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Centrally Coupled Circular Array of Optical Waveguides: The Existence of Stable Steady-State Continuous Waves and Breathing Modes

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Abstract

A circular array of optical waveguides collectively coupled with a central core is investigated. Both linear and nonlinear coupling as well as energy transfer within the array elements and with the core are allowed making thus the model ideal for the design of high power stable amplifiers as well as of all-optical data processing devices in optical communications. The existence of stable steady-state continuous wave modes as well as of breathing modes is demonstrated. These properties render the proposed system functionally rich, far more controllable than a planar one and easier to stabilize.

1. Introduction

The continuous complex Ginzburg–Landau (cCGL) equation is encountered in several branches of physics. It is the dominant underlying model in superconductivity, superfluidity, nonequilibrium fluid dynamics, physical chemistry, nonlinear optics, Bose–Einstein condensates, quantum field theories [1–5], etc. In addition to the well-studied and applied in a multitude of disciplines cCGL equation, discrete complex Ginzburg–Landau (dCGL) have also been considered in lattices for modeling vortices in hydrodynamics [6], semi-conductor laser arrays in optics [7–8] as well as the dynamics of an open Bose–Einstein condensate (where dissipation is naturally expected) with a lattice potential created by the interference of two standing optical waves [9] and a gain resulting from the interaction among condensed and uncondensed atoms [10–11]. While in cCGL equation-based systems self-localized (solitary) solutions as well as dissipative solitons have been found [12–13] (among a multitude of possibilities concerning pattern formation and chaotic behavior), no coherent structures have been found in dCGL equation-based systems until quite recently [14]. Unlike conservative discrete solitons [15,16] these Ginzburg–Landau lattice solitons exhibit internal energy flow and are by nature auto-solitons. In addition discrete dissipative structures were also identified in other complex lattices of the Ablowitz–Ladik type [17].

Although the underlying physics in a dCGL equation-based system is complex and not easily amenable to analytical considerations, the applications are potentially far reaching. In this paper the main focus is to explore the properties of high-power fiber array amplifiers where the stability and phase-locking capabilities of the nonlinear modes during evolution is an issue of great importance. The concept is based on an array of optical waveguides, where both linear and nonlinear coupling and energy transfer may enrich the functionality of such a device. It is expected that the analysis of such devices will be of significance especially in geometries other than planar, such as circular [18] which is the subject of this work. In addition to the circular geometry, the waveguide array is linearly and nonlinearly coupled to a central core allowing energy transfer and exchange not only among the various array elements but also collectively with the central core. It is envisioned that this collective coupling will render the whole system far more controllable than a planar one and easier to stabilize. There are numerous indications in the literature (concerning cCGL, however) which are in support of this conjecture [19–20]. The paper is organized as follows: In the next Section the model equations and the stability of zero solution are discussed. In Section 3 the existence of stable nonlinear CW solutions is investigated, while in Section 4 along with the presentation and discussion of various results, the existence of breathing solutions is presented and several conclusions, though tentative, are drawn.

2. The model and the stability of zero solution

The system under investigation consists of a circular array of N identical linearly and nonlinearly coupled (with the nearest neighbors) optical waveguides (active and/or lossy optical fibers, for instance). The array is also linearly as well as nonlinearly coupled (cubic nonlinearity) with a cylindrical substrate (central core). The whole system is embedded in a cladding of infinite extent. We allow for complex linear coupling coefficients between each node of the array and its neighbors as well as for lossy and/or amplifying behavior in each node of the array separately and in the central core. The reference signs of the imaginary parts of all the linear coupling coefficients involved are chosen in such a way that if they all were positive the model would then represent a circular array of optical amplifiers coupled with a lossy central core. The model can equally well describe a modification of a planar semiconductor laser array [8] to a circular one (away from the saturation limit of the lasing medium) lying on a dielectric cylindrical substrate.

By using the formalism of coupled-mode theory (or the tight binding approximation) [8,21] one obtains the

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following equations in the framework of slowly-varying envelope approximation for the discrete complex mode amplitude $\{u_n(z), n = 1, \ldots, N;\ u_0(z)\}$ with z being the propagation distance and "0" referring to the central core,

$$\begin{aligned} &\mathrm{i}du_n/dz - \mathrm{i}\varepsilon_0 u_n + (\kappa_\alpha - \mathrm{i}\varepsilon_\alpha)(u_{n+1} + u_{n-1}) + \kappa_0 u_0 \\ &\quad + \mu_0 |u_n|^2 u_n + \mu_\alpha \left(|u_{n+1}|^2 + |u_{n-1}|^2\right) u_n \\ &\quad + \mu_x |u_0|^2 u_n = 0, \qquad n = 1, \ldots, N, \end{aligned} \tag{1}$$

