Bessel X waves in two- and three-dimensional bidispersive optical systems

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We show that new families of two- and three-dimensional nondiffracting Bessel X waves are possible in linear bidispersive optical systems. These X waves can be observed in both bulk and waveguide configurations as well as in photonic crystal lattices that simultaneously exhibit normal and anomalous dispersive–diffractive properties in different spatial or spatiotemporal coordinates.

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The broadening of a wave packet through the processes of diffraction and dispersion is a ubiquitous phenomenon that is intrinsically tied to the nature of waves themselves. Interestingly, however, there are occasions where a spatial or a spatiotemporal localized optical wave can propagate free of diffraction and (or) dispersion effects.[1–9] What makes this possible is the fact that these beams, albeit weakly localized, convey infinite energy or power as opposed, for example, to strongly (exponentially) confined fields (waveguided, solitonic, etc.) for which these quantities are finite. A prime example of such a nondiffracting wave configuration is the so-called Bessel beam proposed by Durnin *et al.*[1]

Another important family of such localized solutions is that of X waves, first predicted and observed by Lu and Greenleaf within the context of ultrasonics.[2,3] Subsequently, optical spatiotemporal X waves were also obtained in free space.[6,7] In recent studies nonlinearly induced X waves were experimentally[8] and theoretically[9] investigated in both quadratic and Kerr normally dispersive nonlinear media. More specifically, spontaneously generated X-shaped light bullets (spatiotemporally localized waves) were successfully observed in lithium triborate $\chi^{(2)}$ crystals.[8]

It is perhaps interesting to note that so far all known X-wave solutions correspond to (3 + 1)D wave problems (three space coordinates + time). The question, of course, arises as to whether X-wave solutions are also possible in systems with reduced dimensionality [i.e., two-dimensional (2D) systems].

In this Letter we demonstrate that new families of 2D and three-dimensional (3D) nondiffracting Bessel X-wave packets are possible in bidispersive optical systems. These X waves can be obtained in diverse physical settings such as bulk and waveguide configurations as well as in photonic crystal lattices. A necessary condition for these waves to exist is that the system involved should exhibit bidispersive properties, i.e., both normal and anomalous dispersive–diffractive behavior in different spatial or spatiotemporal coordinates. The general properties of this family of X waves are analyzed, and it is shown that they can involve a complex spin or pseudo-vorticity (and angular momenta).

Before we discuss this class of X waves, it might be beneficial to mention some of the physical bidispersive systems that can support their existence. For example, wave propagation accounting for diffraction and dispersion effects in a bulk normally dispersive medium can be effectively described within the paraxial approximation

$$i\,\frac{\partial\psi}{\partial z} + \frac{1}{2k}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - \frac{k''}{2}\,\frac{\partial^2\psi}{\partial\tau^2} = 0\,, \tag{1}$$

where the term associated with the transverse Laplacian operator describes the process of diffraction and the term involving the time coordinate $\tau = t - z/v$ is associated with dispersive effects. k represents the wave number, $k'' = d^2k/\mathrm{d}\omega^2$ is the dispersion coefficient evaluated at the carrier frequency of the wave ω_0 and is positive ($k'' > 0$) for normally dispersive materials.[10] The above (3 + 1)D Eq. (1) can be reduced to a (2 + 1)D spatiotemporal problem if, for example, waveguiding is provided along, say, the y-transverse coordinate in which case diffraction effects in this direction are eliminated ($\partial^2\psi/\partial y^2 = 0$). A mathematically equivalent (2 + 1)D problem can also arise in describing beam diffraction in photonic lattices.[11,12] In such systems the Floquet–Bloch envelope function is found to obey[12]

$$i\,\frac{\partial\psi}{\partial z} + d_x\,\frac{\partial^2\psi}{\partial x^2} + d_y\,\frac{\partial^2\psi}{\partial y^2} = 0\,, \tag{2}$$

