Guarding curvilinear art galleries with edge or mobile guards

Menelaos I. Karavelas*
Department of Applied Mathematics, University of Crete, GR-714 09 Heraklion, Greece, and Institute of Applied and Computational Mathematics, F.O.R.T.H., P.O. Box 1385, GR-711 10 Heraklion, Greece

Abstract

In this paper we consider the problem of monitoring an art gallery modeled as a polygon, the edges of which are arcs of curves. We consider two types of guards: edge guards (these are edges of the polygon) and mobile guards (these are either edges or straight-line diagonals of the polygon). Our focus is on piecewise-convex polygons, i.e., polygons that are locally convex, except possibly at the vertices, and their edges are convex arcs. We reduce the problem of monitoring a piecewise-convex polygon to the problem of 2-dominating a constrained triangulation graph with edges or diagonals, where 2-dominance requires that every triangle in the triangulation graph has at least two of its vertices in the 2-dominating set. We show that, given a triangulation graph \( T_P \) of a polygon \( P \) with \( n \geq 3 \) vertices: (1) \( \lceil \frac{n}{5} \rceil \) diagonal guards are always sufficient and sometimes necessary, and (2) \( \lceil \frac{n}{10} \rceil \) edge guards are always sufficient and \( \lceil \frac{n}{2} \rceil \) edge guards are sometimes necessary, in order to 2-dominate \( T_P \). We also show that a diagonal (resp., edge) 2-dominating set of size \( \lceil \frac{n}{25} \rceil \) (resp., \( \lceil \frac{n}{5} \rceil \)) can be computed in \( O(n) \) time and space, whereas an edge 2-dominating set of size \( \lceil \frac{2n+11}{10} \rceil \) can be computed in \( O(n^2) \) time and \( O(n) \) space. Based on these results we prove that, in order to monitor a piecewise-convex polygon \( P \) with \( n \geq 2 \) vertices: (1) \( \lceil \frac{n}{5} \rceil \) mobile guards or \( \lceil \frac{n}{10} \rceil \) edge guards are always sufficient, and (2) \( \lceil \frac{n}{5} \rceil \) mobile or edge guards are sometimes necessary. A mobile (resp., edge) guard set for \( P \) of size \( \lceil 4n+5 \rceil \) (resp., \( \lceil 4n+11 \rceil \) or \( \lceil 3n \rceil \)) can be computed in \( O(n \log n + T(n)) \) time and \( O(n) \) space, where \( T(n) \) denotes the time for computing a diagonal (resp., edge) 2-dominating set of size \( \lceil \frac{n+11}{5} \rceil \) (resp., \( \lceil \frac{4n+11}{10} \rceil \) or \( \lceil \frac{3n}{2} \rceil \)) for a triangulation graph with \( n \) vertices.


Keywords: art gallery, curvilinear polygons, triangulation graphs, 2-dominance, edge guards, diagonal guards, mobile guards, piecewise-convex polygons

*mkaravel@tem.uoc.gr
locally convex, except possibly at the vertices, and their edges are convex arcs) with \(\lfloor \frac{n}{2} \rfloor\) vertex guards, whereas there exist classes of such polygons that require a minimum of \(\lfloor \frac{n}{2} \rfloor - 1\) vertex or \(\lfloor \frac{n}{2} \rfloor\) point guards. They also show that \(2n - 4\) point guards are always sufficient and sometimes necessary in order to monitor piecewise-concave polygons, i.e., polygons that are locally concave, except possibly at the vertices, and their edges are convex arcs.

Soon after the first results on monitoring polygons with vertex or point guards, other types of guarding models where considered. Toussaint introduced in 1981 the notion of edge guards. A point \(p\) in the interior of the polygon is considered to be monitored if it is visible from at least one point of an edge in the guard set. Edge guards were introduced as a guarding model in which guards were allowed to move along the edges of the polygon. Another variation, dating back to 1983, is due to O’Rourke: guards are allowed to move along the edges or diagonals of the polygon. This type of guards is called mobile guards. Toussaint conjectured that, except for a few polygons, \(\lfloor \frac{n}{2} \rfloor\) edge guards are always sufficient.

There are only two known counterexamples to this conjecture, with \(n = 7, 11\), due to Paige and Shermer Shemer [1992], requiring \(\lfloor \frac{n}{2} \rfloor\) edge guards, whereas there exists a family of polygons that require \(\lfloor \frac{n}{4} \rfloor\) edge guards. The first step towards Toussaint’s conjecture was made by O’Rourke O’Rourke [1983] O’Rourke [1987] who proved that \(\lfloor \frac{n}{2} \rfloor\) mobile guards are always sufficient and occasionally necessary in order to monitor any polygon with \(n\) vertices.

The technique of O’Rourke, for proving the upper bound, amounts to reducing the problem of monitoring a simple polygon to that of dominating a triangulation graph of the polygon. A triangulation graph is a maximal outerplanar graph, all internal faces of which are triangles. Dominance means that at least one of the vertices of each triangle in the triangulation graph is incident to a mobile guard. The reduction to the problem of triangulation graph dominance is applicable to the case of edge guards. Shermer Shemer [1992] settled the problem of dominating triangulation graphs with edge guards by showing that \(\lfloor \frac{n}{2} \rfloor\) edge guards are always sufficient and sometimes necessary, except for \(n = 3, 6, 13\), in which case one extra edge guard may be necessary; this, also, constitutes the best known upper bound on the number of edge guards that are sufficient in order to monitor an \(n\)-vertex polygon.

