in mind, \(M, \vartheta\) become
\[
M(x) = \begin{cases} 
[ Ai^2(x) + Bi^2(x) ]^{1/2}, & x \leq c_* \\
[ 2Ai(x)Bi(x) ]^{1/2}, & x > c_*, 
\end{cases}
\]
and \(\vartheta(x) = \begin{cases} 
\arctan \frac{Ai(x)}{Bi(x)}, & x \leq c_* \\
\frac{\pi}{4}, & x > c_*. 
\end{cases}\)
where the branch of the inverse tangent is continuous and equal to \(\frac{\pi}{4}\) at \(x = c_*\). For these functions the asymptotics for large \(|x|\) read
\[
E(x) \sim \begin{cases} 
1, & x \to -\infty \\
\sqrt{2} \exp \left(\frac{2}{3}x^{3/2}\right), & x \to +\infty 
\end{cases}
\]
\[
M(x) \sim \pi^{-1/2}|x|^{-1/4}, \quad |x| \to +\infty
\]
\[
\vartheta(x) = \begin{cases} 
\frac{2}{3}|x|^{3/2} + \frac{\pi}{4} + O(\frac{1}{2}|x|^{-3/2}), & x \to -\infty \\
\frac{\pi}{4}, & x \to +\infty 
\end{cases}
\]
APPENDIX B. PARABOLIC CYLINDER FUNCTIONS

The result of the main theorem found in section §6 involves parabolic cylinder functions (cf. §1). So in this section we state a few properties which will be in heavy use, especially about their asymptotic character, wronskians and zeros. We prove none of them. For a rigorous exposition on parabolic cylinder functions one may consult §5 of [16] or §12 of [18] and the references therein.

Consider Weber’s equation

$$\frac{d^2 w}{dx^2} = (\frac{1}{4} x^2 + b)w.$$  (B.1)

The behavior of the solutions depends on the sign of $b$. When $b$ is negative then there exist two turning points $\pm 2\sqrt{-b}$. The solutions are of oscillatory type in the interval between these points but not in the exterior intervals. When $b > 0$ there are no real turning points and there are no oscillations at all. Since only the case $b \leq 0$ will be of interest to us, from now on we seldom mention properties having to do with the other case.

Standard solutions of (B.1) are $U(\pm x; b)$ and $\overline{U}(\pm x; b)$ defined by

$$U(\pm x; b) = \frac{\pi^{\frac{1}{2}} 2^{-\frac{1}{4}(2b+1)}}{\Gamma(\frac{3}{4} + \frac{1}{2}b)} e^{-\frac{1}{4} x^2} I_1(\frac{1}{4} + \frac{1}{2}b; \frac{1}{2}; \frac{1}{2} x^2) + \frac{\pi^{\frac{1}{2}} 2^{-\frac{1}{4}(2b-1)}}{\Gamma(\frac{3}{4} + \frac{1}{2}b)} xe^{-\frac{1}{4} x^2} I_1(\frac{1}{4} + \frac{1}{2}b; \frac{1}{2}; \frac{1}{2} x^2)$$

$$\overline{U}(\pm x; b) = \frac{\pi^{\frac{1}{2}} 2^{-\frac{1}{4}(2b+1)}}{\Gamma(\frac{3}{4} - \frac{1}{2}b)} \sin(\frac{3}{4} \pi - \frac{1}{2} b \pi) e^{-\frac{1}{4} x^2} I_1(\frac{1}{4} + \frac{1}{2}b; \frac{1}{2}; \frac{1}{2} x^2)$$

where $I_1$ denotes the confluent hypergeometric function (again cf. §1). The pair $U(x; b), \overline{U}(x; b)$ is a numerically satisfactory set of solutions (in the sense of [13]) when $x \geq 0$ and $b \leq 0$; both are continuous in $x$ and $b$ in this region.

