A posteriori estimates for approximations of time-dependent Stokes equations

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In this paper, we derive a posteriori error estimates for space-discrete approximations of the time-dependent Stokes equations. By using an appropriate Stokes reconstruction operator, we are able to write an auxiliary error equation, in pointwise form, that satisfies the exact divergence-free condition. Thus, standard energy estimates from partial differential equation theory can be applied directly, and yield a posteriori estimates that rely on available corresponding estimates for the stationary Stokes equation. Estimates of optimal order in $L^\infty(L^2)$ and $L^\infty(H^1)$ for the velocity are derived for finite-element and finite-volume approximations.

Keywords: a posteriori error estimators; finite elements; finite volumes; time-dependent Stokes problem; discrete divergence-free spaces.

1. Introduction

We consider the nonstationary Stokes problem for incompressible flow:

$$\begin{align*}
\mathbf{u}_t - \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \times [0, T], \\
\text{div } \mathbf{u} &= 0 \quad \text{in } \Omega \times [0, T], \\
\mathbf{u} &= \mathbf{0} \quad \text{on } \partial \Omega \times [0, T], \\
\mathbf{u}(\cdot, 0) &= \mathbf{u}_0(\cdot) \quad \text{in } \Omega,
\end{align*}$$

(1.1)

where $\Omega$ is a bounded domain of $\mathbb{R}^d$ ($d = 2, 3$) with a sufficiently smooth boundary for our purposes. The above equation is discretized in space by either ‘finite elements’ or ‘finite volumes’. We are interested in proving a posteriori estimates for these approximations. The problem at hand is still open, despite the fact that it is directly related to the important problem of error control for the time-dependent Navier–Stokes equations. This is partly due to the fact that even in the case of linear parabolic problems, the development of the theory of a posteriori error control is still in progress (see, e.g. Eriksson & Johnson, 1991; Picasso, 1998; Makridakis & Nochetto, 2003; Bergam et al., 2005; Chen & Feng, 2004;
Additional technical difficulties appear in the case of the Stokes problem (1.1). A significant problem arises from the fact that spatially discrete approximations are rarely divergence-free functions. Also, the finite-dimensional spaces which are used in the spatial discretization are very often nonconforming. Very recently, the ideas of Bergam et al. (2005) were appropriately extended to the Stokes case by Bernardi & Verfürth (2004).

In this paper, we show that one can bypass the above technical problems, and derive a posteriori error estimates for finite-element or finite-volume approximations to the Stokes system. In particular, we address in a very natural way the problem that arises from the fact that although $u$ is divergence free, this is not necessarily true of its approximation $u_h$, and therefore the error $u - u_h$ is not divergence free. Our approach allows the treatment of nonconforming elements, the freedom of choosing various stationary (elliptic) estimators and leads to estimates of optimal order in various norms by using energy as well as duality techniques.

The main tool in our analysis is a ‘Stokes reconstruction’ operator, as defined below in Definition 1.2, the solution operator of an appropriate ‘stationary’ Stokes problem. This definition is an appropriate extension of the ‘elliptic reconstruction’ operator in the Stokes case, introduced by Makridakis & Nochetto (2003), for the a posteriori analysis of parabolic problems. Note that a similar operator was used in the a priori error analysis of finite-element approximations of the Navier–Stokes equation by Heywood and Rannacher (1982, Corollary 4.3) for different purposes. In Lemma 1.3, we show that the stationary finite-element approximation of the solution $(U, P)$ of the Stokes reconstruction problem is $(u_h, p_h)$, i.e. the approximation of the time-dependent problem. Then, the derivation of the error estimates is reduced to deriving estimates for $e = U - u$. Theorem 1.4 shows that $e$ satisfies a continuous, time-dependent Stokes problem, and is, in fact, ‘divergence free’. Thus, partial differential equation (PDE) techniques can be applied to derive the final estimates in various norms. Our approach is built on the available a posteriori bounds for the stationary problem, so additional issues such as alternative boundary conditions, totally discontinuous elements, etc. can be handled provided that there exist satisfactory stationary estimators. To focus on the main ideas, in this paper we have chosen to consider the spatially discrete case. The same techniques can be extended to fully discrete schemes with backward Euler time discretization along the lines of Lakkis & Makridakis (2006). The problem of deriving a posteriori bounds when considering more appropriate time-discretization schemes for the Stokes and Navier–Stokes equations requires new ideas. In the case of the heat equation considered in Lakkis & Makridakis (2006), extensive computational experiments highlight the efficiency of the estimators based on the reconstruction technique and indeed show that the resulting a posteriori bounds are of optimal order.

