JOURNAL OF MATHEMATICAL PHYSICS **50**, 104101 (2009)

Comment on "Existence and regularity for an energy maximization problem in two dimensions" [S. Kamvissis and E. A. Rakhmanov, J. Math. Phys. 46, 083505 (2005)]

Spyridon Kamvissis[a)]

Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig, Germany and Department of Applied Mathematics, University of Crete, 71409 Iraklion Greece

(Received 6 July 2009; accepted 10 September 2009; published online 26 October 2009)

A revision of the last appendix of the paper "Existence and regularity for an energy maximization problem in two dimensions" by S. Kamvissis and E. A. Rakhmanov [J. Math. Phys. **46**, 083505 (2005)]. © *2009 American Institute of Physics.* [doi:10.1063/1.3250831]

DROPPING ASSUMPTION (A) IN SEC. 5 OF REF. 2

In Sec. 5 of Ref. 2, we have assumed that the solution of the problem of the maximization of the equilibrium energy is a continuum, say F, which does not intersect the linear segment $[0,iA]$ except of course at $0_+,0_-$. We also prove that F does not touch the real line, except of course at 0 and possibly ∞. This enables us to take variations in Sec. 6 of Ref. 2, keeping fixed a finite number of points, and thus arrive at the identity of Theorem 5 of Ref. 2, from which we derive the regularity of F and the fact that F is, after all, an S-curve.

In general, it is conceivable that F intersects the linear segment $[0,iA]$ at points other than $0_+,0_-$. If the set of such points is finite, there is no problem, since we can always consider variations keeping fixed a finite number of points and arrive at the same result (see the remark after the proof of Theorem 5 of Ref. 2).

If, on the other hand, this is not the case, we have a different kind of problem, because the function V introduced in Sec. 6 of Ref. 2 (the complexification of the field) is not analytic across the segment $[-iA,iA]$.

What is true, however, is that V is analytic in a Riemann surface consisting of infinitely many sheets, cut along the line segment $[-iA,iA]$. So, the appropriate, underlying space for the (doubled up) variational problem should now be a noncompact Riemann surface, say $\mathbb{L}$.

Compactness is crucial in the proof of a maximizing continuum. But we can compactify the Riemann surface $\mathbb{L}$ by compactifying the complex plane. Let the map $\mathbb{C}\to\mathbb{L}$ be defined by

$$y=\log(z-iA)-\log(z+iA).$$

The point $z=iA$ corresponds to infinitely many y-points, i.e., $y=-\infty+i\theta$, $\theta\in\mathbb{R}$, which will be identified. Similarly, the point $z=-iA$ corresponds to infinitely many points $y=+\infty+i\theta$, $\theta\in\mathbb{R}$, which will also be identified. The point $0\in\mathbb{C}$ corresponds to the points $k\pi i$, k odd.

By compactifying the plane we then compactify the Riemann surface $\mathbb{L}$. The distance between two points in the Riemann surface $\mathbb{L}$ is defined to be the corresponding stereographic distance between the images of these points in the compactified $\mathbb{C}$.

With these changes, the proof of the existence of the maximizing continuum in Secs. 1, 3, and 4 of Ref. 2 goes through virtually unaltered. In Sec. 6 of Ref. 2, we would have to consider the

a)Electronic mail: spyros@tem.uoc.gr.

complex field V as a function defined in the Riemann surface $\mathbb{L}$ and all proofs go through. The corresponding result of Sec. 7 of Ref. 2 will give us an S-curve C in the Riemann surface $\mathbb{L}$. We then have the following facts.

Consider the image $\mathbb{D}$ of the closed upper half-plane under

$$y = \log(z - iA) - \log(z + iA).$$

Consider continua in $\mathbb{D}$ containing the points $y = \pi i$ and $y = -\pi i$. Define Green's potential and Green's energy of a Borel measure by (4)–(6) of Ref. 2 and the equilibrium measure by (7) of Ref. 2. Then there exists a continuum F maximizing the equilibrium energy, for the field given by (3) of Ref. 2 with conditions (1) of Ref. 2. F does not touch $\partial\mathbb{D}$ except at a finite number of points. By taking variations as in Sec. 6 of Ref. 2, one sees that F is an S-curve. In particular, the support of the equilibrium measure on F is a union of analytic arcs and at any interior point of supp μ,

$$\frac{d}{dn_+}(\phi + V^{\lambda^F}) = \frac{d}{dn_-}(\phi + V^{\lambda^F}),$$

where the two derivatives above denote the normal derivatives.

We then have the following.

Theorem (Theorem 9 of Ref. 2) Consider the semiclassical limit ($\hbar \to 0$) of the solution of (9) and (10) of Ref. 2 (that is, the initial value problem for the focusing NLS with parameter $\hbar$) with bell-shaped initial data. Replace the initial data by the so-called soliton ensembles data (as introduced in Ref. 1) defined by replacing the scattering data for $\psi(x,0) = \psi_0(x)$ by their WKB approximation. Assume, for simplicity, that the spectral density of eigenvalues satisfies conditions (1) of Ref. 2.

Then, asymptotically as $h \to 0$, the solution $\psi(x,t)$ admits a "finite genus description," in the sense of Theorem A.1 of Ref. 2.

Proof:

(i) The proof of the existence of an S-curve F in $\mathbb{L}$ follows as above.