In this paper we consider the problem of monitoring piecewise-convex polygons with edge or mobile guards; in our context, a mobile guard is either an edge or a straight-line diagonal of the polygon. Our proof technique capitalizes on the technique used by O’Rourke to prove tight bounds on the number of mobile guards that are sufficient for monitoring straight-line polygons O’Rourke [1987]. Unlike O’Rourke’s paradigm, where the solution for the dominance problem is trivially a solution for the geometric guarding problem, in our paradigm we first reduce the geometric problem to a combinatorial problem, and then map of the solution for the combinatorial problem to a solution for the geometric problem. More precisely, in order to monitor piecewise-convex polygons with mobile or edge guards, we first reduce the problem of monitoring a piecewise-convex polygon \(P\) to the problem of 2-dominating a constrained triangulation graph. Given a triangulation graph \(T_p\) of a polygon \(P\), a set of edges and/or diagonals of \(T_p\) is a 2-dominating set of \(T_p\) if every triangle in \(T_p\) has at least two of its vertices incident to an edge or diagonal in the 2-dominating set. We prove that \(\lfloor \frac{2n}{3} \rfloor\) diagonal guards (i.e., edges or diagonals of \(T_p\)) are always sufficient and sometimes necessary in order to 2-dominate \(T_p\), whereas \(\lfloor \frac{2n}{3} \rfloor\) edge guards are always sufficient and \(\lfloor \frac{n}{2} \rfloor\) edge guards are always sufficient and sometimes necessary in order to 2-dominate \(T_p\). The proofs of sufficiency are inductive on the number of vertices of \(P\). In the case of diagonal 2-dominance, our proof yields a linear time and space algorithm. In the case of edge 2-dominance, the inductive step incorporates edge contraction operations, thus yielding an \(O(n^2)\) time and \(O(n)\) space algorithm, where \(n\) is the size of \(P\). A linear time and space algorithm can be attained by slightly relaxing the size of the edge 2-dominating set. More precisely, we have shown that we can 2-dominate \(T_p\) with \(\lfloor \frac{2n}{3} \rfloor\) edges; the proof does not make use of edge contractions and is analogous, though more complicated, to the proof, presented in this paper, for the case of diagonal 2-dominance.

Focusing back to the geometric guarding problem, the triangulation graph \(T_p\) of the piecewise-convex polygon \(P\) is a constrained triangulation graph: we require that certain diagonals of \(T_p\) are present. The remaining non-triangular subpolygons of \(T_p\) are straight-line polygons and may be triangulated arbitrarily. For the edge guarding problem, any edge 2-dominating set computed for \(T_p\) is also an edge guard set for \(P\). A diagonal 2-dominating set \(D\) of \(T_p\), however, may contain diagonals of \(T_p\) that are not embeddable as straight-line diagonals of \(P\). To produce a mobile guard set for \(P\), we keep all edges and straight-line diagonals of \(P\) in \(D\) and map non-straight-line diagonals in \(D\) to edges of \(P\). In summary, we can compute: (1) a mobile guard set for \(P\) of size at most \(\lfloor \frac{2n}{3} \rfloor\) in \(O(n \log n)\) time and \(O(n)\) space; (2) an edge guard set for \(P\) of size at most \(\lfloor \frac{2n}{3} \rfloor\) in \(O(n^2)\) time and \(O(n)\) space; (3) an edge guard set for \(P\) of size at most \(\lfloor \frac{2n}{4} \rfloor\) in \(O(n \log n)\) time and \(O(n)\) space. Finally, we show that \(\lfloor \frac{n}{2} \rfloor\) edge or mobile guards are sometimes necessary in order to monitor \(P\).

The rest of the paper is structured as follows. Section 2 is devoted to 2-dominance of triangulation graphs using diagonal or edge guards. In Section 3 we discuss the problem of monitoring piecewise-convex polygons with mobile or edge guards. Finally, in Section 4 we conclude with a discussion of our results and open problems.

## 2 2-dominance of triangulation graphs

Given a polygon \(P\) with \(n\) vertices, its triangulation graph \(T_p\) is a maximal outerplanar graph, i.e., a Hamiltonian planar graph consisting of \(n\) vertices and \(2n - 3\) edges, all internal faces of which are triangles (cycles of size \(3\)). The triangulation graph of a straight-line polygon, i.e., a polygon the edges of which are line segments, is the planar graph we get when the polygon has been triangulated.

A dominating set \(D\) of a triangulation graph \(T_p\) is a set of vertices, edges or diagonals of \(T_p\) such that at least one of the vertices of each triangle in \(T_p\) belongs to \(D\). An edge (resp., diagonal) dominating set of \(T_p\) is a dominating set of \(T_p\) consisting of only edges (resp., edges or diagonals) of \(P\). A 2-dominating set \(D\) of \(T_p\) is a dominating set of \(T_p\) that has the property that every triangle in \(T_p\) has at least two of its vertices in \(D\). In a similar manner, an edge (resp., diagonal) 2-dominating set of \(T_p\) is a 2-dominating set of \(T_p\) consisting only of edges (resp., edges or diagonals) of \(T_p\).

Before proceeding with the main results of this section, we state the following lemma, which is a direct generalization of Lemmas 1.1 and 3.6 in O’Rourke [1987].

**Lemma 1** Consider an integer \(\lambda \geq 2\). Let \(P\) be a polygon of \(n \geq 2\lambda\) vertices, and \(T_p\) a triangulation graph of \(P\). There exists a diagonal \(d\) in \(T_p\) that partitions \(T_p\) into two pieces, one of which contains \(\lambda\) edges corresponding to edges of \(P\), where \(\lambda \leq k \leq 2(\lambda - 1)\).