In the rest of this section, we introduce the necessary notation and the class of approximations that we will use. Next, we introduce the Stokes reconstruction and discuss its main properties. In Section 2, we show a posteriori error estimates based on the abstract setting introduced in Section 1 and we derive estimates by using energy as well as duality techniques. In Section 3, we apply the abstract theory to the classical nonconforming Crouzeix–Raviart pair of lowest order and we show a posteriori estimates of optimal order in $L^\infty(L^2)$ and $L^\infty(H^1)$ for the velocity error. In Section 4, we consider finite-volume schemes for discretizing (1.1). Still the approximations belong to Crouzeix–Raviart spaces. We appropriately modify the definition of the Stokes reconstruction for the finite-volume case and we show estimates of optimal order in $L^\infty(L^2)$ and $L^\infty(H^1)$. Note the interesting fact that the stationary finite-volume approximation of the solution $(U, P)$ of the Stokes reconstruction problem is still $(u_h, p_h)$ (the approximation of the time-dependent finite-volume problem) in analogy to the finite-element case.

In Bernardi & Verfürth (2004), optimal a posteriori error bounds in $L^2(H^1)$ are derived for fully discrete approximations of (1.1) based on backward Euler time discretization combined with conforming
finite elements for the spatial discretization. To deal with the problem which arises in the analysis due to
the fact that the finite-element spaces are not divergence free, a special stationary problem with nonzero
divergence for the velocity is introduced, and, in addition, certain conditions are required to hold for the
mesh.

1.1 Preliminaries—main definitions.

Let \( \mathbf{H} := (L^2(\Omega))^d \) be the usual Lebesgue space equipped with the inner product
\[ \langle f, g \rangle = \int_{\Omega} f(x) \cdot g(x) \, dx, \]
\( \mathbf{V} := (H^1_0(\Omega))^d, \Pi := \{ \phi \in L^2(\Omega) : \int_{\Omega} \phi(x) \, dx = 0 \} \) and \( \mathbf{V}^\ast := (H^{-1}(\Omega))^d \) be the dual of \( \mathbf{V} \). We
denote the norms on \( \mathbf{H}, \Pi, \mathbf{V} \) and \( \mathbf{V}^\ast \) by \( \| \cdot \|_\mathbf{H}, \| \cdot \|_\Pi, \| \cdot \|_\mathbf{V} \) and \( \| \cdot \|_{\mathbf{V}^\ast} \), respectively. Let \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \)
be the bilinear forms defined as
\[ a(u, v) = \int_{\Omega} \sum_{i=1}^d \nabla u_i \cdot \nabla v_i \, dx, \quad u, v \in \mathbf{V}, \tag{1.2} \]
and
\[ b(u, q) = -\int_{\Omega} \left( \text{div} \, u \right) q \, dx, \quad u \in \mathbf{V}, \quad q \in \Pi. \tag{1.3} \]

We assume that \( f \in L^2(0, T; \mathbf{V}^\ast) \) and \( u_0 \in \mathbf{H} \), and that (1.1) admits a unique weak solution \( (u, p) \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}) \times L^\infty(0, T; \Pi) \) satisfying
\[ \langle u_t(t), v \rangle + a(u(t), v) + b(v, p(t)) = \langle f(t), v \rangle \quad \forall v \in \mathbf{V}, \quad \text{a.e. } t \in [0, T], \]
\[ b(u(t), q) = 0 \quad \forall q \in \Pi, \quad \text{a.e. } t \in [0, T]. \tag{1.4} \]

In the sequel, we will assume that the data of the problem have sufficient (additional) regularity for our
results to hold. For detailed smoothness results and corresponding regularity requirements on the data of
(1.4), see Temam (2001) and Dautray & Lions (1993).

In the sequel, the ‘divergence-free’ subspaces of \( \mathbf{V} \) and \( \mathbf{H} \) will be useful. Following, e.g. Dautray &
Lions (1993), we let \( \mathcal{V} := \{ v \in \mathcal{D}(\Omega) : \text{div} \, v = 0 \} \). We then define \( \mathbf{Z} \) and \( \mathbf{J} \) to be the closure of \( \mathcal{V} \) in \( \mathbf{V} \)
and \( \mathbf{H} \), respectively. Then, the following characterization holds (Dautray & Lions, 1993, Section 1.4):
\[ \mathbf{Z} = \{ v \in \mathbf{V} : \text{div} \, v = 0 \}, \quad \mathbf{J} = \{ v \in \mathbf{H} : \text{div} \, v = 0 \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \}. \tag{1.5} \]

