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Exact X-wave solutions of the hyperbolic nonlinear Schrödinger equation with a supporting potential

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ABSTRACT

We find exact solutions of the two- and three-dimensional nonlinear Schrödinger equation with a supporting potential. We focus in the case where the diffraction operator is of the hyperbolic type and both the potential and the solution have the form of an X-wave. Following similar arguments, several additional families of exact solutions can also can be found irrespectively of the type of the diffraction operator (hyperbolic or elliptic) or the dimensionality of the problem. In particular we present two such examples: The one-dimensional nonlinear Schrödinger equation with a stationary and a "breathing" potential and the two-dimensional nonlinear Schrödinger with a Bessel potential.

1. Introduction

The nonlinear Schrödinger equation (NLS) is one of the fundamental nonlinear wave equations with applications in many branches of science. In the case of one spatial dimension it is integrable and it can be exactly solved using the inverse scattering transform [1]. However, in two or three dimensions, and irrespectively of the type of the diffraction operator, no exact solutions have been found although similarity transformations can transform solutions of the NLS into different solutions of the NLS [2,3]. In such higher dimensional settings exact localized solutions of non-diffracting beams exist in the linear limit. These include for example the Bessel beams supported by the wave equation and the Schrödinger equation with an elliptic diffraction operator [4,5]. In addition, X-wave solutions of the wave equation were proposed and experimentally observed in acoustic waves [6,7]. Algebraic [8] and Bessel [9] X-wave solutions have also been predicted as solutions of the Schrödinger equation. In addition, in [8,10] nonlinear X-waves were suggested and observed. Nonlinear X-waves have also been spontaneously generated in waveguide arrays [11]. In [12] discrete X, Y and Z-shaped patterns were proposed. Methods similar to those used here for constructing exact solutions were suggested in [13–15].

In this Letter we show that families of exact stationary solutions of higher dimensional NLS equations can be found by including an additional potential term. We focus in the cases of $2+1$ and $3+1$ dimensions with a hyperbolic diffraction operator. In optical settings such a diffraction operator is possible if the dispersion is normal. The solutions as well as the supporting potentials have the form of an X-wave. Such solutions can be considered as examples of several other classes of exact solutions. In particular, we present two additional examples: an exact solution of the one-dimensional NLS with a potential that can be stationary along z or periodic along z (breathing) and an exact solution of the two-dimensional NLS equation with a Bessel potential.

2. Two-dimensional solutions

The NLS equation in $2+1$ dimensions with a hyperbolic diffraction operator, a potential term, and a cubic (Kerr) nonlinearity reads

$$i\psi_z + \psi_{xx} - \psi_{yy} + V(x, y)\psi + |\psi|^2\psi = 0. \tag{1}$$

Eq. (1) is normalized, ψ is the amplitude of the wave, and $V(x, y)$ is the potential. Notice that the sign of the nonlinearity in Eq. (1) can change from self-focusing to self-defocusing via the transformation $z \to -z$, $x \to y$, $y \to x$, $V \to -V$. Thus in Eq. (1) without loss of generality the nonlinearity can be chosen to be self-focusing. The conserved quantities of Eq. (1) are the total power