$$\begin{aligned} &\mathrm{i}du_0/dz + \mathrm{i}\Gamma u_0 + \kappa_0 \sum_{n=1}^{N} u_n \\ &\quad + \left(M_0 |u_0|^2 + M_x \sum_{n=1}^{N} |u_n|^2 \right) u_0 = 0 \end{aligned} \tag{2}$$

where μ_0, μ_a, μ_x, M_0 and M_x are real nonlinear coupling coefficients with "0" referring to the self-phase modulation and the rest to the various cross-phase modulation terms (node-to-neighbor, node-to-central core, central core-to-node). The real part of linear coupling with the central core and the nearest neighbors are respectively denoted by K_0 and K_a and account for the respective energy tunneling. On the other hand, $\varepsilon_0, \varepsilon_a$ and Γ are respectively the amplification (or dissipation) rate in each node separately, the gain (or losses) due to coupling with the nearest neighbors and the dissipation (or amplification) rate in the central core. The system of Eqs. (1) and (2) has the form of a cubic dCGL coupled with a discrete nonlinear Schroedinger (dNLS) equation.

In order to investigate the stability of the zero solution on may choose to investigate the linear approximation of Eqs. (1) and (2). The stability of the zero solution is of great importance in the investigation of the existence of stable localized modes (the optical power concentrated in a subset of nodes) or solitary and/or breather type of structures. The reason is that localized structures residing on the circular array have their intensity dropping to zero at their tails. The centrally uncoupled version of the linear approximation of the model ($K_0 = \mu_0 = \mu_a = \mu_x = M_0 = M_x = 0$) is a Toeplitz type of a problem with known eigenmode structure [18]. In the coupled case of the same approximation the resulting eigenvalue problem is modified. By straightforward algebraic manipulations it can be shown that the first $N-1$ eigenvalues coincide with those in the uncoupled case (Toeplitz problem), that is,

$$\begin{aligned} &\lambda_k = -\mathrm{i}\varepsilon_0 + 2(\kappa_\alpha - \mathrm{i}\varepsilon_\alpha)\cos(2\pi k/N), \\ &k = 1, , N-1 \end{aligned} \tag{3}$$

while the remaining two additional eigenvalues are given by the following expression

$$\begin{aligned} \lambda_{N,N+1} &= \kappa_\alpha - i(\varepsilon_T - \Gamma)/2 \\ &\quad \pm \sqrt{N\kappa_0^2 + [\kappa_\alpha - i(\varepsilon_T + \Gamma)/2]^2} \end{aligned} \tag{4}$$

where "+" and "−" stand for the Nth and $N+1$th eigenvalue respectively and $\varepsilon_T \equiv \varepsilon_0 + 2\varepsilon_a$. The general linear solution for the array and the central core can easily be expressed as a superposition of normal modes,

$$\begin{aligned} u_n &= a_N u^{(N)} \mathrm{e}^{\mathrm{i}\lambda_N z} + a_{N+1} u^{(N+1)} \mathrm{e}^{\mathrm{i}\lambda_{N+1} z} + \sum_{k=1}^{N-1} a_k \mathrm{e}^{\mathrm{i}2\pi kn/N + i\lambda_k z} / \sqrt{N}, \\ u_0 &= \mathrm{b}_N U^{(N)} \mathrm{e}^{\mathrm{i}\lambda_N z} + b_{N+1} U^{(N+1)} \mathrm{e}^{\mathrm{i}\lambda_{N+1} z} \end{aligned} \tag{5}$$

where the two new pairs of Nth and $N+1$th eigenfunctions are given by,

$$\begin{aligned} &u^{(j)} = (\lambda_j - \mathrm{i}\Gamma) / \sqrt{N^2\kappa_0^2 + N|\lambda_j - \mathrm{i}\Gamma|^2}, \\ &U^{(j)} = \sqrt{N\kappa_0^2 / \left[N\kappa_0^2 + |\lambda_j - \mathrm{i}\Gamma|^2\right]}, \qquad j = N, N+1 \end{aligned} \tag{6}$$

and a_m, b_m are free (specified by the launching conditions) complex coefficients.

In the following we introduce normalized physical quantities [denoted by the subscript (n)] in all coupling coefficients involved (symbolized by a) as well as in the propagation distance (the coupling coefficient k_0 is excluded), that is, $a^{(n)} \equiv a/(|K_0|N^{1/2}), z^{(n)} \equiv z(|K_0|N^{1/2})$. Separating the real and imaginary parts of the $N+1$ eigenvalues leads to,

$$\begin{aligned} &\mathrm{Im}(\lambda_k^{(n)}) = -\varepsilon_0^{(n)} - 2\varepsilon_\alpha^{(n)}\cos(2\pi k/N), \\ &\mathrm{Re}(\lambda_k^{(n)}) = 2\kappa_\alpha^{(n)}\cos(2\pi k/N), \qquad k = 1, \ldots, N-1 \end{aligned} \tag{7}$$