Then, naturally, Problem (1.4) is reduced to the following two problems: find \( u \in \mathbf{Z} \) such that
\[ \langle u_t(t), v \rangle + a(u(t), v) = \langle f(t), v \rangle \quad \forall v \in \mathbf{Z}, \tag{1.6} \]
and \( p \in \Pi \) such that
\[ b(v, p(t)) = -\langle u_t(t), v \rangle - a(u(t), v) + \langle f(t), v \rangle \quad \forall v \in \mathbf{V}. \tag{1.7} \]

It is known that the well-posedness of Problem (1.6) follows from the coercivity of the bilinear form
\( a(\cdot, \cdot) \), namely,
\[ \alpha \| v \|^2_{\mathbf{V}} \leq a(v, v) \quad \forall v \in \mathbf{V}, \tag{1.8} \]
where $\alpha > 0$, and the well-posedness of Problem (1.7) follows from the continuous inf–sup condition

$$
\beta \| p \|_{\Pi} \leq \sup_{w \in V} \frac{b(w, p)}{\| w \|_{V}} \quad \forall p \in \Pi,
$$

where $\beta > 0$.

Now, let $(V_h, \Pi_h)$ be an appropriate pair of finite-dimensional spaces that is chosen for the discretization of the stationary Stokes problem. Then, the space-discrete time-dependent counterpart of (1.4) is: find $(u_h, p_h): [0, T] \rightarrow V_h \times \Pi_h$ such that

$$
(u_{h,t}(t), \varphi) + a(u_h(t), \varphi) + b(\varphi, p_h(t)) = \langle f(t), \varphi \rangle \quad \forall \varphi \in V_h,
$$

Note that we do not assume that $V_h \subset V$, but only that $a(\cdot, \cdot)$ can be extended to $(V_h + V) \times (V_h + V)$ and $b(\cdot, \cdot)$ can be extended to $(V_h + V) \times \Pi$. In addition, for $v \in V_h$, with a slight abuse of notation, we still denote its norm in $V_h$ by $\| v \|_V$ (understood in the elementwise sense).

Then, indeed, similarly to the continuous case, (1.10) is reduced to: find $u_h \in Z_h$ such that

$$
(u_{h,t}(t), \varphi) + a(u_h(t), \varphi) = \langle f(t), \varphi \rangle \quad \forall \varphi \in Z_h,
$$

and then find $p_h \in \Pi_h$ such that

$$
b(\varphi, p_h(t)) = -(u_{h,t}(t), \varphi) - a(u_h(t), \varphi) + \langle f(t), \varphi \rangle \quad \forall \varphi \in V_h,
$$

where $Z_h \subset V_h$ is the ‘discrete divergence-free’ subspace of $V_h$

$$
Z_h = \{ \varphi \in V_h: b(\varphi, q) = 0 \quad \forall q \in \Pi_h \}.
$$

The existence and the uniqueness of $(u_h, p_h)$ is well known (Brenner & Scott, 1994, p. 248; Girault & Raviart, 1986, p. 59) under the following conditions:

$$
\alpha^* \| v \|_{\tilde{V}}^2 \leq a(v, v) \quad \forall v \in V_h,
$$

where $\alpha^* > 0$ and

$$
0 < \beta^* := \inf_{q \in \Pi_h} \sup_{v \in V_h} \frac{b(v, q)}{\| v \|_V \| q \|_{\Pi}}.
$$

In the a priori analysis, it is usually assumed that $\alpha^*$ and $\beta^*$ are independent of $h$, but this is not required in this paper.

We proceed now to define the ‘a posteriori estimator functions’ for the stationary problem. For $g \in V^*$, let $(w, q) \in V \times \Pi$ be the unique solution of the stationary Stokes equation

$$
-\Delta w + \nabla q = g \quad \text{in } \Omega, \quad \text{div } w = 0 \quad \text{in } \Omega,
$$

or in weak form

$$
a(w, v) + b(v, q) = \langle g, v \rangle \quad \forall v \in V, \quad b(w, \tilde{q}) = 0 \quad \forall \tilde{q} \in \Pi.
$$
Let \((w_h, q_h) \in V_h \times \Pi_h\) be the corresponding finite-element solution, i.e.
\[
\begin{align*}
a(w_h, \varphi) + b(\varphi, q_h) &= \langle g, \varphi \rangle \quad \forall \varphi \in V_h, \\
b(w_h, \bar{q}) &= 0 \quad \forall \bar{q} \in \Pi_h.
\end{align*}
\]

(1.17)

We thus assume the availability of a posteriori estimators for this problem, as expressed by the following assumption.

