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A note on perfect revivals in finite waveguide arrays

Ioannis D. Chremmos *, Nikolaos K. Efremidis

Department of Applied Mathematics, University of Crete, Heraklion 71409, Greece

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ABSTRACT

We propose a simple and algorithmic method for designing finite waveguide arrays capable of diffractionless transmission of arbitrary discrete beams by virtue of perfect revivals. Our approach utilises an inverse matrix eigenvalue theorem published by Hochstadt in 1974, which states that the Jacobi matrix, describing the system's discrete evolution equations, is uniquely determined by its eigenvalues and the eigenvalues of its largest leading principal submatrix, as long as the two sets of eigenvalues interlace. It is further shown that, by arranging the two sets of eigenvalues symmetrically with respect to zero, the resulting Jacobi matrix has zero diagonal elements. Therefore, arrays with arbitrary revival lengths can be obtained by engineering only the inter-waveguide couplings.

1. Introduction

Over the past 15 years, a distinct and exciting research domain of integrated optics has been formed that engages in the study of the so-called *discrete diffraction* of light [1,2]. We so refer to the characteristic propagation and coupling phenomena observed inside periodic arrays of evanescently coupled optical waveguides, briefly termed waveguide arrays (WGAs). A primary study of such configurations, conceived as arrays of parallel optical fibers, was conducted as early as 1965 by Jones [3], without however shedding much light on their diffraction properties. These were analysed several years later and it was further found that discrete diffraction can be counterbalanced by nonlinearity to create discrete, self-trapped states that propagate without diffraction along a WGA, thereafter termed *discrete solitons* [4]. The first successful experimental observation of these predictions came in the late 1990s [5], essentially marking the onset of a new field in optics. A few years later, lattice solitons were also predicted [6] and observed [7] in two-dimensional configurations. In the years that followed, discrete diffraction in WGAs was found to enable a diversity of phenomena, in addition to discrete solitons, that are absent in continuous media. Among them are Bloch-momentum dependent diffraction [8], optical Bloch oscillations [9], surface optical Bloch oscillations [10,11], Rabi oscillations [12], Zener tunnelling [13] and dynamic localisation [14].

What clearly sets WGAs apart is the ability to manipulate diffraction. In uniform media, the spatial spread of light beams due to (continuous) diffraction is unavoidable and can only be delayed using specially structured waveforms, such as Bessel [15] or Airy beams [16]. By contrast, a WGA is a one-dimensional photonic crystal in which discrete diffraction can be manipulated in a number of ways in order to achieve diffractionless propagation or transmission of discrete optical beams. Perhaps the most obvious way for diffractionless *propagation* through a WGA is to impart to the beam a transverse (Bloch) momentum corresponding to a phase difference $\pi/2$ between adjacent waveguides. Then the spectrum of the beam lies within the middle of the first Brillouin zone where the second-order diffraction vanishes [8]. The limitation of this method is that the beam must not be too narrow, otherwise the third-order diffraction will inevitably distort it. More elaborate techniques, inspired by dispersion compensation in optical fibers, have suggested to cascade short WGA sections with alternating signs of diffraction so that the total average diffraction is zero [17]. This is achieved if the successive waveguide segments have opposite inclinations. Another approach is to employ defect modes, i.e. diffractionless states that are trapped by local perturbations in the inter-waveguide couplings or the waveguide propagation constants [18]. However, the profile of defect modes is uniquely determined by the perturbation parameters, and therefore such modes cannot be used to transmit arbitrary beams.

In some applications, one may not be interested in cancelling diffraction all along the path of an optical beam (diffractionless *propagation*), but only in receiving the input beam without distortion after it has propagated for some distance along the WGA (diffractionless *transmission*). For example, in the discrete Talbot effect [19], a periodic input beam propagates with periodic recurrences, termed *perfect revivals*, at intervals z_T because its

* Corresponding author.
E-mail address: jochremm@central.ntua.gr (I.D. Chremmos).