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi|^2\, dx\, dy \tag{2}$$

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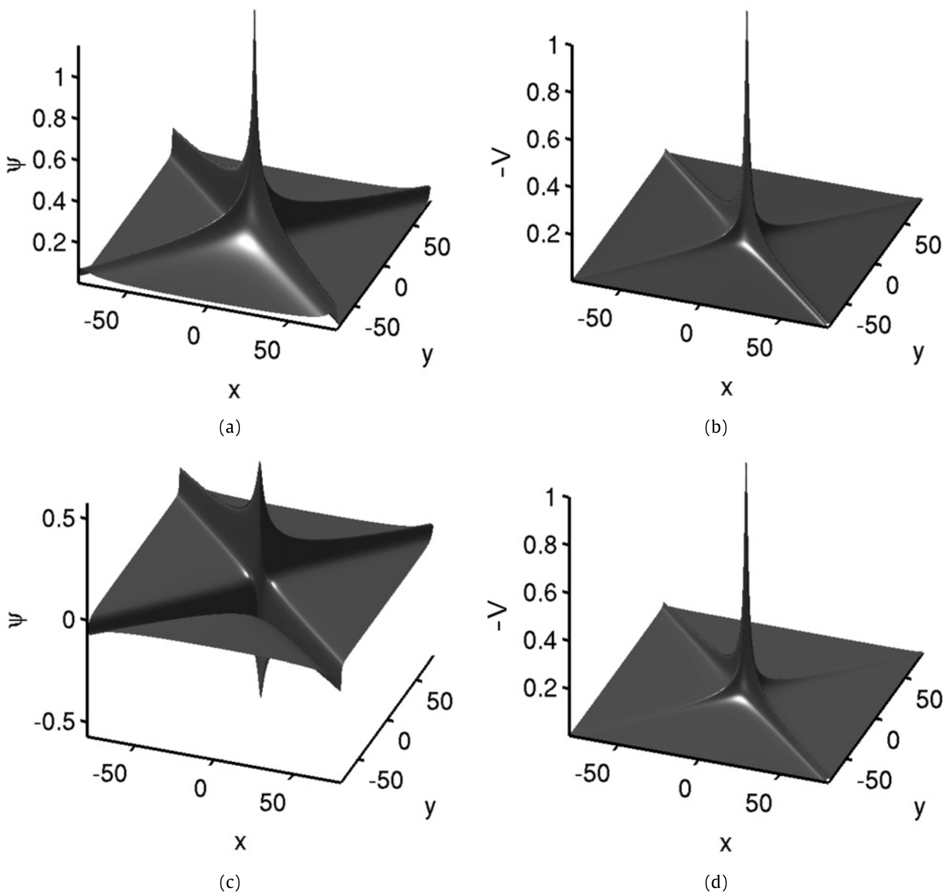

Fig. 1. Amplitude of the two-dimensional X-wave solutions (left column) and the corresponding negative of the potentials (right column) for (a), (b) $\alpha = 2$, $\beta = 1$ and (c), (d) $\alpha = 1$, $\beta = 2$.

and the Hamiltonian

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[|\psi_x|^2 - |\psi_y|^2 - \frac{1}{2}|\psi|^4 - V(x,y)|\psi|^2\right]dx\,dy. \tag{3}$$

By direct substitution it can be proved that Eq. (1) supports solutions of the form

$$\psi = \frac{\sqrt{2}}{(u^2+4v^2)^{1/4}}\left[1 - \frac{u}{(u^2+4v^2)^{1/2}}\right]^{1/2}, \tag{4}$$

if $\alpha^2 > \beta^2$ and

$$\psi = \frac{\sqrt{2}\,\mathrm{sgn}(v)}{(u^2+4v^2)^{1/4}}\left[1 - \frac{u}{(u^2+4v^2)^{1/2}}\right]^{1/2}, \tag{5}$$

if $\alpha^2 < \beta^2$, where the functions u and v are given by

$$u = x^2 + \beta^2 - y^2 - \alpha^2, \tag{6}$$

and

$$v = \alpha x + \beta y. \tag{7}$$

In both cases the supporting potential is

$$V = \frac{-3}{(u^2+4v^2)^{1/2}}. \tag{8}$$

The solutions (4)–(5) as well as the potential (8) are not singular provided that $|\beta| \neq |\alpha|$. If $\beta = \pm\alpha$ then along the line $x \pm y = 0$ (respectively) both $u(x,y)$ and $v(x,y)$ are equal to zero. Thus, both $\psi(x,y)$ and $V(x,y)$ go to infinity as (x,y) approach this line.