(ii) We want to deform the original discrete Riemann–Hilbert problem to the set $\hat{F}$ consisting of the projection of F to the complex plane. It is clear, however, that $\hat{F}$ may not encircle the spike $[0, iA]$. It is possible, on the other hand, to append S-loops (considered in $\mathbb{L}$) and end up with a sum of S-loops, such that the amended $\hat{F}$ *does* encircle the spike $[0, iA]$, meaning that $[0, iA]$ is a subset of the closure of the union of the interiors of the loops of which $\hat{F}$ consists. A little thought shows that this is all we need. (Indeed, within each of the loops we use the same pole-removing transformation as in Ref. 1. Eventually of course we have to use different interpolations, according to the sheet of each piece of F.)

To see that we can always append the needed S-loop, suppose there is an open interval, say $(i\alpha, i\alpha_1)$, which lies in the exterior of $\hat{F}$, while $i\alpha, i\alpha_1 \in \hat{F}$. Let us assume, for example, that $\hat{F}$ crosses $[0, iA]$ along bands at $i\alpha, i\alpha_1$; call these bands S, S_1 and assume they lie in the principal sheet. Let β^-, β^+ be points (considered in $\mathbb{C}$) lying on S to the left and right of $i\alpha$, respectively, and at a small distance from $i\alpha$. Similarly, let β_1^-, β_1^+ be points lying on S_1 to the left and right of $i\alpha_1$, respectively, and at a small distance from $i\alpha_1$. We will show that there exists a "gap" region including the preimages of β^-, β_1^- lying in the Nth sheet for $-N$ large enough, and similarly there exists a "gap" region including the preimages of β^+, β_1^+ lying in the Mth sheet for M large enough, both being regions for which the gap inequalities hold *a priori*, irrespectively of the actual S-curve, depending only on the external field! Indeed, note that the quantity $\mathrm{Re}(\tilde{\phi}^{\sigma}(z))$ (which defines the variational inequalities) is *a priori* bounded above by $-\phi(z)$. For this, see (8.8) in Chapter 8 of Ref. 1; there is actually a sign error: the right formula is

$$\mathrm{Re}(\tilde{\phi}^{\sigma}(z)) = -\phi(z) + \int G(z,\eta)\rho^{\sigma}(\eta)d\eta.$$

Next note [see, for example, (5.8) of Ref. 1 with K varying along the natural numbers according to the relevant sheet of $\mathbb{L}$] that the difference of the values of the function $\mathrm{Re}(\tilde{\phi}^{\sigma}(z))$ in consecutive sheets is $\delta\,\mathrm{Re}(\tilde{\phi}^{\sigma}) \sim \pm 2\,\mathrm{Im}\,\rho(z)\pi\,\mathrm{Re}\,z$ near the spike $[0,iA]$ [remember $\mathrm{Im}\,\rho(z) > 0$ there] and hence the difference of the values at points on consecutive sheets whose image under the projection to the complex plane is $i\eta+\epsilon$, where η is real and ϵ is a small (negative or positive) real, is $\delta(\mathrm{Re}\,\tilde{\phi}^{\sigma}) \sim \pm 2\pi\,\mathrm{Im}\,\rho(z)\epsilon$. This means that on the left (right) side of the imaginary semiaxis, the inequality $\mathrm{Re}(\tilde{\phi}^{\sigma}(z)) < 0$ will be eventually (depending on the sheet) be valid at any given small distance to it.

We now connect the preimages of β^- and β_1^- (under the projection of $\mathbb{L}$ to $\mathbb{C}$) lying in the Nth sheet to the preimages of β^- and β_1^- lying in the principal sheet respectively, using the results of Ref. 2. Similarly we join the preimages of β^+ and β_1^+ lying in the Mth sheet to the preimages of β^- and β_1^- lying in the principal sheet respectively.

Then, we join the the preimages of β^- and β_1^- (under the projection of $\mathbb{L}$ to $\mathbb{C}$) lying in the Nth sheet and the preimages of β^+ and β_1^+ lying in the Mth sheet, along the according gap regions.

It is easy to see that (together with the bands S and S_1) we end up with an S-loop (in $\mathbb{L}$) whose projection is covering the "lacuna" $(i\alpha, i\alpha_1)$.

The original discrete Riemann–Hilbert problem can be trivially deformed to a discrete Riemann–Hilbert on the resulting (projection of the) union of S-loops. All this is possible even in the case where $\hat{F}$ self-intersects.

(iii) We deform the discrete Riemann–Hilbert problem to the continuous one with the right band/gap structure (on $\hat{F}$; according to the equilibrium measure on F), which is then explicitly solvable via theta functions exactly as in Ref. 1. Both the discrete-to-continuous approximation and the opening of the lenses needed for this deformation are justified as in Ref. 1 and therefore the technical details will not be repeated here. It is important to notice that our construction has ensured the analytic continuation of the jump matrix along $\hat{F}$ (oriented according to F). The g-function is defined by the same Thouless-type formula with respect to the equilibrium measure (cf. Sec. 2(iii)). It satisfies the same conditions as in Ref. 1 (measure reality and variational inequality) on bands and gaps. The equilibrium measure lives in $\mathbb{L}$ but the Riemann–Hilbert problem lives in $\mathbb{C}$.

[1] S. Kamvissis, K. T.-R. McLaughlin, and P. D. Miller, *Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation*, Annals of Mathematics Studies Vol. 154 (Princeton University Press, Princeton, 2003).

[2] S. Kamvissis and E. A. Rakhmanov, J. Math. Phys. **46**, 083505 (2005).