**Proof.** Choose \(d\) to be a diagonal of \(T_p\) that separates off a minimum number of polygon edges that is at least \(\lambda\). Let \(k \geq \lambda\) be this minimum number, and label the vertices of \(P\) with the labels 0, 1, \ldots, \(n - 1\), such that \(d\) is \((0, k)\). The diagonal \(d\) supports a triangle whose apex is at vertex \(t\), \(0 \leq t \leq k\). Since \(k\) is minimal \(t \leq \lambda - 1\) and \(k - t \leq \lambda - 1\). Thus, \(\lambda \leq k \leq 2(\lambda - 1)\). \(\square\)
Diagonal guards Using Lemma 11 for $\lambda = 4$, yields the following theorem concerning the worst-case number of diagonals that are sufficient and necessary in order to 2-dominate a triangulation graph. The inductive proof that follows is not the simplest possible. The interested reader may find a much simpler alternative proof in [Karavelas 2008]. The simpler proof, however, makes use of edge contractions, which make it unsuitable as a basis for a linear time and space algorithm. On the other hand, the proof presented below can be implemented in linear time and space, as will be discussed below. The proof that follows is a detailed, rather technical, case-by-case analysis; we present it, however, uncondensed, so as to illustrate the details that pertain to our linear time and space algorithm.

**Theorem 2** Every triangulation graph $T_P$ of a polygon $P$ with $n \geq 3$ vertices can be 2-dominated by $\lfloor \frac{4n-1}{3} \rfloor$ diagonal guards. This bound is tight in the worst-case.

**Proof.** The proof for $3 \leq n \leq 7$ is straightforward and is omitted. Let us now assume that $n \geq 8$ and that the theorem holds for all $n'$ such that $3 \leq n' < n$. By means of Lemma 11 with $\lambda = 4$, there exists a diagonal $d$ that partitions $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k$ boundary edges of $T_P$ with $4 \leq k \leq 6$. Let $v_1, 0 \leq i \leq k$, be the $k + 1$ vertices of $T_1$, as we encounter them while traversing $P$ counterclockwise, and let $v_0v_k$ be the common edge of $T_1$ and $T_2$. For each value of $k$ we are going to define a diagonal 2-dominating set $D$ for $T_P$ of size $\lfloor \frac{4n-1}{3} \rfloor$. In what follows $d_{ij}$ denotes the diagonal $v_iv_j$, whereas $e_i$ denotes the edge $v_iv_{i+1}$. Consider each value of $k$ separately.

$k = 4$. In this case $T_2$ contains $n - 3$ vertices. By our induction hypothesis we can 2-dominate $T_2$ with $f(n - 3) = \lfloor \frac{4n-1}{3} \rfloor - 1$ diagonal guards. Let $D_2$ be the diagonal 2-dominating set for $T_2$. At least one of $v_0$ and $v_4$ is in $D_2$. The cases are symmetric, so we can assume without loss of generality that $v_0 \in D_2$.

Consider the following cases (see Fig. 1): $d_{13} \in T_1$. Set $D = D_2 \cup \{d_{13}\}$. $d_{24} \in T_1$. Set $D = D_2 \cup \{d_{24}\}$. $d_{02}, d_{03} \in T_1$. Set $D = D_2 \cup \{e_2\}$.

$k = 5$. The presence of diagonals $d_{03}$ and $d_{15}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. The apex $v$ of this triangle can either be $v_3$ or $v_5$. The two cases are symmetric, so we can assume, without loss of generality, that the apex of $t$ is $v_3$. Consider the triangulation graph $T' = T_2 \cup \{t\}$. It has $n - 3$ vertices, hence, by our induction hypothesis, it can be 2-dominated with $f(n - 3) = \lfloor \frac{4n-1}{3} \rfloor - 1$ diagonal guards. Let $D'$ be the 2-dominating set for $T'$. Consider the following cases (see Fig. 2).

$d_{02} \in D_2$. Set $D = D' \cup \{e_3\}$. $d_{02} \not\in D_2$. If $d_{25} \not\in D'$, set $D = (D' \setminus \{d_{25}\}) \cup \{d_{02}, e_4\}$. Otherwise, $v_2$ cannot belong to $D'$ (both edges of $T'$ incident to $v_2$ do not belong to $D'$). However, the triangle $t$ is 2-dominated in $T'$, which implies that both $v_0$ and $v_5$ belong to $D'$. Hence, set $D = D' \cup \{e_2\}$.

$k = 6$. The presence of diagonals $d_{04}$, $d_{05}$, $d_{16}$ and $d_{26}$ would violate the minimality of $k$. Let $t$ be the triangle supported by $d$ in $T_1$. The apex $v$ of this triangle must be $v_5$. Let $t'$ be the second triangle in $T_1$. Let $t'$ be supported by the diagonal $d_{26}$, and let $v'$ be its vertex opposite to $d_{26}$. Consider the triangulation graph $T'' = T_2 \cup \{t, t'\}$. It has $n - 3$ vertices, hence, by our induction hypothesis, it can be 2-dominated with $f(n - 3) = \lfloor \frac{4n-1}{3} \rfloor - 1$ diagonal guards. Let $D''$ be the 2-dominating set for $T''$.

Let us first consider the case $v' \equiv v_2$. Let $d''$ be the unique diagonal of the quadrilateral $v_3v_4v_5v_6$. Consider the following cases (see Fig. 3).