where d_x and d_y represent the effective diffraction coefficients in this lattice as obtained from the corresponding curvatures of the spatial dispersion relation. Of particular interest here are certain locations within the first Brillouin zone [for example, the X point corresponding to a square lattice with $p = 0$, $q = \pi$ (Ref. 12)], where $d_x d_y < 0$. In this case the diffraction is, for instance, normal in the x direction and anomalous along y. Both Eqs. (1) and (2) can be judiciously scaled by normalizing the independent variables involved. In this case they assume the form

$$i\frac{\partial \psi}{\partial Z} + \frac{\partial^2 \psi}{\partial X^2} - \frac{\partial^2 \psi}{\partial Y^2} = 0\,, \tag{3a}$$

$$i\frac{\partial \psi}{\partial Z} + \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} - \frac{\partial^2 \psi}{\partial T^2} = 0\,. \tag{3b}$$

The (2 + 1)D Eq. (3a) is a scaled version of Eq. (2), and similarly (3 + 1)D Eq. (3b) corresponds to Eq. (1). The independent variables X, Y, Z, and T represent dimensionless spatial or spatiotemporal coordinates, depending on the nature of the underlying problem (as per our previous discussion).

Note that Eqs. (3) are bidispersive since they exhibit simultaneously both normal and anomalous ($\pm$) diffraction–dispersion behavior along different coordinates. To better appreciate the outcome of bidispersion, let us consider the evolution of an initially circular 2D Gaussian beam resting on a constant background, i.e., $\psi(X, Y, Z = 0) = 1 + B\exp[-(X^2 + Y^2)]$. Figure 1(a) depicts the intensity profile of this beam at the origin, whereas Fig. 1(b) shows the beam intensity at $Z = 2$ when $B = 1$ [after analytically solving Eq. (3a)]. In this case an X pattern is generated as a result of bidispersion. Notice that in the absence of this effect ($d_x d_y > 0$) the beam would have evolved into a circular or an elliptical object.

We first seek analytical X-wave solutions for the 2D case, e.g., Eq. (3a). A straightforward calculation shows that Eq. (3a) admits of the following 2D Bessel X-wave packet solution:

$$\begin{aligned}\psi &= AK_m\{a[X^2 + (iY + c)^2]^{1/2}\}\\ &\quad\times \exp\left[im\tan^{-1}\left(\frac{iY + c}{X}\right)\right]\exp(ia^2 Z)\,,\\ &= AK_m\{a[X^2 + (iY + c)^2]^{1/2}\}\\ &\quad\times\left[\frac{i(X - Y) - c}{i(X + Y) + c}\right]^{m/2}\exp(ia^2 Z)\,,\end{aligned} \tag{4}$$

where A is a constant wave amplitude, $K_m(z)$ is a modified Bessel function of the second type, the integer order m ($m = 0, 1, 2\ldots$) accounts for an intrinsic pseudo-vorticity of the beam, and a is related to the beam width. Finally, the arbitrary constant c determines an inner spatial scale at which the singularity of the K_m function is suppressed, so the field attains a maximum, $AK_m(ac)$, at the central point $X = Y = 0$.[13] In obtaining the second form of Eq. (4) we used the relation $\tan^{-1}(z) = (-i/2)\ln[(1 + iz)/(1 - iz)]$.