$$\begin{aligned} &\mathrm{Im}(\lambda_{N,N+1}^{(n)}) = (\Gamma^{(n)} - \varepsilon_T^{(n)})/2 \mp \sqrt{(\sqrt{P^2 + D^2} - D)/2}, \\ &\mathrm{Re}(\lambda_{N,N+1}^{(n)}) = \kappa_\alpha^{(n)} \pm \mathrm{sgn}[\kappa_\alpha^{(n)}(\Gamma^{(n)} + \varepsilon_T^{(n)})] \\ &\qquad \times \sqrt{(\sqrt{P^2 + D^2} + D)/2} \end{aligned} \tag{8}$$

where $P \equiv K_a^{(n)}(\Gamma^{(n)} + \varepsilon_{T^{(n)}})$ and $D \equiv 1 + K_{a^{(n)2}} - (\Gamma^{(n)} + \varepsilon_{T^{(n)}})^2/4$. Stability of the zero solution requires all $N+1$ eigenvalues to possess positive imaginary parts simultaneously. That in turn leads to the following conditions for stability:

$$\begin{aligned} &\max\left(\varepsilon_T^{(n)}\right) = \begin{cases} 2\varepsilon_\alpha^{(n)}[1 - \cos(2\pi/N)] & \text{for } \varepsilon_\alpha^{(n)} \geq 0, \\ 2\varepsilon_\alpha^{(n)}\{1 - \cos[(2\pi/N)\|N/2\|]\} & \text{for } \varepsilon_\alpha^{(n)} < 0, \end{cases} \\ &(\Gamma^{(n)} - \varepsilon_T^{(n)})/2 \geq \sqrt{(\sqrt{P^2 + D^2} - D)/2} \end{aligned} \tag{9}$$

where "$\|\ \|$" symbolizes for the integer part. The first condition in (9) assures the stability of the first $N-1$ modes existing in the uncoupled case as well, while the second one assures the stability of both the remaining modes.

3. Existence and stability of continuous wave (CW) nonlinear modes

Looking for CW nonlinear modes, that is, for modes of the same amplitude in every node of the circular array, one may easily find that the first $N-1$ of these modes are identical with those in the uncoupled case,

$$u_{n,\ell} = U_0 e^{i(\nu_\ell^{(n)} z^{(n)} + 2\pi\ell/N)}, \qquad n = 1, \ldots, N;$$

$$u_{0,l} = 0, \ell = 1, \ldots N-1 \tag{10}$$

where, now, the wave-number $\nu_\ell^{(n)}$ is given by,

$$\nu_\ell^{(n)} = 2\kappa_\alpha^{(n)} \cos(2\ell/N) + \mu_T^{(n)} |U_0|^2 - i\left[\varepsilon_0^{(n)} + 2\varepsilon_\alpha^{(n)} \cos(2\pi\ell/N)\right], \qquad \ell = 1, \ldots, N-1 \tag{11}$$

with $\mu_T \equiv \mu_0 + 2\mu_a$. Since $|U_0|^2 > 0$, from the real part of $\nu_\ell^{(n)}$ one can obtain the bifurcation behavior of these modes. It is quite interesting to notice that each one of these modes bifurcates from the zero solution for a different value of $\mathrm{Re}(\nu_\ell^{(n)})$. The bifurcation values are given by $\mathrm{Re}(\nu_\ell^{(n)}) = 2k_a \cos(2\pi\ell/N)$. This is an interesting *cascade* type of bifurcation. As far as the stability of these modes is concerned, it becomes obvious from Eq. (11) that the condition is identical to the first condition, Eq. (9), for stability of the zero solution. If this condition is satisfied the corresponding CW solution will at most decay to zero as it propagates.

As far as the rest of the modes are concerned, one may proceed as follows: By substituting in Eqs. (1) and (2) the values,

$$u_n = U_0 e^{i\nu^{(n)} z^{(n)}}, \qquad n = 1, \ldots, N; \qquad u_0 = V_0 e^{i\nu^{(n)} z^{(n)}} \tag{12}$$

the following set ($\pm$) of solutions is obtained:

$$U_0/V_0\big|_\pm = e\left[\pm\sqrt{1/e - t(\varepsilon_T^{(n)} + \Gamma^{(n)})^2/(1+e)^2} - i\,\mathrm{sgn}(\kappa_0)(\varepsilon_T^{(n)} + \Gamma^{(n)})/(1+e)\right]/\sqrt{N}, \tag{13}$$

$$|V_0|^2\big|_\pm = \left[2\kappa_\alpha^{(n)} \pm \mathrm{sgn}(\kappa_0)(1-e) \times \sqrt{1/e - (\varepsilon_T^{(n)} + \Gamma^{(n)})^2/(1+e)^2}\right]/(M^{(n)} - \mu^{(n)}) \tag{14}$$

along with the following expression for the real part of the wave-number $\nu^{(n)}$,

$$\mathrm{Re}(\nu_\pm^{(n)}) = \left[2\kappa_\alpha^{(n)} M^{(n)} \pm \mathrm{sgn}(\kappa_0)(M^{(n)} - e\mu^{(n)}) \times \sqrt{1/e - (\varepsilon_T^{(n)} + \Gamma^{(n)})^2/(1+e)^2}\right]/(M^{(n)} - \mu^{(n)}).$$