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constituent Bloch modes have propagation constants that differ by integer multiples of $2\pi/z_T$. Unfortunately, unlike Talbot effect in continuous media, discrete diffraction limits the allowed periodicities to only 1, 2, 3, 4 or 6 waveguides, hence the discrete Talbot effect cannot be used to transmit arbitrary beams.

Of higher interest are Bloch oscillations [9,20] where the waveguides are subjected to a linear index gradient, causing light beams to propagate along sinusoidal paths, which is the analogue of the free electrons' oscillatory motion inside a crystal under the influence of a dc electric field. Similar to the solid-state case, the "biased" WGA supports Wannier–Stark states [9] whose eigenvalues are equally separated (Wannier–Stark ladder) by the propagation constant gradient $\Delta\beta$. The beating of these states leads to perfect periodic revivals of the input condition at intervals $2\pi/\Delta\beta$. Therefore, Bloch oscillations can be used for diffractionless transmission of arbitrary discrete beams through a WGA. Attention must be paid to the fact that ideal Bloch oscillations occur in infinite lattices. In finite WGAs with a linear index gradient, the eigenvalues of the modes that are confined close to the array edges deviate from the ladder, thus destroying the perfect periodic recurrence of input beams that excite these modes. Fortunately, as we have recently shown [10], this problem can be overcome by appropriately engineering the parameters (couplings and effective indices) of the WGA close to the edges. Using these "matched" terminations, the eigenvalues of the edge modes are restored back to the ladder thus facilitating perfect (surface) Bloch oscillations. The method is especially practical since the parameters of only few (less than 10) waveguides near the edges of the array have to be controlled.

It should be noted that another approach for obtaining Bloch oscillations in finite WGAs was presented in [21]. In this work, properties of Kac matrices [22] were utilised to derive an analytic expression for the coupling coefficients of a finite WGA with constant propagation constants, that results in equidistant mode eigenvalues and, hence, perfect periodic revivals of any input condition. This can be a useful approach since it only requires to control the inter-waveguide couplings, which can be done by adjusting their separations. A disadvantage is that the ratio of the maximum (centre) to the minimum (edge) coupling coefficient required is equal to $(N+1)/2\sqrt{N}$ (N being the number of waveguides), which may be difficult to achieve with increasing N. By contrast, the short array terminations of [10] can be used to restore Bloch oscillations in finite WGAs of any size larger than twice the size of the terminations.

In its most general form, the problem of revivals in engineered WGAs arrays was treated in [23] in a rigorous mathematical context. It was indeed shown that, by proper selection of the waveguide propagation constants and/or couplings, the eigenstates of the system can be made to beat at regular intervals, repeatedly reconstructing arbitrary input conditions. Analytical formulas were given for lattices of up to 5 waveguides, with the algebraic complexity becoming formidable for bigger lattices. In the same work, mirrored rebirths of the input condition occurring at fractions of the revival length (termed fractional revivals) were also considered.

In this brief communication, we revisit perfect revivals in finite WGAs, proposing a simple and algorithmic method for designing the involved parameters. In essence, we bring to light a theorem of matrix inverse eigenvalue theory, published by Hochstadt [24], which states that a real, symmetric, tridiagonal matrix with positive off-diagonal elements (called Jacobi matrix), is uniquely determined by its eigenvalues and the eigenvalues of its largest leading principal submatrix (the matrix obtained by deleting the last row and column), as long as the two sets of eigenvalues interlace. Among the several methods that have been proposed for constructing a Jacobi matrix from its spectral data, we here opt for that proposed by Hald [25], which involves a straightforward recurrent procedure. But before, let us briefly formulate the problem under consideration.