$d_{02} \in D'$. Set $D = D' \cup \{d''\}$. $d_{02} \not\in D'$. We further distinguish between the following two cases:

$d_{36} \in D'$. If $v_6 \in D'$, simply set $D = (D' \setminus \{d_{36}\}) \cup \{e_2, e_5\}$. If $v_6 \not\in D'$, the diagonal $d_{36}$ cannot belong to $D'$. Therefore, in order for the triangle $t'$ to be 2-dominated by $D'$, we must have that $v_2 \not\in D'$. Thus, set $D = (D' \setminus \{d_{36}\}) \cup \{e_0, e_5\}$. $d_{36} \not\in D'$. In order for $t'$ to be 2-dominated by $D'$ we must have that either $d_{03} \not\in D'$ or $e_2 \not\in D'$. If $d_{03} \in D'$, set $D = (D' \setminus \{d_{03}\}) \cup \{e_0, d_{02}, d''\}$; otherwise, set $D = (D' \setminus \{e_2\}) \cup \{d_{02}, d''\}$.

The situation is entirely symmetric if $v'' \equiv v_4$. Hence, the only remaining case is the case where $v' \equiv v_1$ and $v'' \equiv v_5$. Consider the following cases (see Fig. 4).

$d_{13} \in D'$. Set $D = (D' \setminus \{d_{13}\}) \cup \{e_0, d_{35}\}$. $d_{13} \not\in D'$. We further distinguish between the following two cases:

$d_{30} \in D'$. Set $D = (D' \setminus \{d_{30}\}) \cup \{e_0, e_{35}\}$. $d_{30} \not\in D'$. It is assumed in this subfigure that $d'' \equiv d_{35}$. Middle left: $d_{02} \not\in D'$ and $d_{36} \in D'$ and $v_6 \in D'$. Middle right: $d_{02} \not\in D'$ and $d_{30} \in D'$ and $v_6 \not\in D'$. Right: $d_{13} \not\in D'$ and $d_{03} \in D'$; also $d_{13}, d_{03} \not\in D'$ and $e_0 \in D'$.
Edge/diagonal of $H$ edge/diagonal in $C$. Values 1, 2, 3, respectively. All three triangulation graphs require at least $1 + 1 + 1$ diagonal guards in order to be 2-dominated.

If $d_{3b} \notin D'$, if $e_b \in D'$, set $D = D' \cup \{d_{3b}\}$. Otherwise, i.e., if $e_b \notin D'$, and $v_3$ cannot be in $D'$. Since the triangle $t'$ is 2-dominated in $D'$, both $v_0$ and $v_3$ have to belong to $D'$. Since the diagonal $d_{3a}$ does not belong to $D'$, the diagonal $d_{3b}$ has to belong to $D'$ in order for $v_3$ to be in $D'$. Thus, set $D = (D' \setminus \{d_{3a}\}) \cup \{d_{13}, e_5\}$.

Let us now turn our attention to establishing the lower bound. Consider the triangulation graphs $T_i$, $i = 1, 2, 3$, with $n = 3m + i - 1$ vertices, shown in Fig. 5, and let $D_i$ be the diagonal 2-dominating set of $T_i$. The central part of $T_i$ is triangulated arbitrarily. Notice that each subgraph of $T_i$, shown in either light or dark gray, requires at least one among its edges or diagonals to be in $D_i$ in order to be 2-dominated. This observation immediately establishes a lower bound of $\left\lceil \frac{n}{3} \right\rceil$.

Let us now assume that $|D_3| = \left\lceil \frac{n}{3} \right\rceil$. Under this assumption, each shaded subgraph in $T_3$ has to have exactly one among its edges/diagonals in $D_3$. Moreover, none of the diagonals in the central part of $T_3$ (not shown in Fig. 5(bottom)) can belong to $D_3$, since then we would have $|D_3| > \left\lceil \frac{n}{3} \right\rceil$. Consider the triangulated hexagon $H := \{v_0, v_1, v_2, v_3, v_4, v_5\}$, with $i = 1, 2, 3$ and $j = 1, 2, 3$. In order for $H$ to be 2-dominated with exactly one of its edges/diagonals, both $v_0$ and $v_{3m-3}$ have to be in $D_3$ due to edges/diagonals in the neighboring shaded subgraphs, while the unique edge or diagonal of $H$ in $D_3$ must be the diagonal $d_{3m-2, 3m}$. Since we require that $v_{3m-3}$ must belong to $D_3$ via an edge/diagonal of the quadrilateral $v_{3m-6}v_{3m-5}v_{3m-4}v_{3m-3}$, and at the same time we require that exactly one of the edges/diagonals of $v_{3m-9}v_{3m-8}v_{3m-7}v_{3m-6}$ to be in $D_3$, $v_{3m-4}$ must belong to $D_3$ and $v_{3m-6}$ must be in $D_3$ due to an edge/diagonal in the quadrilateral $v_{3m-9}v_{3m-8}v_{3m-7}v_{3m-6}$. Cascading this argument, we conclude that, since $v_0$ must belong to $D_3$ due to an edge/diagonal of the quadrilateral $v_0v_1v_2v_3$, and at the same time exactly one of the edges/diagonals of $v_0v_1v_2v_3$ must be in $D_3$, $v_2$ must belong to $D_3$ and $v_3$ must belong to $D_3$ due to an edge/diagonal in $H$. This yields a contradiction, since the unique edge/diagonal of $H$ in $D_3$ is $d_{3m-2, 3m}$, which is, obviously, not incident to $v_0$. □

The proof of Theorem 2 can almost immediately be transformed into a linear time and space algorithm. The triangulation graph $T_P$ of $P$ is assumed to be represented via a half-edge representation. Half-edges and vertices in our representation are assumed to have additional flags for indicating whether a half-edge is a boundary edge of the polygon, or whether a half-edge or a vertex of $T_P$ is marked as being in the diagonal 2-dominating set of $T_P$. Under these assumptions, adding or removing a half-edge or a vertex from the sought-for 2-dominating set, querying a half-edge or a vertex for membership in the 2-dominating set, as well as forming the triangulation graph for the recursive calls, all take $O(1)$ time.