In Eqs. (13–15) the non-negative (necessary requirement for the existence of these CW solutions) parameter e was introduced, which is expressed in terms of the imaginary part of the wave-number $\nu^{(n)}$ (crucial for the existence of CW solutions as we will shortly see) given by $e = [\Gamma^{(n)} - \mathrm{Im}(\nu_\pm^{(n)})]/[\mathrm{Im}(\nu_\pm^{(n)}) + \varepsilon_T^{(n)}] \geq 0$ as well as the new normalized collective nonlinear coefficients, $\mu^{(n)} \equiv \mu_x^{(n)} + e\mu_T^{(n)}/N$ and $M^{(n)} \equiv M_0^{(n)} + eM_x^{(n)}$. One may easily show that Eq. (15) reduces to the second one in Eq. (8) in the linear limit $U_0 \to 0$, $V_0 \to 0$ while $U_0/V_0 \neq 0$, that is, $\mathrm{Re}(\nu_\pm^{(n)}) = \mathrm{Re}(\lambda_{N+1,\ldots,N}^{(n)})$.

In order for steady-state CW solutions to exist (neither growing, nor decaying, only oscillating) the imaginary part of the wave-number $\nu^{(n)}$ must be set equal to zero. In this case, the steady state CW solutions can then easily be obtained by substituting in Eqs. (13–14) the parameter e by $e = \Gamma^{(n)}/\varepsilon_T^{(n)} \geq 0$. The wave-number associated with the oscillatory behavior of these CW solutions is then provided by the same substitution in Eq. (15). Bifurcation diagrams for these steady-state CW solutions can then easily be obtained. Depending upon the choice of the various normalized coupling coefficients one may obtain one, two or none solutions for $|U_0|$ and $|V_0|$ as functions of the normalized amplification (or loss) rate $\varepsilon_0^{(n)}$.

An important issue for the existence of these steady state CW solutions is their stability. For these nonlinear modes to be stable the zero solution does not necessarily need to be stable. Thus, one can look for stable steady-state CW solutions even in the regimes where the zero solution is unstable (several or all the $N+1$ linear modes discussed in the previous Section). As far as the stability of the first $N-1$ nonlinear CW modes are concerned we already mentioned that if $\varepsilon_T^{(n)}$ does not exceeds the maximum value set by the first condition in Eq. (9) these modes are decaying. The second condition in Eq. (9) which assures the stability of two additional linear modes does not necessarily need to be satisfied in this case. On the other hand, the stability of the additional nonlinear modes expressed by Eqs. (13–14) with the imaginary part of the wave-number set to zero, cannot be analytically addressed. Instead, one may linearize Eqs. (1) and (2) around a particular choice for $|U_0|$ and $|V_0|$ taken from the respective bifurcation diagram. This procedure renders a solvable linear system of $2N+2$ equations for the $N+1$ (randomly chosen) small perturbations of the respective CWs, δu_n ($n = 1, \ldots, N$) and δu_0 and their complex conjugates, expressed as a $(2N+2) \times 1$ column vector $\delta\boldsymbol{u}(z^{(n)})$. The solution can straightforwardly cast in the following form,

$$\delta\boldsymbol{u}(z^{(n)}) = \exp(z^{(n)}\mathbf{A}) \circ \delta\boldsymbol{u}(z^{(n)} = 0),$$

$$\mathbf{A} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b}^* & \mathbf{a}^* \end{pmatrix}, \qquad \mathbf{a} \equiv \begin{pmatrix} \mathbf{T_a} & \mathbf{C_a} \\ \mathbf{R_a} & S_\mathbf{a} \end{pmatrix},$$

$$\mathbf{b} \equiv \begin{pmatrix} \mathbf{T_b} & \mathbf{C_b} \\ \mathbf{R_b} & S_\mathbf{b} \end{pmatrix} \tag{16}$$

where $\mathbf{A}$ is a $(2N+2) \times (2N+2)$ block matrix. Its block sub-matrices, $\mathbf{a}$ and $\mathbf{b}$ (the asterisk denotes complex conjugate) are also block-type $(N+1) \times (N+1)$ matrices and each one consists of one Toeplitz $N \times N$ sub-matrix (**T**), one $N \times 1$ column vector (**C**), one $1 \times N$ row vector (**R**) and a scalar element (S). The constituents of the matrix **A** depend upon the choice of steady-state CW and the coupling coefficients. The stability is then obviously related directly to the asymptotic behavior (annihilation as $z^{(n)} \to \infty$) of the respective exponential matrix $\mathbf{exp}(\boldsymbol{z}^{(n)}\mathbf{A})$. This issue will be discussed in the next Section.