**Assumption 1.1** Let \((w, q) \in Z \times \Pi\) and \((w_h, q_h) \in Z_h \times \Pi_h\) be the exact solution and its finite-element approximation given in (1.16) and (1.17) above. For the space \(X\) equal to \(H\), \(V\) or \(V^*\), we assume that there exist a posteriori estimator functions \(E = E((w_h, q_h), g; X)\) and \(E_{\text{pres}} = E_{\text{pres}}((w_h, q_h), g; \Pi)\), which depend on \((w_h, q_h), g\) and the corresponding norm, such that
\[
\|w - w_h\|_X \leq E((w_h, q_h), g; X), \quad X \text{ equal to } H, V \text{ or } V^*,
\]
and
\[
\|q - q_h\|_\Pi \leq E_{\text{pres}}((w_h, q_h), g; \Pi).
\]

(1.18)

(1.19)

Next, we will define the ‘Stokes reconstruction’. Let \(\Delta_h: H^2 \cap Z \subset J \rightarrow J\) be the ‘Stokes operator’, namely, the \(L^2\)-projection of the Laplace operator onto \(J\) (Heywood & Rannacher, 1982). We then introduce a discrete version of the Stokes operator \(\Delta_h: Z_h \rightarrow Z_h\) by
\[
\langle \Delta_h v, \chi \rangle = -a(v, \chi) \quad \forall \chi \in Z_h.
\]

(1.20)

By \(f_h\) we denote the \(L^2\)-projection of \(f\) onto \(Z_h\), i.e.
\[
\langle f, \chi \rangle = \langle f_h, \chi \rangle \quad \forall \chi \in Z_h.
\]

(1.21)

Then, we have (compare to Makridakis & Nochetto, 2003) the following definition.

**Definition 1.2** (Stokes reconstruction) For fixed \(t \in [0, T]\), let \((U, P) \in V \times \Pi\) be the solution of the stationary Stokes problem
\[
\begin{align*}
a(U, v) + b(v, P) &= \langle g_h(t), v \rangle \quad \forall v \in V, \\
b(U, q) &= 0 \quad \forall q \in \Pi,
\end{align*}
\]

(1.22)

where
\[
g_h := -\Delta_h u_h - f_h + f.
\]

(1.23)

We call \((U, P) = (U(t), P(t))\) the Stokes reconstruction of \((u_h(t), p_h(t))\).

As before, \(U \in Z\) and \(P \in \Pi\) are, respectively, the solutions of the following problems:
\[
a(U, v) = \langle g_h, v \rangle \quad \forall v \in Z,
\]

(1.24)

and
\[
b(v, P) = -a(U, v) + \langle g_h, v \rangle \quad \forall v \in V.
\]

(1.25)
LEMMA 1.3 Assume that \((U, P)\) is the unique solution of the stationary Stokes problem (1.22). For fixed \(t \in [0, T]\), let \((U_h, P_h) = (U_h(t), P_h(t)) \in V_h \times \Pi_h\) be the finite-element solution of (1.22), namely,

\[
a(U_h, \varphi) + b(\varphi, P_h) = \langle g_h, \varphi \rangle \quad \forall \varphi \in V_h, \\
b(U_h, q) = 0 \quad \forall q \in \Pi_h.
\] (1.26)

Then,

\[
U_h(t) = u_h(t) \quad \text{and} \quad P_h(t) = p_h(t),
\] (1.27)

where \((u_h, p_h)\) is the solution of (1.10).

Proof. Let \(v_h \in Z_h\), then \(b(v_h, P_h) = 0\) and \(a(U_h, v_h) = \langle g_h, v_h \rangle\). Now,

\[
\langle g_h, v_h \rangle = -\langle \tilde{\Delta}h u_h, v_h \rangle - \langle f_h, v_h \rangle + \langle f, v_h \rangle = a(u_h, v_h),
\] (1.28)

i.e.

\[
a(U_h - u_h, v_h) = 0 \quad \forall v_h \in Z_h.
\] (1.29)

Since \(U_h, u_h \in Z_h\), we get \(U_h(t) = u_h(t)\). Also, according to (1.11), we obtain

\[
\langle u_{ht}, \tilde{\Delta}h u_h - f_h, \varphi \rangle = 0 \quad \forall \varphi \in Z_h,
\]
so \(u_h\) satisfies the following relation in \(L^2\):

\[
u_{ht} - \tilde{\Delta}h u_h - f_h = 0.
\] (1.30)