2. Methods

Under the validity of coupled-mode theory and nearest-neighbour interactions, the evolution of the power-normalised mode amplitudes in a WGA of N single-mode waveguides is expressed by the evolution equations

$$\left(i\frac{d}{dz}+a_n\right)\psi_n+\kappa_{n-1}\psi_{n-1}+\kappa_n\psi_{n+1}=0, \tag{1}$$

where $n=1,2,\ldots,N$, z is the propagation distance, κ_n is the field coupling coefficient between waveguides n and $n+1$ and a_n is the propagation constant detuning of waveguide n from an average value β_0. The finiteness of the lattice requires that the boundary couplings are set to zero, i.e. $\kappa_0=\kappa_N=0$. By arranging the mode amplitudes in a vector $\mathbf{\Psi}=(\psi_1,\psi_2,\ldots,\psi_N)^{\mathrm{T}}$, Eq. (1) is compactly written in the matrix form $i\,d\mathbf{\Psi}/dz+\mathbf{A\Psi}=\mathbf{0}$, where $\mathbf{A}$ is the Jacobi matrix with diagonal elements $\{a_n\}$ and off-diagonal elements $\{\kappa_n\}$. We note that, for standard total-internal-reflection waveguides and fibers, where the coupling is through the evanescent field, the coupling coefficients are positive ($\kappa_n>0$). The evolution of an arbitrary input $\mathbf{\Psi}(0)$ is written $\mathbf{\Psi}(z)=\exp(i\mathbf{A}z)\mathbf{\Psi}(0)$. The latter can be computed by noting that real and symmetric $\mathbf{A}$ is diagonalisable by an orthogonal matrix $\mathbf{Q}$, i.e. $\mathbf{A}=\mathbf{Q\Lambda Q}^{\mathrm{T}}$, from which we get $\exp(i\mathbf{A}z)=\mathbf{Q}\exp(i\mathbf{\Lambda}z)\mathbf{Q}^{\mathrm{T}}$, where $\mathbf{\Lambda}=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$ is in Jordan form, i.e. is the diagonal matrix of the real eigenvalues with the corresponding eigenvectors being the columns of $\mathbf{Q}$. Notice also that, since $\mathbf{A}$ is Hermitian, $\exp(i\mathbf{A}z)$ is unitary, hence total power $\mathbf{\Psi}^{\dagger}\mathbf{\Psi}$ is conserved along z († denotes the Hermitian transpose).

In perfect revivals, the input excitation is repeatedly reproduced, up to an accumulating phase, at intervals L, denoting the revival length. As shown in [23], this requires that the eigenvalues of $\mathbf{A}$ (which are distinct [23]) are separated by integer multiples of $2\pi/L$ namely

$$\lambda_m=\lambda_0+N_m\frac{2\pi}{L}, \tag{2}$$

where $m=1,2,\ldots,N$, N_m is integer and λ_0 is an arbitrary real number. This happens because an arbitrary input condition can be decomposed into the set of eigenvectors of matrix $\mathbf{A}$, which form a complete and orthogonal set. Every eigenvector propagates along the WGA with its own propagation constant which is an eigenvalue of $\mathbf{A}$. Due to the condition of Eq. (2), the eigenmodes interfere in phase at intervals L, thus reconstructing the input condition times a common phase factor. Our problem is therefore equivalent to constructing a Jacobi matrix $\mathbf{A}$, whose eigenvalues satisfy Eq. (2). After Hochstadt's theorem [24], we know that this is always possible, provided that the $N-1$ eigenvalues $\{\mu_n\}$ of the largest leading principal submatrix of $\mathbf{A}$ (hereafter called $\mathbf{A}'$) are chosen so as to interlace $\{\lambda_n\}$. Obviously, this can be done in an infinite number of ways, implying infinite possible realisations of $\mathbf{A}$. Also note that λ_0 can be absorbed by β_0, by replacing $\mathbf{A}$ by $\mathbf{A}-\lambda_0\mathrm{I}$, which is equivalent to a common detuning in the propagation constants of all waveguides. In addition, the revival length can be scaled out and set to 2π by multiplying $\mathbf{A}$ by $L/2\pi$.