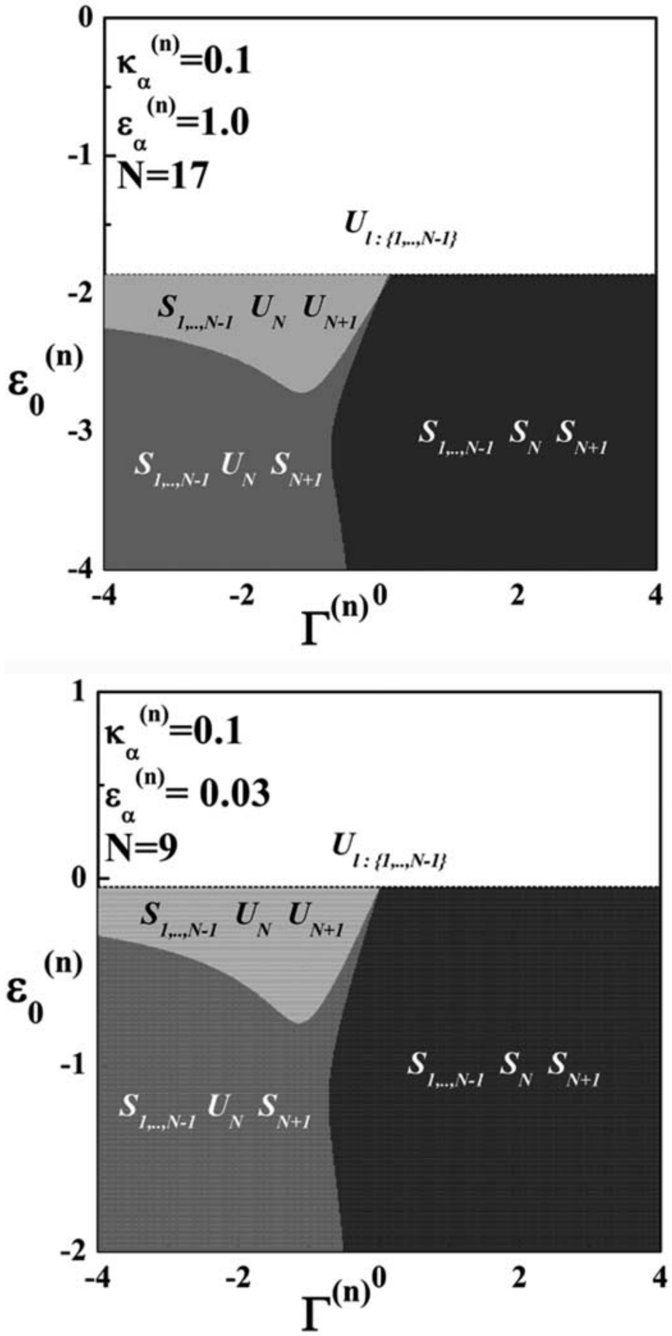

Fig. 1. Stability diagrams on ($\varepsilon_0^{(n)} - \Gamma^{(n)}$)-plane for the zero solutions for $K_a^{(n)} = 0.1$ and: (a) $\varepsilon_a^{(n)} = 1, \ldots, N = 17$; (b) $\varepsilon_a^{(n)} = 0.03, N = 9$. At least one of the first $N-1$ linear modes is unstable (white), all the first $N-1$ linear modes are stable and both of the remaining linear modes are unstable (light gray), all but one of the linear due to coupling modes are stable (gray), the zero solution is completely stable (dark gray).

4. Discussion of various examples and conclusions

The stability of the zero solutions is best depicted in area diagrams such as the ones in Fig. 1. The ($\varepsilon_0^{(n)} - \Gamma^{(n)}$) diagrams are parameterized by only three parameters, namely the number of nodes (it becomes not necessary for $\varepsilon_a^{(n)} = 0$), and the values of the normalized coupling coefficients (real and imaginary) to the nearest neighbors. In the white area *at least one* of the first $N-1$ linear modes (which also exist in the no-core case) is unstable. In the light gray area *all* the first $N-1$ linear modes are stable, while both of the remaining linear modes (due to coupling) are unstable. In the gray area all but one of the linear due to coupling modes are stable and finally in the dark gray area the zero solution is completely stable. It is easy to observe that in every case the stability is assured only if the array elements are lossy ($\varepsilon_0^{(n)} < 0$). However, there could be energy channeling to the neighbors ($\varepsilon_a^{(n)} > 0$) and amplifying core ($\Gamma^{(n)} < 0$) in such a case. This fact leads to the conclusion that a conceptual device with an optically amplifying central core coupled with an array on its circumference will be linearly stable as far as the noise level is concerned.