For $b \in \mathbb{R}$, their values at $x = 0$ obey

$$U(0; b) = \pi^{-\frac{1}{2}} 2^{-\frac{1}{4}(2b+1)} \Gamma(\frac{1}{4} - \frac{1}{2}b) \sin(\frac{\pi}{4} - \frac{1}{2} b \pi)$$

$$U'(0; b) = -\pi^{-\frac{1}{2}} 2^{-\frac{1}{4}(2b-1)} \Gamma(\frac{1}{4} - \frac{1}{2}b) \sin(\frac{3\pi}{4} - \frac{1}{2} b \pi)$$

$$\overline{U}(0; b) = \pi^{-\frac{1}{2}} 2^{-\frac{1}{4}(2b+1)} \Gamma(\frac{3}{4} - \frac{1}{2}b) \sin(\frac{3\pi}{4} - \frac{1}{2} b \pi)$$

$$\overline{U}'(0; b) = -\pi^{-\frac{1}{2}} 2^{-\frac{1}{4}(2b-1)} \Gamma(\frac{3}{4} - \frac{1}{2}b) \sin(\frac{3\pi}{4} - \frac{1}{2} b \pi).$$

Those values of $b$ that make the Gamma functions in the definitions of $U$ and $\overline{U}$ infinite (the Gamma function has simple poles at the non-positive integers), are called \textit{exceptional values}. For a fixed $b \in \mathbb{R}$ other than an exceptional value, the behaviors of $U$ and $\overline{U}$ as $x \to +\infty$ satisfy

$$U(x; b) \sim x^{-\frac{1}{2} b - \frac{1}{4}} e^{-\frac{1}{4} x^2}$$

$$U'(x; b) \sim \frac{1}{2} x^{-\frac{1}{2} b - \frac{1}{4}} e^{-\frac{1}{4} x^2}$$

$$\overline{U}(x; b) \sim \sqrt{\frac{2}{\pi}} \Gamma(\frac{1}{2} - b) x^{b-\frac{1}{2}} e^{\frac{1}{4} x^2}$$

$$\overline{U}'(x; b) \sim (2\pi)^{-\frac{1}{4}} \Gamma(\frac{1}{2} - b) x^{b+\frac{1}{2}} e^{\frac{1}{4} x^2}.$$
These estimates are uniform in \( b \) when \( b \) takes values over a fixed compact interval not containing exceptional values.

For the wronskian of \( U(\cdot; b), \overline{U}(\cdot; b) \) we have

\[
\mathcal{W}[U(\cdot; b), \overline{U}(\cdot; b)](x) = 2^{\frac{3}{2}}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2} - b\right), \quad x \in \mathbb{R}.
\]  

(B.3)

When \( b = 0 \) the standard solutions of equation (B.1) are related to the modified Bessel functions \( K_{\frac{1}{4}} \) and \( I_{\frac{1}{4}} \) in the following way. For \( x \geq 0 \) we have

\[
U(x; 0) = (2\pi)^{-\frac{1}{4}} x^\frac{1}{2} K_{\frac{1}{4}}(\frac{1}{4}x^2)
\]

\[
\overline{U}(x; 0) = (\pi x)^{\frac{1}{4}} I_{\frac{1}{4}}(\frac{1}{4}x^2) + (2\pi x)^{-\frac{1}{4}} x^\frac{1}{2} K_{\frac{1}{4}}(\frac{1}{4}x^2).
\]

In order to express the character of these standard solutions for large negative \( b \), we need some preparations first. Take \( \mu \gg 1 \) to be a large positive number and set \( b = -\frac{1}{2} \mu^2\) and \( x = \mu y \sqrt{2} \) where \( y \geq 0 \). If we consider the function \( \eta \) to be

\[
\eta(y) = \begin{cases} 
-\frac{3}{2} \int_0^1 (1-s^2)^{\frac{1}{2}} ds, & 0 \leq y \leq 1 \\
\frac{3}{2} \int_1^y (s^2 - 1)^{\frac{1}{2}} ds, & y \geq 1 
\end{cases}
\]

(B.4)

then as \( \mu \to +\infty \) we have

\[
U(\mu y \sqrt{2}; -\frac{1}{2} \mu^2) = 2^{\frac{3}{4}}\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2} + \frac{1}{2} \mu^2\right)^{\frac{1}{2}} \left(\frac{\eta}{\mu y^2 - 1}\right)^{\frac{1}{2}} \left[ Ai(\mu^{\frac{4}{3}} \eta) + \frac{M(\mu^{\frac{4}{3}} \eta)}{E(\mu^{\frac{4}{3}} \eta)} O(\mu^{-2}) \right]
\]

(B.5)

\[
\overline{U}(\mu y \sqrt{2}; -\frac{1}{2} \mu^2) = 2^{\frac{3}{4}}\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2} + \frac{1}{2} \mu^2\right)^{\frac{1}{2}} \frac{\eta^\frac{1}{2}}{\mu (y^2 - 1)^{\frac{1}{4}}} \left[ Bi(\mu^{\frac{4}{3}} \eta) + M(\mu^{\frac{4}{3}} \eta) E(\mu^{\frac{4}{3}} \eta) O(\mu^{-2}) \right]
\]

(B.6)

where \( Ai, Bi, E \) and \( M \) are the standard Airy functions’ terminology (cf. section A in the appendix).