Consider a diagonal $d$ that separates $T_P$ into two triangulation graphs $T_1$ and $T_2$, where $T_1$ contains $k = 4, 5$ or 6 edges of $P$; recall from the proof of Lemma 1 for $\lambda = 4$ that the value of $k$ is minimal. Let $\Delta$ be the dual tree of $T_P$, $\Delta_1$ (resp., $\Delta_2$) the dual tree of $T_1$ (resp., $T_2$) and $\Delta_1 = \Delta_1 \cup \{d\}$, where $d$ is the dual edge of $\Delta$. $\Delta_i$ consists of a subtree of $\Delta$ with 2, 3 or 4 edges of $\Delta$, connected with the rest of $\Delta$ via a degree-2 or a degree-3 node (see Fig. 6). Moreover, for $n \geq 13$, the subtrees $\Delta_i$ corresponding to different diagonals $d$ of $T_P$ must be edge disjoint (otherwise the number of vertices of $P$ would be less than 13).

Having made these observations we can now describe the algorithm for computing the diagonal 2-dominating set $D$ for $T_P$. We first describe the initialization steps: (1) initialize $D$ to be empty; (2) create a queue $Q$, and initialize it to be empty. $Q$ will consist of diagonals of $T_P$; (3) for each diagonal $d$ of $T_P$ determine whether it separates off 4, 5 or 6 edges of $P$ in $T_P$ and its size is minimal. In other words, determine if the dual edge $d'$ of $d$ in $\Delta$ is adjacent to subtrees of the form shown in Fig. 6. If so, put $d$ in $Q$.

The recursive part of the algorithm is as follows:

1. If the number of vertices of $T_P$ is less than 13, find a diagonal 2-dominating set $D$ and return.

2. If $Q$ is not empty:
   (a) Pop a diagonal $d$ out of $Q$.
   (b) If $T_2$ has less than 13 vertices, empty the queue $Q$ and find a 2-dominating set $D_2$ for $T_2$. Based on $D_2$, and according to the cases in the proof of Theorem 2, compute $D$ and return.
   (c) Determine the case in the proof of Theorem 2 to which $d$ corresponds. Let $T$ be the triangulation graph for which we are supposed to find the 2-dominating set recursively, and let $\Delta$ be the dual tree of $T$. Let $V'$ be the set of vertices in $\left(\Delta \cap \Delta_1\right)$. For any $v \in V$ determine if $v$ is a leaf-node to a subtree of $\Delta$ like the subtrees in Fig. 6. If so, add the corresponding diagonal to $Q$.
   (d) Recursively, find a diagonal 2-dominating $\hat{D}$ for $\hat{T}$, using $Q$ as the queue.
   (e) Construct from $\hat{D}$ a diagonal 2-dominating set $D$ for $T_P$ and return.

It is straightforward to verify that the time $T(n)$ spent for the recursive part of our algorithm satisfies the recursion $T(n) = T(n-3) + O(1)$, which gives $T(n) = O(n)$. Since initialization takes linear time, and our space requirements are obviously linear in the size of $P$ (we do not duplicate parts of $T_P$ for the recursive calls, but rather set appropriately the boundary flags for some half-edges), we arrive at the following theorem.

**Theorem 3** Given the triangulation graph $T_P$ of a polygon $P$ with...
n ≥ 3 vertices, we can compute a diagonal 2-dominating set for $T_P$ of size at most $\left\lfloor \frac{n+1}{2} \right\rfloor$ in $O(n)$ time and space.

**Edge guards.** Applying Lemma 1 for $\lambda = 5$ we can prove that $\left\lfloor \frac{n+1}{2} \right\rfloor$ edge guards are sufficient in order to 2-dominate the triangulation of an $n$-vertex piecewise-convex polygon. The proof is similar to the proof of Theorem 4, however, exactly like the simple (omitted) proof of Theorem 2 it makes use of edge contractions, yielding an $O(n^2)$ time and $O(n)$ space algorithm. A linear time and space algorithm is feasible by relaxing the requirement on the size of the edge 2-dominating set. More precisely, applying Lemma 1 for $\lambda = 6$, we have shown that we can 2-dominate the triangulation graph of a piecewise-convex polygon with $\left\lfloor \frac{n+1}{2} \right\rfloor$ edge guards. Although this result is weaker, it does not use edge contractions. We can, thus, devise a linear time and space algorithm for computing an edge 2-dominating set of size at most $\left\lfloor \frac{n+1}{2} \right\rfloor$, in exactly the same manner as in the case of diagonal 2-dominance. The following theorem summarizes our results, including our worst-case lower bound on the number of edge guards required to 2-dominate the triangulation graph of a piecewise-convex polygon.