Figures 2(a)–2(d) show the intensity profiles of such 2D Bessel X waves when $a = 1$, $c = 1$, and their vorticity is $m = 0, 1, 2, 3$. The phase associated with the first two solutions ($m = 0, 1$) is also shown in Fig. 3 for the same set of parameters. We stress that, in contrast with ordinary vortices of order m, which are distinguished by phase circulation $2m\pi$ about the center,[14] the circulation (i.e., true vorticity) of the present solutions is zero for all values of m. This becomes apparent if one notices that the fractional term in the last expression of Eq. (4) has no poles or zeros for any finite c. It is important to note that, for $m \neq 0$, the vorticity function $\tan^{-1}[(iY + c)/X]$ is complex[13] and greatly affects the intensity symmetries of the X-wave solution, as clearly shown in Figs. 2(b)–2(d). Moreover, in spite of the fact that the modified Bessel functions asymptotically tend to zero at infinity [since $K_m(z) \approx (\pi/2z)^{1/2}\exp(-z)$ for large $|z|$], these waves, like their regular Bessel counterparts,[1] are found to convey infinite power. To demonstrate this we note that Eq. (3a) exhibits the following two invariants: $P = \iint \mathrm{d}X\mathrm{d}Y|\psi|^2$ and $H = \iint \mathrm{d}X\mathrm{d}Y(|\psi_X|^2 - |\psi_Y|^2)$, representing the power and the Hamiltonian of the system. When the solution of Eq. (3a) is static (nondiffracting), e.g., $\psi = \exp(i\lambda Z)G(X, Y)$, the relation $H = -\lambda P$ is true, provided that $G(X, Y)$ is zero at infinity. If one defines a moment $M = \iint \mathrm{d}X\mathrm{d}Y(X^2 - Y^2)|\psi|^2$, then it follows that $\mathrm{d}^2M/\mathrm{d}Z^2 = 8H$. Therefore if a nondiffracting beam exists with finite power or a

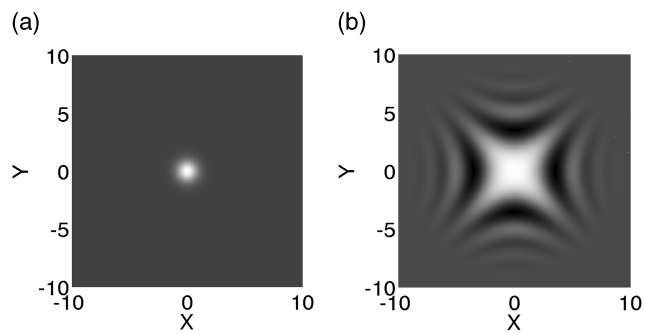

Fig. 1. (a) Intensity profile of an initial Gaussian beam on top of a constant background and (b) the resulting X-shaped diffraction pattern at $z = 2$.

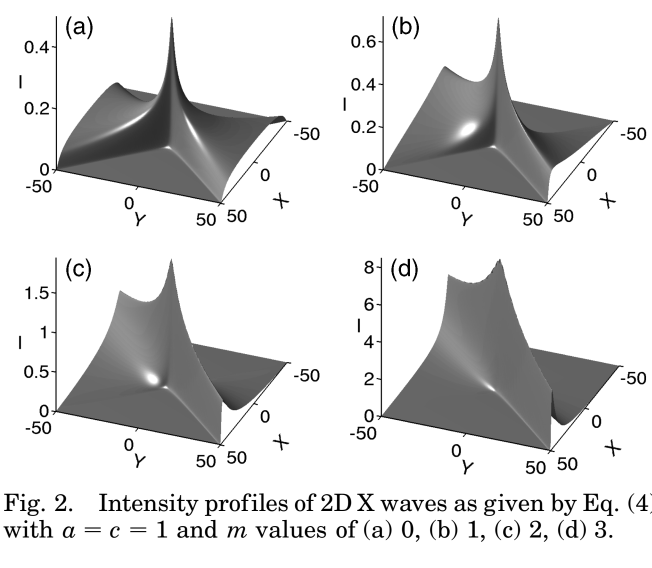

Fig. 2. Intensity profiles of 2D X waves as given by Eq. (4) with $a = c = 1$ and m values of (a) 0, (b) 1, (c) 2, (d) 3.