For \( b \leq 0 \), the number of zeros of \( U(\cdot; b) \) in the interval [0, +\( \infty \)) is \( \lfloor \frac{1}{4} - \frac{1}{2} b \rfloor \) while \( \overline{U}(\cdot; b) \) has \( \lfloor \frac{3}{4} - \frac{1}{2} b \rfloor \) zeros in [0, +\( \infty \)). Actually, the zeros of \( U(\cdot; b) \) and \( \overline{U}(\cdot; b) \) do not cross each other. They interlace, with the largest one belonging to \( \overline{U}(\cdot; b) \). For sufficiently large \( |b| \), all the real zeros of these two functions lie to the left of \( 2\sqrt{-b} \), the positive turning point of Weber’s equation \( ^{10} \).

\(^{10}\)For \( U(\cdot; b) \), this result holds for all \( b \leq 0 \).
Figure 5. An example of Parabolic Cylinder Functions $U(\cdot; b)$ (continuous) and $\overline{U}(\cdot; b)$ (dashed) for some $b < 0$. This is Figure 5.1 in [10].

To express the errors for the approximations of our problem, we need to define some auxiliary functions having to do with the nature of $U(\cdot; b)$ and $\overline{U}(\cdot; b)$ for negative $b$. In this case the character of each is partly oscillatory and partly exponential, so we introduce one weight function $E$, two modulus functions $M$ and $N$, and finally two phase functions $\theta$ and $\omega$.

We denote by $\rho(b)$ the largest real root of the equation

$U(x; b) = \overline{U}(x; b)$.

We know (cf. §13 of [16] and the references therein) that $\rho(0) = 0$ and $\rho(b) > 0$ for $b < 0$. Also, $\rho$ is continuous when $b \in (-\infty, 0]$. An asymptotic estimate for large negative $b$ is

$\rho(b) = 2(-b)^{\frac{1}{2}} + c_\ast(-b)^{-\frac{1}{4}} + O(b^{-\frac{5}{6}})$ as $b \to -\infty$ (B.7)

where $c_\ast \approx -0.36605$ is the smallest in absolute value root of the equation $Ai(x) = Bi(x)$.

For $b \leq 0$ we define

$$E(x; b) = \begin{cases} 1, & 0 \leq x \leq \rho(b) \\ \left(\frac{U(x; b)}{\overline{U}(x; b)}\right)^{1/2}, & x > \rho(b). \end{cases}$$

It is seen that $E$ is continuous in the region $[0, +\infty) \times (-\infty, 0]$ of the $(x, b)$-plane and for $b \leq 0$ fixed, $E(\cdot; b)$ is non-decreasing in the interval $[0, +\infty)$. Again for $b \leq 0$
and $x \geq 0$ we set
\[
U(x; b) = \frac{1}{E(x; b)} M(x; b) \sin \theta(x; b), \quad \overline{U}(x; b) = E(x; b) M(x; b) \cos \theta(x; b)
\]
and
\[
U'(x; b) = \frac{1}{E(x; b)} N(x; b) \sin \omega(x; b), \quad \overline{U}'(x; b) = E(x; b) N(x; b) \cos \omega(x; b).
\]
Thus
\[
M(x; b) = \begin{cases} 
\left[ U(x; b)^2 + \overline{U}(x; b)^2 \right]^{1/2}, & 0 \leq x \leq \rho(b) \\
2U(x; b)\overline{U}(x; b)^{1/2}, & x > \rho(b)
\end{cases}
\]
and
\[
\theta(x; b) = \begin{cases} 
\arctan \left[ \frac{U(x; b)}{\overline{U}(x; b)} \right], & 0 \leq x \leq \rho(b) \\
\frac{\pi}{2}, & x > \rho(b)
\end{cases}
\]
where the branch of the inverse tangent is continuous and equal to $\frac{\pi}{2}$ at $x = \rho(b)$.