Notice that the expression inside the square brackets in Eqs. (4)–(5) becomes zero if $v(x,y) = 0$ and in addition $u(x,y) > 0$. Assuming that $\beta \neq 0$ we can combine these two conditions to

$$g(x) = u\left(x, -\frac{\alpha x}{\beta}\right) = x^2\left(1 - \frac{\alpha^2}{\beta^2}\right) + \beta^2 - \alpha^2 > 0. \tag{9}$$

The derivative $g'(x)$ is zero when $x = 0$. If $\beta^2 > \alpha^2$ then the minimum of $g(x)$ is given by $g(0) = \beta^2 - \alpha^2 > 0$ $[g(x) \geqslant g(0) > 0]$. On the other hand if $\alpha^2 > \beta^2$ we find that $g(x) \leqslant g(0) = \beta^2 - \alpha^2 < 0$. We conclude that along the line $v(x,y) = 0$ the relation $\mathrm{sgn}(u(x,y)) = \mathrm{sgn}(\beta^2 - \alpha^2)$ holds true. Thus the solution given by Eq. (4) is always positive, whereas ψ in Eq. (5) is zero along the line $v = 0$. The expression $\mathrm{sgn}(v)$ in Eq. (5) takes care of the continuity of the derivative of the solution along this line. Typical examples of these solutions and the corresponding potentials are depicted in Fig. 1.

We would like to point out that these $(2+1)$-dimensional solutions of the NLS equation with a potential are the result of the balance between diffraction, nonlinearity and the potential term.

The intensity of the X-wave solutions is given by

$$I = |\psi|^2 = \frac{2}{(u^2+4v^2)^{1/2}}\left|1 - \frac{u}{(u^2+4v^2)^{1/2}}\right|. \tag{10}$$

It can be proved that the total power P carried out by the X-wave is infinite. We first note that the solution is not singular and thus the total power is finite inside any circular disk $\rho = \sqrt{x^2+y^2} < \rho_0$ with center the origin $(0,0)$ and a finite radius ρ_0. Thus the solution carries infinite power if and only if the integral (2) is infinite outside this disk. In polar coordinates we can find a continuous function $\Theta(\theta)$ with the properties (i) that its maximum is given

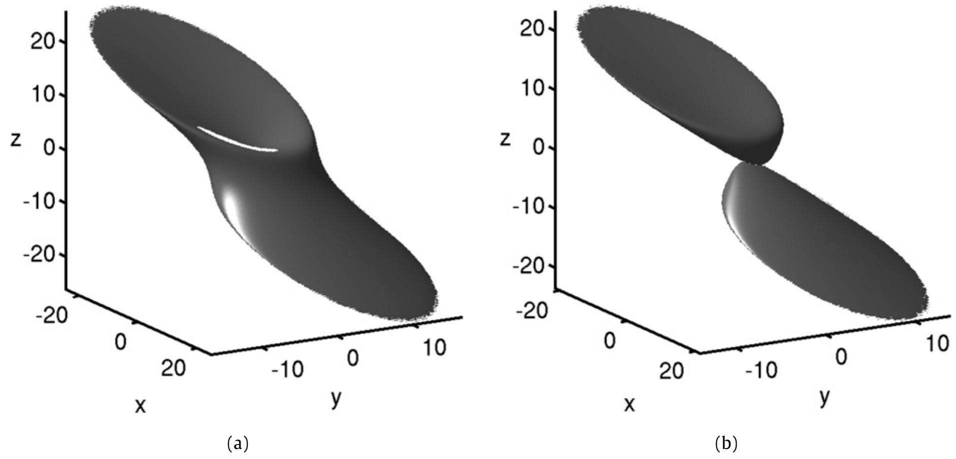

Fig. 2. Iso-surface plot depicting the absolute value of the amplitude for $|\psi| = 0.26$ of the three-dimensional X-wave solutions. In particular, in (a) and (b) the solutions ψ^+ and ψ^- are shown, respectively, for $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_3 = 3$.