The existence and stability of the nonlinear CW modes is depicted in an example in Fig. 2. The bifurcation diagrams expressed in terms of $|U_0|$ and $|V_0|$ versus $\varepsilon_0^{(n)}$ are respectively given by Fig. 2(a) and (b) and Fig. 2(c) and (d) for a particular choice of parameters ($K_a^{(n)} = 0.1$, $\varepsilon_a^{(n)} = 1.0$, $N = 17$, $\Gamma^{(n)} = 0.4$, $\mu_0^{(n)} = M_0^{(n)} = 1$, $\mu_x^{(n)} = \mu_a^{(n)} = M_x^{(n)} = 2$). The dotted circle in these diagrams denotes the starting non-zero values for the CW amplitudes. The shaded area is the area where *at least one* of the first $N-1$ linear modes is unstable and the vertical dotted line is an asymptote. One can easily observe that the CW modes are mainly emerging from zero. Choosing a steady-state CW value close to the left non-zero emerging point ($\varepsilon_0^{(n)} = -1.90$) leads to a stable situation as one can easily observe in Fig. 2(e) and (f) where the propagation of the respective CWs in the array and the central core are depicted in terms of the associated spectral power density. Upon moving in $\varepsilon_0^{(n)}$ to the right ($\varepsilon_0^{(n)} = -1.88$) the CW becomes unstable as it shown in Fig. 2(g) and (h). Extensive numerical simulation leads to the tentative conclusion that stable steady-state CWs reside near the left bound of the bifurcation diagrams.

The investigation has also been focused on the existence of stable localized solutions. In the model at hand all the nonlinear coupling coefficients are real, that is nonlinear losses do not exist. In such a case, at least when the central coupling is absent, there is evidence in the literature [14] that a highly confined discrete soliton solution cannot exist. This is because, if this were the case, the spectral energy density of this structure should scale inversely proportional to the (positive) imaginary part of the coefficient of self-phase modulation term, which is set by default to zero in our model, rendering the existence of a localized mode of immense power improbable. However, broader solutions may possibly exist. There is evidence in the literature [17] that there exist breathing solutions in the discrete Ginzburg–Landau equation which is the planar-periodic no-core limit of the model under investigation. Our investigation is thus turned into the existence of breathing solutions, that is, solutions whose intensity exhibit azimuthally periodic behavior as the light propagates along z. Different input amplitudes should also converge to the same breathing solution locking-in itself to this breathing behavior. One such example is presented in Fig. 3: For the particular choice of this example for the parameters involved ($K_a^{(n)} = 0.296$, $\varepsilon_a^{(n)} = 0.423$, $N = 35$, $\Gamma^{(n)} = 0.062$, $\mu_0^{(n)} = M_0^{(n)} = 0.423$, $\mu_x^{(n)} = 0$, $\mu_a^{(n)} = M_x^{(n)} = 0.845$) from the respective bifurcation diagram for the steady-state CW solutions (not shown here) it is evident that steady state CWs with $|U_0| \approx 1$ and $|V_0| \approx 0$ certainly reside near the