In Hald's algorithm [25], one starts by defining the eigenvalues $\{\lambda_n\}$ and $\{\mu_n\}$ of matrices $\mathbf{A}$ and $\mathbf{A}'$, respectively. Then the characteristic polynomials of these matrices, $p_N(\lambda)$ and $p_{N-1}(\lambda)$, follow immediately by the factor theorem

$$p_N(\lambda)=\prod_{n=1}^{N}(\lambda_n-\lambda),\quad p_{N-1}(\lambda)=\prod_{n=1}^{N-1}(\mu_n-\lambda). \tag{3}$$

Subsequently, the recurrence relation between the characteristic polynomials of the leading principal submatrices

$$p_n(\lambda)=(a_n-\lambda)p_{n-1}(\lambda)-\kappa_{n-1}^2p_{n-2}(\lambda), \tag{4}$$

is applied for $n=N,N-1,\ldots,1$ with $p_0\equiv 1, p_{-1}\equiv 0$. At each step, the parameters a_n,κ_{n-1} are determined so that the polynomial $p_{n-2}(\lambda)$ is of degree $n-2$ with leading coefficient $(-1)^{n-2}$. At the last step $(n=1)$, a_1 is determined from the relation $p_1(\lambda)=a_1-\lambda$. In the end of the procedure, all the parameters of the WGA, namely the N propagation constants $\{a_n\}$ and the $N-1$ coupling coefficients $\{\kappa_n\}$, have been determined.

3. Examples

We here present some examples of designing WGAs capable of perfect revivals using the outlined method. For example, consider a lattice of $N=7$ waveguides and assume that the eigenvalues of A have the minimum spacing required for a normalised revival length $L=2\pi$, i.e. $\{\lambda_n\}=\{-3,-2,-1,0,1,2,3\}$. Subsequently, the eigenvalues of $\mathbf{A}'$ are chosen randomly so as to interlace $\{\lambda_n\}$, say $\{\mu_n\}=\{-2.9,-1.5,-0.2,0.9,1.3,2.2\}$. The detunings and couplings resulting from Hald's algorithm are given in the caption of Fig. 1. Parts (a) and (b) of that figure show the simulated evolution of the field amplitude under excitation of a single waveguide at the boundary of the WGA. Perfect revivals with period 2π are clearly verified. Notice that the parameters of the array do not exhibit any symmetry, hence the field patterns are quite different, when the first or the last waveguide is excited. In (a) the periodic breathing of optical power is slower because the coupling coefficients at the left side of the array are lower.

The realisation of a WGA with varying a_n may be difficult because the propagation constant of each waveguide must be independently tuned. Therefore, designs involving waveguides with a common effective index $(a_n=0)$ would be preferable. This can be achieved if the eigenvalues of $\mathbf{A}$ and $\mathbf{A}'$ are arranged symmetrically with respect to zero. In other words, if λ_n is an eigenvalue of $\mathbf{A}$ then $-\lambda_n$ is also an eigenvalue and similarly for matrix $\mathbf{A}'$. The proof of $a_n=0$ for all n can be made by induction. Indeed, in this case, the characteristic polynomial $p_N(\lambda)$ contains only terms $\lambda^N,\lambda^{N-2},\ldots$ and, similarly, $p_{N-1}(\lambda)$ contains only terms $\lambda^{N-1},\lambda^{N-3},\ldots$, as can be shown from Eq. (3). Subsequently, from