**Theorem 4 (Karavelas 2008)** Given the triangulation graph $T_P$ of a polygon $P$ with $n \geq 3$ vertices, we can either compute: (1) an edge 2-dominating set for $T_P$ of size at most $\left\lfloor \frac{n+1}{2} \right\rfloor$ (except for $n = 4$, where one additional edge is required) in $O(n^2)$ time and $O(n)$ space, or (2) an edge 2-dominating set for $T_P$ of size at most $\left\lfloor \frac{n+1}{2} \right\rfloor$ (except for $n = 4$, where one additional edge is required) in $O(n)$ time and space. Finally, there exists a family of triangulation graphs with $n \geq 3$ vertices that require $\left\lfloor \frac{n+1}{2} \right\rfloor$ edge guards in order to be 2-dominated.

### 3 Piecewise-convex polygons

Let $v_1, \ldots, v_n$, $n \geq 2$, be a sequence of points and $a_1, \ldots, a_n$ a set of curvilinear arcs, such that $a_i$ has as endpoints the points $v_i$ and $v_{i+1}$. We will assume that the arcs $a_i$ and $a_j$, $i \neq j$, do not intersect, except when $j = i + 1$ or $j = i + 1$, in which case they intersect only at the points $v_i$ and $v_{i+1}$, respectively. We define a **curvilinear polygon** $P$ to be the closed region of the plane delimited by the arcs $a_i$. The points $v_i$ are called the vertices of $P$. An arc $a_i$ is a convex arc if every line on the plane intersects $a_i$ at most two points or along a line segment. A polygon $P$ is called a **locally convex polygon** if $P$ is locally convex except possibly at its vertices (see Fig. 7(left)). A polygon $P$ is called a **piecewise-convex polygon**, if it is locally convex and its edges are convex arcs (see Fig. 7(right)).

Let $a_i$ be an edge of a piecewise-convex polygon $P$ with endpoints $v_i$ and $v_{i+1}$. We call the convex region $r_i$ delimited by $a_i$ and $\overline{v_i v_{i+1}}$ a **room**, where $\overline{v_i v_{i+1}}$ denotes the line segment from $x$ to $y$. A room is called degenerate if the arc $a_i$ is a line segment. For $p, q \in a_i$, $\overline{pq}$ is called a **chord** of $a_i$; the chord of $r_i$ is $\overline{v_i v_{i+1}}$. An empty room is a non-degenerate room that does not contain any vertex of $P$ in the interior of $r_i$ or in the interior of $\overline{v_i v_{i+1}}$. A non-empty room is a non-degenerate room that contains at least one vertex of $P$ in the interior of $r_i$ or in the interior of $\overline{v_i v_{i+1}}$. We say that a point $p$ in the interior of a piecewise-convex polygon $P$ is visible from a point $q$ if $\overline{pq}$ lies in the closure of $P$. We say that $P$ is monitored by a guard set $G$ if every point in $P$ is visible from at least one point belonging to some guard in $G$. An edge (resp., mobile) guard is an edge (resp., edge or diagonal) of $P$ belonging to a guard set $G$ of $P$. An edge (resp., mobile) guard set is a guard set that consists of only edge (resp., mobile) guards.

Let $P$ be a piecewise-convex polygon with $n \geq 3$ vertices. Consider a convex arc $a_i$ of $P$, with endpoints $v_i$ and $v_{i+1}$, and let $r_i$ be the corresponding room. If $r_i$ is a non-empty room, let $X_i$ be the set of vertices of $P$ that lie in the interior of $\overline{v_i v_{i+1}}$, and let $R_i$ be the set of vertices of $P$ in the interior of $r_i$ or in $X_i$. If $R_i \neq X_i$, let $C_i$ be the set of vertices in the convex hull of the vertex set $(R_i \setminus X_i) \cup \{v_i, v_{i+1}\}$; if $R_i = X_i$, let $C_i = X_i \cup \{v_i, v_{i+1}\}$. Finally, let $C_i^* = C_i \setminus \{v_i, v_{i+1}\}$. If $r_i$ is an empty room, let $C_i = \{v_i, v_{i+1}\}$ and $C_i^* = \emptyset$. Let $T_P$ be the sought-for triangulation graph of $P$. The vertex set of $T_P$ is the set of vertices of $P$. The edges and diagonals of $T_P$, as well as their embedding, are defined as follows (see also Fig. 8):

- If $a_i$ is a line segment or $r_i$ is an empty room, the edge $(v_i, v_{i+1})$ is an edge in $T_P$, and is embedded as $\overline{v_i v_{i+1}}$.
- If $r_i$ is a non-empty room, the following edges or diagonals belong to $T_P$:
  1. $(v_i, v_{i+1})$,
  2. $(c_{i,j}, c_{i,j+1})$, for $1 \leq j \leq K_i - 1$, where $K_i = |C_i|$, $c_{i,1} \equiv v_i$ and $c_{i,K_i} \equiv v_{i+1}$. The remaining $c_{i,j}$’s are the vertices of $P$ in $C_i^*$ as we encounter them when walking inside $r_i$ and on the convex hull of the point set $C_i$ from $v_i$ to $v_{i+1}$, and
  3. $(v_i, c_{i,j})$, for $3 \leq j \leq K_i - 1$, provided that $K_i \geq 4$. We call these diagonals weak diagonals. The diagonals $(c_{i,j}, c_{i,j+1})$, $1 \leq j \leq K_i - 1$ are embedded as $c_{i,j}, c_{i,j+1}$, whereas the diagonals $(v_i, c_{i,j})$, $3 \leq j \leq K_i - 1$, are embedded as curvilinear segments. Finally, the edges $(v_i, v_{i+1})$ are embedded as curvilinear segments, namely, the arcs $a_i$.