by $\Theta(\pi/4) = 1$ and (ii) that away from the maximum it decays fast enough to zero so that the inequality $I(\rho,\theta) \geqslant I(\rho,\pi/4)\Theta(\theta)$ holds true. Then

$$P(\rho_0 \leqslant \rho \leqslant R) \geqslant 2\pi A \int_{\rho_0}^{R} \rho I(\rho,\pi/4)\,d\rho \tag{11}$$

where $A = \int_0^{2\pi} \Theta(\theta)\,d\theta$. Using inequalities it can be shown that for $\theta = \pi/4$ and $\rho > \rho_0$ the intensity is bounded from below according to

$$I > \frac{1}{2\rho^2(\alpha^2 + \beta^2)}. \tag{12}$$

Thus, we find the following lower bound

$$P(\rho_0 \leqslant \rho \leqslant R) > \frac{1}{\alpha^2 + \beta^2} \ln\left(\frac{R}{\rho_0}\right) \tag{13}$$

for the X-wave power. Taking the limit $R \to \infty$ this lower bound diverges to infinity.

3. Three-dimensional solutions

In the case of three dimensions the nonlinear Schrödinger equation with a hyperbolic diffraction operator, potential, and cubic nonlinearity takes the following form

$$i\psi_t + \psi_{xx} + \psi_{yy} - \psi_{zz} + V(x,y,z)\psi + |\psi|^2\psi = 0 \tag{14}$$

in normalized coordinates. By direct substitution it can be shown that Eq. (14) supports the exact solutions

$$\psi^+ = \frac{1}{(u^2 + 4v^2)^{1/4}}\left[1 + \frac{u}{(u^2 + 4v^2)^{1/2}}\right]^{1/2} \tag{15}$$

and

$$\psi^- = \frac{\operatorname{sgn}(v)}{(u^2 + 4v^2)^{1/4}}\left[1 - \frac{u}{(u^2 + 4v^2)^{1/2}}\right]^{1/2} \tag{16}$$

where in the previous expressions

$$u(x,y,z) = x^2 + y^2 - z^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2, \tag{17}$$

$$v(x,y,z) = \alpha_1 x + \alpha_2 y + \alpha_3 z \tag{18}$$

and $\alpha_3^2 > \alpha_1^2 + \alpha_2^2$. The corresponding potential for each one of these solutions is different and is given by $V^\pm = -(\psi^\pm)^2$. Note that $\psi^\pm$ are solutions of the linear Schrödinger equation without a potential. In Eq. (14) the nonlinearity is exactly balanced by the potential term. In a similar manner, real exact solutions of the linear Schrödinger equation can become solutions of the NLS equation with a potential. In contrast the two-dimensional solution presented in the previous section does not satisfy the linear Schrödinger equation.

The solutions (15)–(16) are not singular provided that $\Delta = \alpha_3^2 - \alpha_1^2 - \alpha_2^2 > 0$. This can be proved as following: The functions $\psi^\pm$ have singularities in the points (x,y,z) of the three-dimensional space that satisfy $u(x,y,z) = 0$ along with $v(x,y,z) = 0$. Assuming that $\alpha_3 \neq 0$ these conditions are equivalent to

$$f(x,y) = x^2\left(1 - \frac{\alpha_1^2}{\alpha_3^2}\right) + y^2\left(1 - \frac{\alpha_2^2}{\alpha_3^2}\right) - 2\frac{\alpha_1\alpha_2}{\alpha_3^2}xy + \Delta = 0. \tag{19}$$

The function $f(x,y)$ is elliptic if $\Delta > 0$, parabolic if $\Delta = 0$ and hyperbolic if $\Delta < 0$. In the hyperbolic case the origin $(0,0)$ is a saddle point and the condition $f(x,y) = 0$ is satisfied along the two hyperbolic curves. In the parabolic case equation $f(x,y) = 0$ holds along the straight line $x\alpha_2 = y\alpha_1$ that passes from the origin. In the elliptic case the function $f(x,y)$ has a minimum $f(0,0) = \Delta > 0$. Thus, equation $f(x,y) = 0$ cannot be satisfied and the solutions $\psi^\pm$ are not singular. Iso-surface plots of such solutions and the corresponding potentials are depicted in Fig. 2.