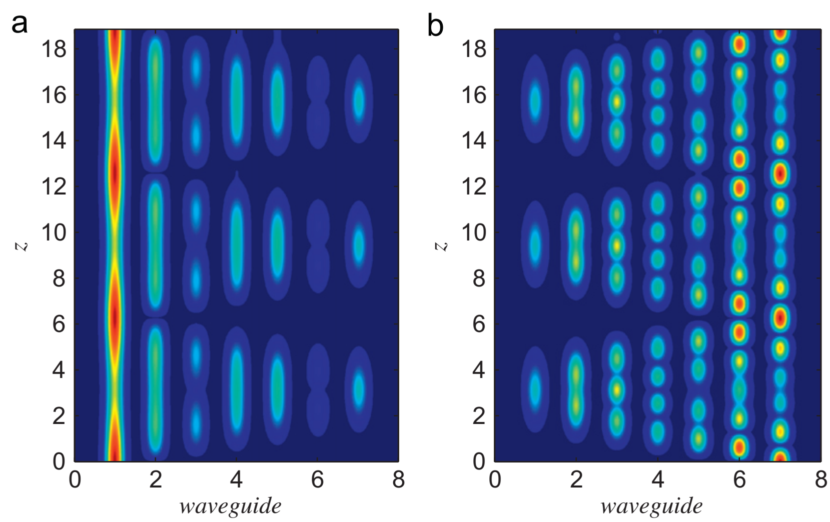

Fig. 1. Perfect revivals under excitation of a single boundary waveguide in a WGA with seven elements. The array parameters are $\{a_n\}=\{0.76,-0.2,-0.33,-0.17,-1.01,0.75,0.2\}$ and $\{\kappa_n\}=\{0.56,1.06,1.71,1.43,1.21,2.23\}$.

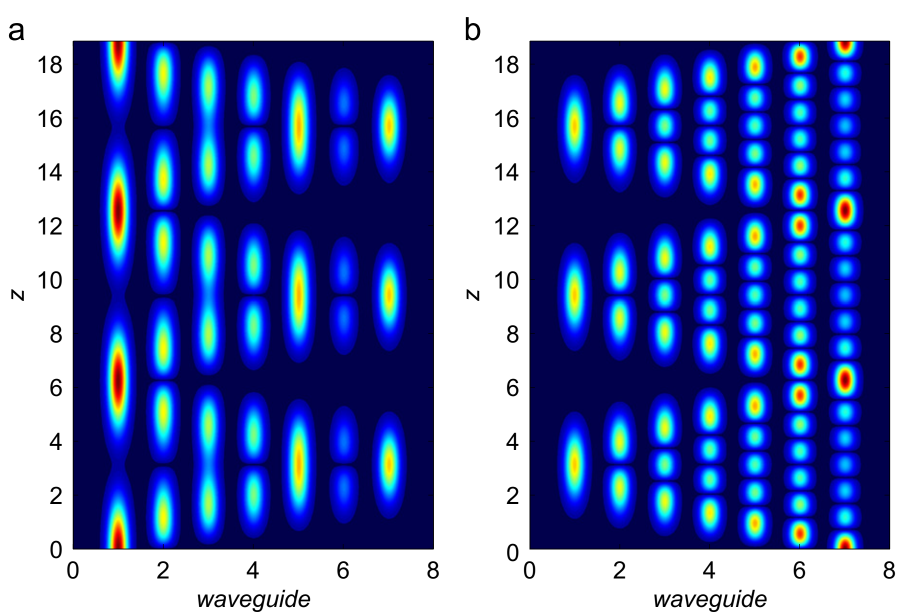

Fig. 2. Perfect revivals under excitation of a single boundary waveguide in a WGA with seven elements. All waveguides have the same propagation constant $(a_n=0)$ while the coupling coefficients are $\{\kappa_n\}=\{0.87,1.17,1.37,1.50,1.58,2.29\}$.