The edges $(v_i, v_{i+1})$, along with the diagonals $(c_{i,j}, c_{i,j+1})$, $1 \leq j \leq K_i - 1$, partition $P$ into subpolygons of two types: (1) subpolygons that lie entirely inside a non-empty room, called **crecents**, and (2) subpolygons delimited by edges of the polygon $P$, as well as diagonals of the type $(c_{i,j}, c_{i,j+1})$, called **stars**. In general, a piecewise-convex polygon may only have crescents, or only stars, or both. The crescents are triangulated by means of the diagonals $(v_i, c_{i,j})$, $3 \leq j \leq K_i - 1$. To finish the definition of the triangulation graph $T_P$, we simply need to triangulate all stars inside $P$. Since stars are straight-line polygons, any polygon triangulation algorithm may be used to triangulate them.

In direct analogy to the types of subpolygons we can have inside $P$, we have two possible types of triangles in $T_P$: (1) triangles inside stars, called **star triangles**, and (2) triangles inside a crescent, called **crescent triangles**. Crescent triangles have at least one edge that is
a weak diagonal, except when the number of vertices of $P$ in the interior of the corresponding room $r$ is exactly one, in which case none of the three edges of the unique crescent triangle in $r$ is a weak diagonal. A crescent triangle that has at least one weak diagonal among its edges is called a weak triangle.

**Mobile guards** Let $G_{T_P}$ be a diagonal 2-dominating set of $T_P$. Based on $G_{T_P}$, we define a set $G$ of edges or straight-line diagonals of $P$ as follows (see also Fig. 9): (1) add to $G$ every non-weak diagonal of $G_{T_P}$, and (2) for every weak diagonal in $G_{T_P}$, add to the edge of $P$ delimiting the crescent that contains the weak diagonal. Clearly, $|G| \leq |G_{T_P}|$.

**Lemma 5** Let $P$ be a piecewise-convex polygon with $n \geq 3$ vertices, $T_P$ its constrained triangulation graph, and $G_{T_P}$ a diagonal 2-dominating set of $T_P$. The set $G$ of mobile guards, defined by mapping every non-weak diagonal of $G_{T_P}$ to itself, and every weak diagonal $d$ of $G_{T_P}$ to the corresponding convex arc of $P$ delimiting the crescent that contains $d$, is a mobile guard set for $P$.

**Proof.** Let $q$ be a point in the interior of $P$. $q$ is either inside: (1) an empty room $r_1$ of $P$; (2) a star triangle $t_s$ of $T_P$; (3) a non-weak crescent triangle $t_{cw}$ of $T_P$, or (4) a weak crescent triangle $t_m$ of $T_P$. In any of the four cases, $q$ is visible from at least two vertices $v_1$ and $v_2$ of $T_P$ that are connected via an edge or a diagonal in $T_P$. In the first case, $q$ is visible from the two endpoints $v_1$ and $v_{i+1}$ of $t_s$. In the second case, $q$ is visible from all three vertices of $t_s$. The third case arises when $q$ is inside a non-empty room $r_1$ with $|C_r^1| = 1$ ($t_{cw}$ is the unique crescent triangle in $r_1$), in which case $q$ is visible from at least two of the three vertices $v_j$, $v_{j+1}$, and $v_{j+2}$ of $C_r^1$. Finally, in the fourth case, $q$ has to lie inside the crescent of a non-empty room of $T_P$ with $|C_r^1| \geq 2$, and is visible from at least two consecutive vertices $v_{j,k}$ and $v_{j+k+1}$ of $C_j$.

Since $G$ is a diagonal 2-dominating set for $T_P$, and $(u_1, u_2) \in T_P$, at least one of $u_1$ and $u_2$ belongs to $G_{T_P}$. Without loss of generality, let us assume that $u_1 \in G_{T_P}$. If $u_1 \notin G_{T_P}$, $q$ is monitored by $u_1$. If $u_1 \notin G_{T_P}$, $u_1$ has to be an endpoint of a weak diagonal $d_w$ in $G_{T_P}$. Let $r_T$ be the room, inside the crescent of which lies $d_w$. Since $d_w \in G_{T_P}$, we have that $a \in G$. If $q$ lies inside the crescent of the room $r_T$ (this can only happen in case (4) above), $q$ is visible from $a$, and thus monitored by $a$. Otherwise, $u_1$ cannot be an endpoint of $d_w$ (the edges of $G_{T_P}$, whereas $u_2 \notin G$), which implies that $u_1 \in C_r^1$, i.e., $u_1 \equiv e_{r,m}$, with $2 \leq m \leq K_r - 1$. But then $q$ lies inside the cone with apex $e_{r,m}$, delimited by the rays $e_{r,m+1}$ and $e_{r,m-1}$, and containing at least one of $u_1$ and $v_{i+1}$ in its interior. Since, $q$ is visible from the intersection point of the line $q_1u_1$, with $a \in G$, $q$ is monitored by $a$.