Similarly
\[
N(x; b) = \begin{cases} 
\left[ U'(x; b)^2 + \overline{U}'(x; b)^2 \right]^{1/2}, & 0 \leq x \leq \rho(b) \\
\frac{U'(x; b)^2\overline{U}(x; b)^2 + \overline{U}'(x; b)^2U(x; b)^2}{U(x; b)\overline{U}(x; b)^2}^{1/2}, & x > \rho(b)
\end{cases}
\]
and
\[
\omega(x; b) = \begin{cases} 
\arctan \left[ \frac{U'(x; b)}{\overline{U}'(x; b)} \right], & 0 \leq x \leq \rho(b) \\
\arctan \left[ \frac{U'(x; b)}{\overline{U}'(x; b)} \right], & x > \rho(b)
\end{cases}
\]
where the branches of the inverse tangents are chosen to be continuous and fixed by $\omega(x; b) \to -\frac{\pi}{2}$ as $x \to +\infty$.

For large $x$ we have
\[
E(x; b) \sim \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \Gamma(\frac{1}{2} - b) \frac{1}{2} x^{b-\frac{1}{2}} e^{-\frac{1}{2} x^2}
\]
and
\[
M(x; b) \sim \left( \frac{8}{\pi} \right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - b)}{x^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi}}, \quad N(x; b) \sim \frac{\Gamma(\frac{1}{2} - b)}{(2\pi)^{\frac{1}{2}}} x^{\frac{1}{2}}.
\]
Both of these hold for fixed $b$ and are also uniform for $b$ ranging over any compact interval in $(-\infty, 0]$.

**Appendix C. Variational operator**

In this section we present some facts about the variation of a function of one variable and the variational operator that it gives rise to. We begin with a definition. Consider a function $f : (a, b) \to \mathbb{R}$ over a finite or infinite interval $(a, b)$ in the real line. Denote by $P_m$ a partition of this interval that separates it in $n P_m = m$ subintervals; that is $\mathbb{N} \ni m \geq 2$ and there are points $x_k \in (a, b), k = 1, \ldots, m - 1$ so that
\[
a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b.
\]
Also denote by $P$ the set of all possible partitions of $(a,b)$ and set
\[
V_{a,b}(f) := \sup_{P \in P} \sum_{k=0}^{n_P-1} |f(x_{k+1}) - f(x_k)|.
\]

We have the following definition

**Definition C.1.** Let $f : (a,b) \to \mathbb{R}$ be a function. If $V_{a,b}(f) < +\infty$, then $f$ is said to be of bounded variation and the number $V_{a,b}(f)$ is called its (total) variation.

With the above in mind, we have the following important theorem

**Theorem C.2.** Let $f : (a,b) \to \mathbb{R}$ be a function. If $f$ is continuous in the closure of $(a,b)$, differentiable in $(a,b)$ so that $f''$ is continuous within $(a,b)$ and $f' \in L^1((a,b))$ then $f$ is of bounded variation and furthermore
\[
V_{a,b}(f) = \|f'\|_{L^1((a,b))} = \int_a^b |f'(x)| \, dx < +\infty.
\]

**Proof.** The proof can be found in chapter 1, §11.4 in [17].

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**Appendix D. A Theorem on Integral Equations**

The proofs of theorems about WKB approximation in the case of absence of turning points (like Theorems 2.1 and 2.2 in chapter 6 of [17]), may be adapted to other types of approximate solutions of linear differential equations where turning points may be present. For second-order equations the basic steps consist of

(i) construction of a Volterra integral equation for the error term -say $h$- of the solution, by the method of variation of parameters

(ii) construction of the Liouville-Neumann expansion (a uniformly convergent series) for the solution $h$ of the integral equation in (i) by Picard’s method of successive approximations

(iii) confirmation that $h$ is twice differentiable by construction of similar series for $h'$ and $h''$

(iv) production of bounds for $h$ and $h'$ by majoring the Liouville-Neumann expansion.

It would be tedious to carry out all these steps in every case. But we have the following general theorem which automatically provides (ii), (iii) and (iv) in problems relevant to us.