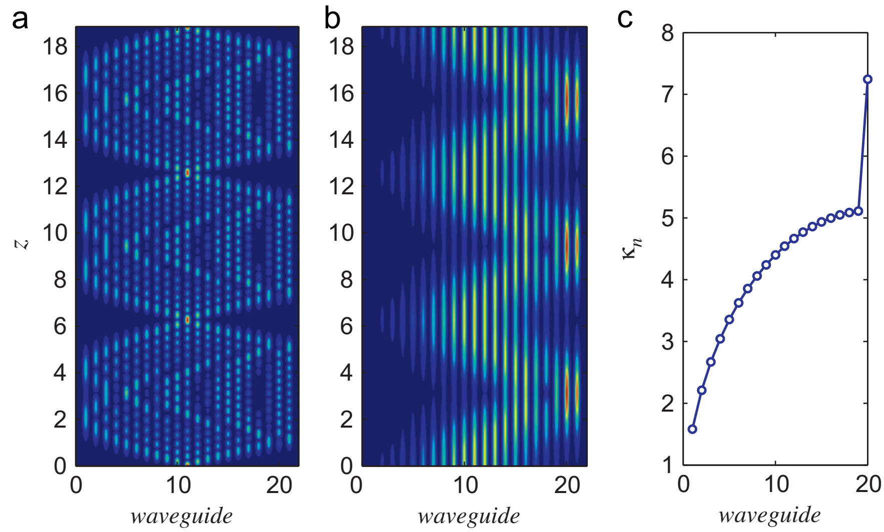

Fig. 3. Perfect revivals in a WGA with 21 elements. All waveguides have the same propagation constant ($a_n=0$) while the coupling coefficients are shown in (c). In (a) only the middle waveguide is excited, while in (b) the input condition is the discrete Gaussian beam $\psi_n(0)=\exp[-((n-11)/5)^2]$.

the recurrence relation of Eq. (4), it follows that $a_N=0$ and that the polynomial $p_{N-2}(\lambda)$ contains only terms $\lambda^{N-2},\lambda^{N-4},\ldots$. By applying Eq. (4) recurrently for $n=N-1,\ldots,1$, all a_n are shown to vanish. For example, assume $\{\lambda_n\}=\{-3,-2,-1,0,1,2,3\}$ and $\{\mu_n\}=\{-2.5,-1.5,-0.5,0.5,1.5,2.5\}$. After applying Hald's algorithm, the resulting detunings are zero while the couplings are given in the caption of Fig. 2. The simulated field evolution for single boundary waveguide excitation, shown in the same Figure, verifies the perfect revivals. Such an array is easier to implement by using identical waveguides and adjusting their mutual separations.

In Fig. 3 an example with a larger array $N=21$ is considered. The assumed eigenvalues are $\lambda_n=n-(N+1)/2$, $n=1,2,\ldots,N$, for matrix $\mathbf{A}$ and $\mu_n=\lambda+0.5$, $n=1,2,\ldots,N-1$, for matrix $\mathbf{A}'$. The resulting coupling coefficients are shown in Fig. 3(c), while $a_n=0$. Fig. 3(a) and (b) show the simulated field amplitude evolution in the case of excitation of the middle waveguide and for a discrete beam with a Gaussian envelope, respectively. Perfect revivals with period 2π are verified. Note in (b) that the evolution pattern resembles Bloch oscillations, which are here the result of a varying coupling coefficient across the lattice rather than of a varying propagation constant as in classic Bloch oscillations.

4. Conclusion

We have presented a simple systematic method for designing WGAs capable of perfect revivals with a desired revival length. The key is Hochstadt's inverse eigenvalue theorem for Jacobi matrices and the associated construction algorithm by Hald. Of special interest is the case in which the Jacobi matrix, describing the system's discrete evolution equations, has zero diagonal elements, because the corresponding WGA can be conveniently fabricated by controlling only the inter-waveguide separations. Our work provides additional contribution to the few other methods so far reported for obtaining perfect revivals in discrete optical lattices, namely Bloch oscillations [9], surface Bloch oscillations [10], direct analytical calculations [23] and closed form solutions [21]. The phenomenon of periodic field revivals offers an indirect path for "beating" diffraction in WGAs (which are inherently diffractive light guiding devices) and retrieving arbitrary discrete optical beams without distortion after propagating over a desired length.

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