Our approach for computing the mobile guard set $G$ of $P$ consists of three major steps: (1) Construct the constrained triangulation $T_P$ of $P$; (2) Compute a diagonal 2-dominating set $G_{T_P}$, for the triangulation graph $T_P$; (3) Map $G_{T_P}$ to $G$. The sets $C_r^1$, needed in order to construct the constrained triangulation $T_P$ of $P$, can be computed in $O(n \log n)$ time and $O(n)$ space (cf. [Karavelas and Tsigaridas 2008]). Once we have the sets $C_r^1$, the constrained triangulation $T_P$ of $P$ can be constructed in linear time and space. By Theorem 3, computing $G_{T_P}$ takes linear time; furthermore $|G_{T_P}| \leq \lfloor \frac{n+1}{2} \rfloor$, which implies that $|G| \leq \lfloor \frac{n+1}{2} \rfloor$. Finally, the construction of $G$ from $G_{T_P}$ takes $O(n)$ time and space: for every diagonal $d$ in $G_{T_P}$ we need to determine if it is a weak diagonal, in which case we need to add the edge of $P$ delimiting the crescent in which $d$ lies to $G$; by appropriate bookkeeping at the time of construction of $T_P$ these operations can take $O(1)$ per diagonal. Summarizing, by Theorem 4, Lemmas 5, and our analysis above, we arrive at the following theorem. The case $n = 2$ can be trivially established.

**Theorem 6** Let $P$ be a piecewise-convex polygon with $n \geq 2$ vertices. We can compute a mobile guard set for $P$ of size at most $\lfloor \frac{n+1}{2} \rfloor$ in $O(n \log n)$ time and $O(n)$ space.

**Edge guards** Let $G_{T_P}$ be and edge 2-dominating set of $T_P$ (see Fig. 10). The set $G$ of edge guards, defined by mapping every edge in $G_{T_P}$ to the corresponding convex arc of $P$, is an edge guard set for $P$ (cf. Karavelas 2008).

By Theorem 4, we can either compute an edge 2-dominating set $G_{T_P}$ of size $\lfloor \frac{n+1}{2} \rfloor$ in $O(n^2)$ time and $O(n)$ space, or an edge 2-dominating set $G_{T_P}$ of size $\lfloor \frac{n}{2} \rfloor$ (except for $n = 4$ where one additional edge is needed) in linear time and space. Since $T_P$ can be computed in $O(n \log n)$ time and $O(n)$ space, and $|G| = |G_{T_P}|$, we arrive at the following theorem. The case $n = 2$ is trivial, since in this case any of the two edges of $P$ is an edge guard set for $P$.

**Theorem 7** Let $P$ be a piecewise-convex polygon with $n \geq 2$ vertices. We can either: (1) compute an edge guard set for $P$ of size $\lfloor \frac{n+1}{2} \rfloor$ (except for $n = 4$, where one additional edge guard is required) in $O(n^2)$ time and $O(n)$ space, or (2) compute an edge guard set for $P$ of size $\lfloor \frac{n}{2} \rfloor$ (except for $n = 2, 4$, where one additional edge guard is required) in $O(n \log n)$ time and $O(n)$ space.

**Lower bound construction** Consider the piecewise-convex polygon $P$ of Fig. 11. Each spike consists of three edges, namely, two line segments and a convex arc. In order for points in the non-empty room of the convex arc to be monitored, either one of the three edges of the spike, or a diagonal at least one endpoint of which is an endpoint of the convex arc, has to be in any guard set of $P$: the chosen edge or diagonal in a spike cannot monitor the non-empty room inside another spike of $P$. Since $P$ consists of $k$ spikes, yielding $n = 3k$ vertices, we need at least $k$ edge or mobile guards to
monitor $P$. We, thus, conclude that $P$ requires at least $\lceil \frac{2n}{3} \rceil$ edge or mobile guards in order to be monitored.

4 Discussion and open problems

As far as the problem of 2-dominance of triangulation graphs is concerned, we have not yet found a way to compute an edge 2-dominating set of size at most $\lceil \frac{2n}{3} \rceil$ in $o(n^2)$ time, whereas we have shown that it is possible to compute an edge 2-dominating set of size at most $\lceil \frac{n}{7} \rceil$ in linear time and space. It, thus, remains an open problem how to compute an edge 2-dominating set of size at most $\lceil \frac{2n}{3} \rceil$ in $o(n^2)$ time and linear space. Moreover, we conjecture that there exist triangulation graphs that require a minimum of $\lceil \frac{2n}{3} \rceil$ edge guards; thus far we have found such triangulation graphs for $n = 7, 12, 17, 22$.

Once a 2-dominating set $D$ has been found for the constrained triangulation graph of a piecewise-convex polygon $P$, we either prove that $D$ is also a guard set for $P$ (this is the case for edge guards) or we map $D$ to a mobile guard set for $P$. In the case of edge guards, the piecewise-convex polygon is actually monitored by the endpoints of the edges in the guard set. In the case of mobile guards, interior points of the edges may also be needed in order to monitor the interior of the polygon. The latter observation should be contrasted against the corresponding results for the class of straight-line polygons, where, for both edge and mobile guards, the polygon is essentially monitored by the endpoints of these guards (cf. [O'Rourke 1987]).

Another important observation, due to the lower bound in Theorem 4 is that the proof technique of this paper cannot possibly yield better results for the edge guarding problem. If we are to close the gap between the upper and lower bounds, a fundamentally different technique will have to be used.

Thus far we have limited our attention to the class of piecewise-convex polygons. It would be interesting to attain similar results for locally concave polygons (i.e., curvilinear polygons that are locally convex polygons except possibly at the vertices), for piecewise-concave polygons (i.e., locally concave polygons the edges of which are convex arcs), or for curvilinear polygons with holes.

Acknowledgements

The author was partially supported by the IST Programme of the EU (FET Open) Project under Contract No IST-006413 – (ACS - Algorithms for Complex Shapes with Certified Numerics and Topology).

References