The solution ψ^+ is always positive because $u > 0$ when $v = 0$. On the other hand ψ^- becomes zero along the plane $v = 0$. In Eq. (16) the function $\operatorname{sgn}(v)$ takes care of the continuity of the first derivative. As in the two-dimensional case, it can be shown that the total power carried out by the nonlinear X-waves is infinite.

Considering the stability of the two- and three-dimensional nonlinear X-waves presented here, our numerical simulations show that the solutions remain stable for relatively long propagation distances. However, the source leading to the destabilization is numerical and related to the domain boundaries. We would like to point out that is it difficult to avoid such boundary effects which generate instabilities because the amplitude of the X-waves in particular directions decays slowly (as $1/r$) at infinity. Thus our simulations are not conclusive to whether the solutions are stable, but show the general trend of the X-waves to at least propagate in a stable fashion for appreciable distances.

4. Conclusions and extensions of this work

In conclusion, we have found exact solutions of the two- and three-dimensional nonlinear Schrödinger equation in the presence of a potential. In the three-dimensional case, the solution presented is also a solution of the linear Schrödinger equation and the potential counteracts the effect of nonlinearity. On the other hand, in the two-dimensional case the exact solution results from the balance between diffraction, nonlinearity, and the potential.

We would like to point out that using similar arguments, several other classes of solutions can be found for the NLS equation in one, two, or three dimensions with a hyperbolic or elliptic diffraction operator. Here we present two such examples. The first one is related with the one-dimensional NLS equation

$$i\psi_z + \psi_{xx} + \gamma_1 V(x)\psi + \gamma_2|\psi|^2\psi = 0, \tag{20}$$

for $\gamma_1 + \gamma_2 = 1$. It can be shown that Eq. (20) is satisfied for

$$V(x) = 2\alpha\,\mathrm{sech}^2(\sqrt{\alpha}x) \tag{21}$$

and

$$\psi(x) = \sqrt{2\alpha}\,\mathrm{sech}(\sqrt{\alpha}x)\exp(i\alpha z). \tag{22}$$

Additional families of exact solutions originating from higher order solitons can be found. In particular Eq. (20) is also satisfied for

$$\psi(x,z) = 4\sqrt{2}e^{iz}\frac{\cosh(3x) + 3e^{8iz}\cosh(x)}{\cosh(4x) + 4\cosh(2x) + 3\cos(8z)} \tag{23}$$

and

$$V(x,z) = 32\frac{\cosh^2(3x) + 9\cosh^2(x) + 6\cosh(x)\cosh(3x)\cos(8z)}{(\cosh(4x) + 4\cosh(2x) + 3\cos(8z))^2}. \tag{24}$$

In the case $\gamma_1 = 0$, $\gamma_2 = 1$ this solution is identical to the two soliton solution of the NLS equation. If $\gamma_1 = 1$, $\gamma_2 = 0$ we obtain a localized mode of the Schrödinger equation with a "breathing" potential. If $\gamma_1 \neq 0, 1$ we find an exact solution of an NLS equation with a periodic z-dependent potential. We would like to point out that the solutions presented above are also valid for self-defocusing nonlinearity (if $\gamma_2 < 0$).

The second example concerns the NLS equation in two spatial dimensions with an elliptic diffraction operator

$$i\psi_z + \psi_{xx} + \psi_{yy} + V(x,y)\psi + |\psi|^2\psi = 0. \tag{25}$$

Eq. (25) supports Bessel solutions of the form

$$\psi = J_0(k\rho)\exp(-ik^2 z) \tag{26}$$

where the potential is given by $V = -|\psi|^2 = -J_0^2(k\rho)$.

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