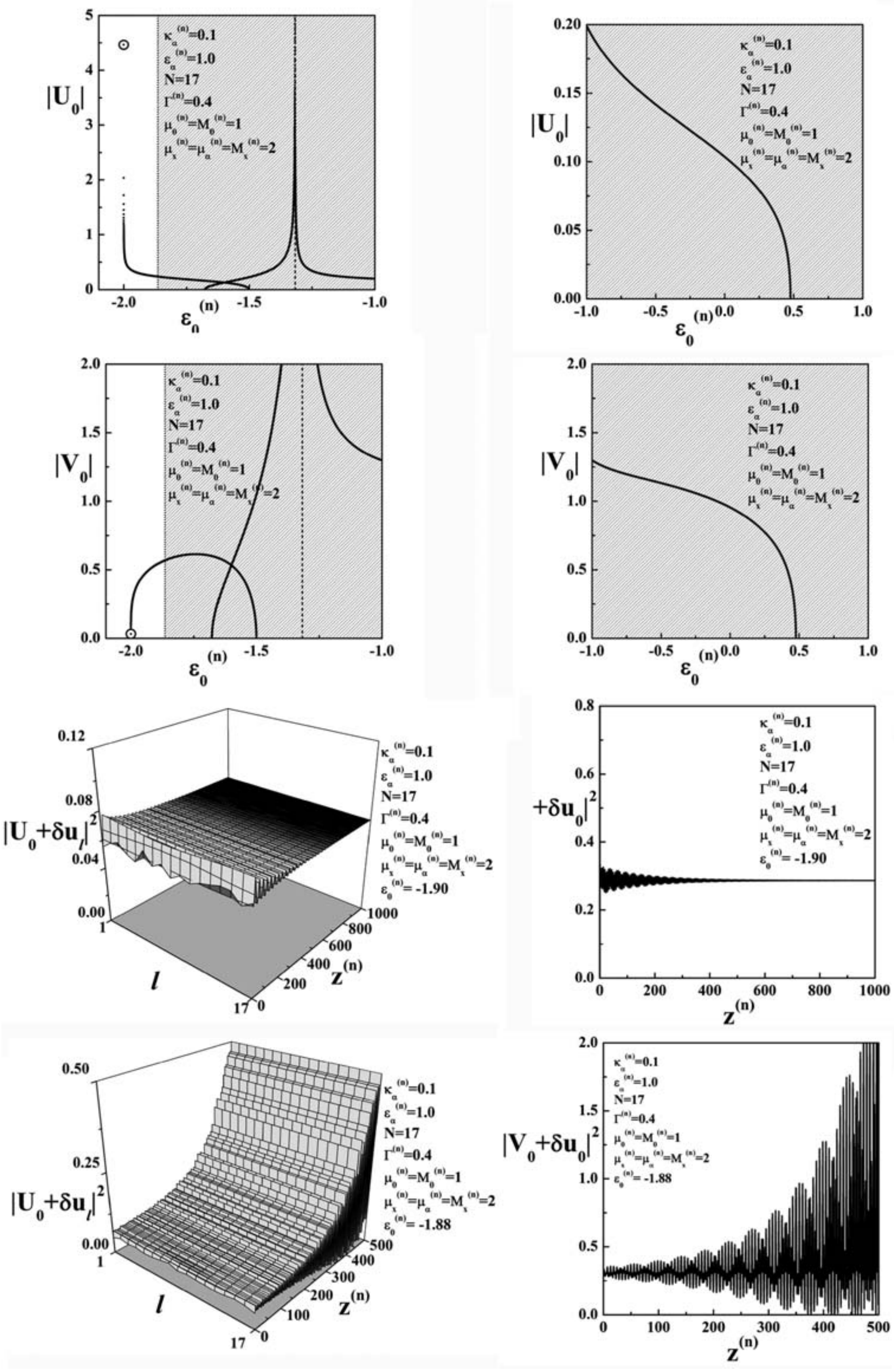

Fig. 2. Bifurcation diagrams of $|U_0|$ (a, b) and $|V_0|$ (c, d) versus $\varepsilon_0^{(n)}$ for $K_a^{(n)} = 0.1$, $\varepsilon_a^{(n)} = 1.0$, $N = 17$, $\Gamma^{(n)} = 0.4$, $\mu_0^{(n)} = M_0^{(n)} = 1$, $\mu_x^{(n)} = \mu_a^{(n)} = M_x^{(n)} = 2$. The dotted circle denotes the starting non-zero values for the CW amplitudes. In the shaded area at least one of the first $N - 1$ linear modes is unstable and the vertical dotted line is an asymptote; (e, f): propagation of CWs in the array and the central core, respectively for $\varepsilon_0^{(n)} = -1.90$; (g, h): propagation of CWs in the array and the central core, respectively for $\varepsilon_0^{(n)} = -1.88$.

left boundary in the bifurcation diagrams. According to our previous (related to Fig. 2) tentative observation, these CWs are stable. With the same choice of parameters we now launch a beam at the 9th node with $u_9 = U_0 = 1 (u_{n\neq 9} = 0)$ and $u_0 = V_0 = 0$. After a short transition distance a breathing behavior is established (Fig. 3(a)). This established behavior is shown in the gray scale diagram in Fig. 3(b) for large distances. In Fig. 3(c), on the other hand, an established periodic behavior in the central core, intuitively expected, is also observed. This asymptotically breathing behavior is robust as far as the input values are concerned as long as the launching is highly localized (preferably in one node of the array if N is small; the specific node is irrelevant since the system is cyclic). Simulations support this conclusion. However, moving away from the left boundary of the bifurcation diagrams of the respective steady-state CWs results into a chaotic short of behavior: For instance, setting only $\mu_a^{(n)} = 0.0$ leads to the behavior shown in Fig. 3(d). Asymptotically periodic behavior is lacking for both the