**Theorem D.1.** Consider the equation
\[
h(\zeta) = \int_\beta^\gamma K(\zeta,t) \phi(t) \{J(t) + h(t)\} \, dt \tag{D.1}
\]
for the function $h$ accompanied by the following assumptions

- the “path” of integration consists of a segment $[\beta, \zeta]$ of the real axis, finite or infinite where $\beta \leq t \leq \zeta \leq \gamma$
- the real functions $J$ and $\phi$ are continuous in $(\beta, \gamma)$ except for a finite number of discontinuities and infinities
- the real kernel $K$ and its first two partial derivatives with respect to $\zeta$ are continuous functions of both variables when $\zeta, t \in (\beta, \gamma)$

---

11 This is Theorem 10.2 found in chapter 6 of [17]. It is a variant of Theorem 10.1 from the same reference.
• $K(\zeta, \zeta) = 0$, $\zeta \in (\beta, \gamma)$
• when $\zeta \in (\beta, \gamma)$ and $t \in (\beta, \zeta]$ we have

$$|K(\zeta, t)| \leq P_0(\zeta)Q(t), \quad \left| \frac{\partial K(\zeta, t)}{\partial \zeta} \right| \leq P_1(\zeta)Q(t), \quad \left| \frac{\partial^2 K(\zeta, t)}{\partial \zeta^2} \right| \leq P_2(\zeta)Q(t)$$

where the $P_j, j = 0, 1, 2$ and $Q$ are continuous real functions, the $P_j, j = 0, 1, 2$ being positive.
• when $\zeta \in (\beta, \gamma)$, the integral

$$\Phi(\zeta) = \int_\beta^\zeta |\phi(t)|dt$$

converges and the following suprema

$$\kappa = \sup_{\zeta \in (\beta, \gamma)} \{Q(\zeta)|J(\zeta)|\}, \quad \kappa_0 = \sup_{\zeta \in (\beta, \gamma)} \{P_0(\zeta)Q(\zeta)\}$$

are finite.

Under these assumptions equation (D.1) has a unique solution $h$ which is continuously differentiable in $(\beta, \gamma)$ and satisfies

$$\frac{h(\zeta)}{P_0(\zeta)} \to 0, \quad \frac{h'(\zeta)}{P_1(\zeta)} \to 0 \quad \text{as} \quad \zeta \downarrow \beta.$$ 

Furthermore,

$$\left| \frac{h(\zeta)}{P_0(\zeta)} \right|, \left| \frac{h'(\zeta)}{P_1(\zeta)} \right| \leq \frac{\kappa}{\kappa_0}[\exp\{\kappa_0\Phi(\zeta)\} - 1]$$

and $h''$ is continuous except at the discontinuities of $\phi, J$.

Proof. The proof is a slight variation of that for Theorem 10.1 of chapter 6 in [17].

We are going to use this theorem to prove the existence and behavior of approximate solutions of the equation

$$\frac{d^2 Y}{d\zeta^2} = \left[h^{-2}(\zeta^2 - \alpha^2) + \psi(\zeta; h, \alpha)\right]Y. \quad (D.2)$$

We have the following

**Theorem D.2.** For each value of $h$, assume that the function $\psi(\zeta; h, \alpha)$ is continuous in the region $[0, Z] \times [0, \delta]$ of the $(\zeta, \alpha)$-plane and that $V_{0, Z}(H)$ (cf. (6.1) and (6.3)) converges uniformly with respect to $\alpha$. Then in this region, equation (D.2) has solutions $Y_1$ and $Y_2$ which are continuous, have continuous first and second partial $\zeta$-derivatives and are given by

$$Y_1(\zeta; h, \alpha) = U(\zeta \sqrt{2h^{-1} - \frac{1}{2}h^{-1}\alpha^2}) + \epsilon_1(\zeta; h, \alpha)$$

$$Y_2(\zeta; h, \alpha) = \overline{U}(\zeta \sqrt{2h^{-1} - \frac{1}{2}h^{-1}\alpha^2}) + \epsilon_2(\zeta; h, \alpha)$$

Here $Z$ is always positive and may depend continuously on $\alpha$ or be infinite and $\delta$ is a positive finite constant.
where

\[ \frac{|\epsilon_1(\zeta; h, \alpha)|}{M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)} \leq \frac{\partial \epsilon_1}{\partial \zeta} (\zeta; h, \alpha) \sqrt{2h^{-1}}N(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) \]

and

\[ \frac{|\epsilon_2(\zeta; h, \alpha)|}{M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)} \leq \frac{\partial \epsilon_2}{\partial \zeta} (\zeta; h, \alpha) \sqrt{2h^{-1}}N(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) \]