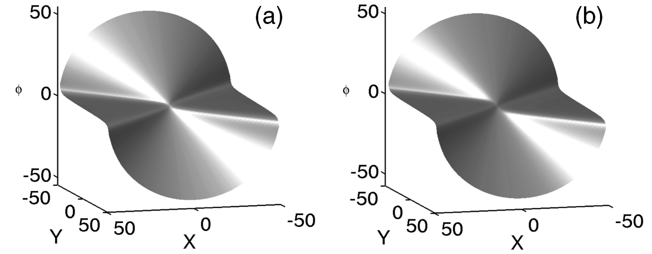

Fig. 3. Phase of a 2D X-wave solution given by Eq. (4) with $a = c = 1$ and m values of (a) 0 and (b) 1.

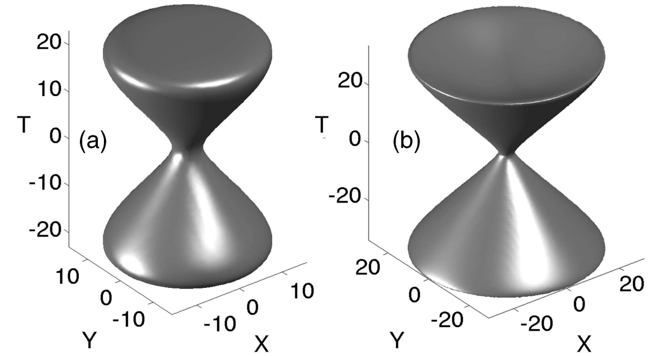

Fig. 4. Isointensity profile of a 3D X-wave solution given by Eq. (6) with $a = c = 1$ and (a) $l = m = 0$ and (b) $l = 1$ and $m = 0$.

Hamiltonian, then according to the latter relation its moment should evolve, and hence the solution itself should, too. This contradiction proves that no stationary X waves exist with finite power or Hamiltonian.

In addition to the 2D Bessel-type X waves of Eq. (4), other X structures can be obtained in z-independent form, $\psi = U(X, Y)$. As follows from Eq. (3a) these zero eigenvalue ($a = 0$) waves obey the D'Alembert equation,

$$U_{XX} - U_{YY} = 0\,. \tag{5}$$

Equation (5) has the general solution $U = F(X - Y) + G(X + Y)$, where F and G are arbitrary functions that are bounded at infinity.

Besides 2D Bessel X waves, 3D Bessel X nondiffracting structures are also possible. These represent solutions of Eq. (3b) and are given by

$$\psi = A\,\frac{K_{l+1/2}(ar)}{r^{1/2}}\,Y_l{}^m(\theta, \phi)\exp(ia^2 z)\,, \tag{6}$$

where $r = [X^2 + Y^2 + (iT + c)^2]^{1/2}$, $\phi = \tan^{-1}(Y/X)$, $\theta = \tan^{-1}[(X^2 + Y^2)^{1/2}/(iT + c)]$. In Eq. (6) $K_{l+1/2}(z)$ represents a modified Bessel function of fractional order and $Y_l{}^m(\theta, \phi)$ are spherical harmonics with complex arguments. In general $Y_l{}^m(\theta, \phi)$ are given by $Y_l{}^m(\theta, \phi) \propto P_l{}^m\,[\cos(\theta)]\exp(im\phi)$, where $P_l{}^m(z)$ are associated Legendre functions. The integer l takes values from the set of $0, 1, 2, 3\ldots$, and the values of the spin integer m lie in the range $|m| \le l$. The modified Bessel functions $K_{l+1/2}(z)$ in Eq. (6) are, in fact, elementary functions; in particular, $K_{1/2}(z) = \sqrt{\pi/2z}\,\exp(-z)$. We stress that these solutions differ from the algebraic ones for 3D X waves found in Ref. 2. Figure 4 depicts the isointensity profiles of such 3D X waves when $a = 1$, $c = 1$. In Fig. 4(a), $l = m = 0$, and in Fig. 4(b) $l = 1$ and $m = 0$.

In conclusion, we have demonstrated that new families of 2D and 3D nondiffracting Bessel X waves are possible in bidispersive optical systems.

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