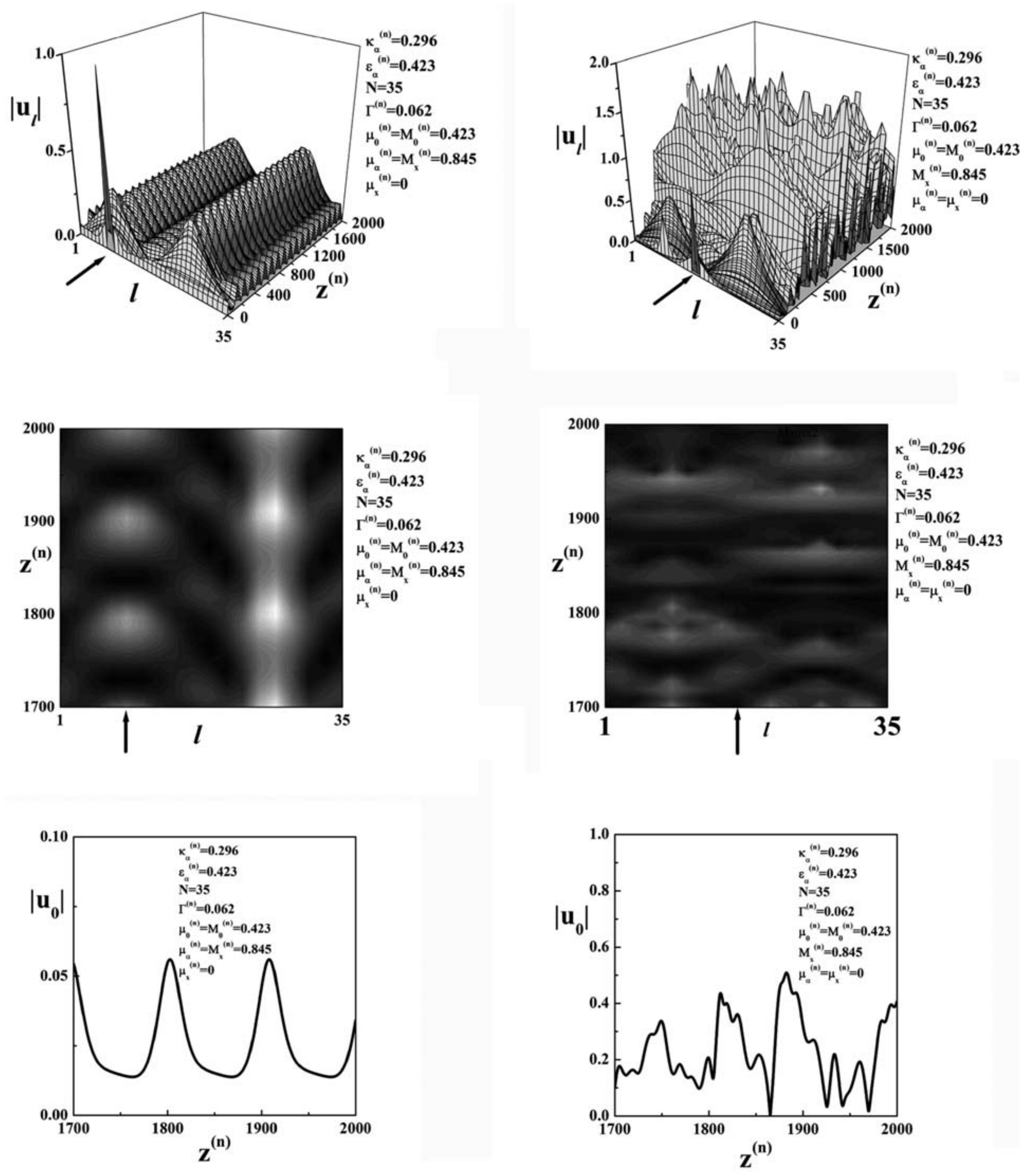

Fig. 3. Breathing modes: (a) launching a beam at the 9th node with $u_9 = U_0 = 1$ ($u_{n\neq 9} = 0$) and $u_0 = V_0 = 0$ and $\mu_a^{(n)} = 0.845$; (b) gray scale diagram of the array amplitudes at large distances; (c) dynamics of the central core amplitude at large distances; (d) launching a beam at the 17th node with $u_{17} = U_0 = 1$ ($u_{n\neq 17} = 0$), $u_0 = V_0 = 0$ and $\mu_a^{(n)} = 0$; (e) gray scale diagram of the array amplitudes at large distances; (f) dynamics of the central core amplitude at large distances. The rest of the parameters are set to $K_a^{(n)} = 0.296$, $\varepsilon_a^{(n)} = 0.423$, $N = 35$, $\Gamma^{(n)} = 0.062$, $\mu_0^{(n)} = M_0^{(n)} = 0.423$, $\mu_x^{(n)} = 0$, $M_x^{(n)} = 0.845$.

circular array (Fig. 3(e)) and the central core (Fig. 3(f)).

In conclusion, the existence of stable steady-state CW solutions has been demonstrated. Furthermore, for a parameters' choice leading to these stable steady-state CW solutions, there also exist breathing semi-localized solutions. The system in hand exhibits complex physics and can be important for novel high power stable amplifiers in optical communications where, for their design, the study of stability and phase-locking during evolution of the nonlinear modes is of great importance.

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