Proof. We will prove the theorem only for the first solution since the proof for the second follows mutatis mutandis. Observe that the approximating function \( U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) \) satisfies \( \frac{d^2U}{d\zeta^2} = h^{-2}(\zeta^2 - \alpha^2)U \). If we subtract this from \( D.2 \) we obtain the following differential equation for the error term

\[ \frac{d^2\epsilon_1}{d\zeta^2} - h^{-2}(\zeta^2 - \alpha^2)\epsilon_1 = \psi(\zeta; h, \alpha)[\epsilon_1 + U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)]. \]

By use of the method of variation of parameters and also \( B.3 \) one arrives at the integral equation

\[ \epsilon_1(\zeta; h, \alpha) = \frac{1}{2} \Gamma(\frac{1}{2} + \frac{1}{h^{-1}\alpha^2}) \int_{\zeta}^{\infty} K(\zeta, t)\psi(t; h, \alpha)[\epsilon_1(t; h, \alpha) + U(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)] dt \]

in which

\[ K(\zeta, t) = U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)U(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) \]

\[ - U(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)U(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2). \]

Bounds for the kernel \( K \) and its first two partial derivatives (with respect to \( \zeta \)) are expressible in terms of the auxiliary functions \( E, M \) and \( N \). We have

\[ |K(\zeta, t)| \leq \frac{E(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{E(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}M(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)M(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) \]

\[ |\frac{\partial K}{\partial \zeta}(\zeta, t)| \leq \sqrt{2h^{-1}} \frac{E(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}{E(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)}N(\zeta \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2)M(t \sqrt{2h^{-1}}; -\frac{1}{2}h^{-1}\alpha^2) \]

and similarly

\[ \frac{\partial^2 K}{\partial \zeta^2}(\zeta, t) = (2h^{-1})^{\frac{3}{2}} K(\zeta, t). \]
All these estimates allow us to solve the equation (D.2) by applying Theorem D.1. Using the notation of that theorem we have

\[ \phi(t) = \frac{\psi(\zeta; \hbar, \alpha)}{\Omega(\zeta \sqrt{2\hbar^{-1}})} \]

\[ \psi_1(t) = 0 \]

\[ J(t) = U(t \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2) \]

\[ K(\zeta, t) = \frac{1}{2} \frac{(\pi \hbar)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + \frac{1}{2} \hbar^{-1} \alpha^2\right)}\Omega(t \sqrt{2\hbar^{-1}})K(\zeta, t) \]

\[ Q(t) = \frac{1}{2} \frac{(\pi \hbar)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + \frac{1}{2} \hbar^{-1} \alpha^2\right)}\Omega(t \sqrt{2\hbar^{-1}})E(t \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2)M(t \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2) \]

\[ P_0(\zeta) = \frac{M(\zeta \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2)}{E(\zeta \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2)} \]

\[ P_1(\zeta) = \sqrt{2\hbar^{-1}} \frac{N(\zeta \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2)}{E(\zeta \sqrt{2\hbar^{-1}}; -\frac{1}{2} \hbar^{-1} \alpha^2)} \]

\[ \Phi(\zeta) = V_{0,2}(H) \]

\[ \kappa_0 \leq \frac{1}{2} (\pi \hbar)^{\frac{1}{2}} l(-\frac{1}{2} \hbar^{-1} \alpha^2) \]

where the role of \( \beta \) is played here by \( Z \) and \( \kappa \) is replaced for simplicity by the upper bound \( \kappa_0 \). Then the bounds (D.3) and (D.4) follow from Theorem D.1.

Finally, observe that all the integrals which occur in the analysis above, converge uniformly when \( \alpha \in [0, \delta] \) and \( \zeta \) lies in any compact interval of \( [0, Z) \); allowing us to state that \( \epsilon_1 \) and its first two partial \( \zeta \)-derivatives are continuous in \( \alpha \) and \( \zeta \). Consequently, the same stands for \( \mathcal{V}_1 \) which signifies the end of the proof.

\[ \square \]

**Data Availability**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**References**


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