Further,

\[
b(\varphi, p_h) - b(\varphi, P_h) = -a(u_h, \varphi) - \langle u_{ht}, \varphi \rangle + \langle f, \varphi \rangle - [-a(U_h, \varphi) + \langle g_h, \varphi \rangle] = a(U_h - u_h, \varphi) - \langle u_{ht}, \tilde{\Delta}h u_h - f_h, \varphi \rangle,
\] (1.31)

for all \(\varphi \in V_h\). According to (1.30) and the fact that \(U_h = u_h\), we get

\[
b(\varphi, p_h - P_h) = 0 \quad \forall \varphi \in V_h.
\]

Due to the discrete inf–sup assumption (1.14), we have that \(p_h = P_h\). So we conclude that \((u_h, p_h) \in Z_h \times \Pi_h\) is the finite-element solution of the stationary Stokes equation whose exact solution is \((U, P)\).

We have the following result.

THEOREM 1.4 (Error equation) Let \((U, P)\) be the Stokes reconstruction and let \((u, p)\) be the solution of the Stokes problem (1.1) which is assumed to be sufficiently regular. If \(e := U - u\) and \(\varepsilon := P - p\), then \((e, \varepsilon)\) is the weak solution of the problem

\[
e_t - \Delta e + \nabla \varepsilon = (U - u_h)_t, \\
\text{div } e = 0.
\] (1.32)
Furthermore, $U - u_h$ and $(U - u_h)_t$ satisfy the estimates
\[
\|\hat{c}_t^{(j)}(U - u_h)\|_X \leq \mathcal{E}(\hat{c}_t^{(j)} u_h, \hat{c}_t^{(j)} p_h, \hat{c}_t^{(j)} g_h; X), \quad j = 0, 1,
\]
where $X$ is $H$, $V$ or $V^*$, and $\mathcal{E}$ is the a posteriori estimator function defined in Assumption 1.1.

**Proof.** The pair $(U, P)$ is the unique solution of the stationary Stokes problem
\[
-\Delta U + \nabla P = g_h \quad \text{in } \Omega,
\]
\[
\text{div } U = 0 \quad \text{in } \Omega.
\]
According to (1.30), we have
\[
U_t - \Delta U + \nabla P = (U - u_h)_t + f,
\]
and the first assertion of this theorem is now obvious.

Next, since $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are time-independent, $(U_t, P_t)$ is the solution of the stationary Stokes problem (1.22) with right-hand side $g_{h,t}$. Respectively, $(U_{h,t}, P_{h,t}) \in V_h \times P_h$ is the finite-element solution of the same problem with right-hand side $g_{h,t}$. Then, Lemma 1.3 implies that $(u_{h,t}, p_{h,t}) = (U_{h,t}, P_{h,t})$, i.e. for fixed $t$, $(u_{h,t}, p_{h,t})$ is the stationary finite-element solution of the Stokes problem with exact solution $(U_t, P_t)$. The second assertion then follows by Assumption 1.1.

2. Error estimates

2.1 Energy estimates

Next, we derive a posteriori estimates by applying energy techniques to the auxiliary problem (1.32). We start with $L^\infty(H)$ and $L^2(V)$ error estimates for the velocity.

**Theorem 2.1** ($L^\infty(H)$- and $L^2(V)$-norm error estimates) Assume that $(u, p)$ is the solution of the time-dependent Stokes problem (1.1) and $(u_h, p_h)$ is its finite-element approximation (1.10). Let $(U, P)$ be the solution of the stationary Stokes problem (1.16) and let $\mathcal{E}$ be as defined in Assumption 1.1. Then, the following a posteriori error bounds hold for $0 < t \leq T$:
\[
\|u(t) - U(t)\|^2_H + \int_0^t \|(u - U)(s)\|^2_V \, ds
\]
\[\leq \|u(0) - U(0)\|^2_H + \int_0^t \mathcal{E}((u_{h,t}, p_{h,t}), g_{h,t}; V^*)^2 \, ds.\]

In addition, we have
\[
\|(u - u_h)(t)\|_H \leq \|u_0 - u_{h0}\|_H + \left(\int_0^t \mathcal{E}((u_{h,t}, p_{h,t}), g_{h,t}; V^*)^2 \, ds\right)^{1/2}
\]
\[+ \mathcal{E}((u_h(0), p_h(0)), g_h(0); H) + \mathcal{E}((u_h(t), p_h(t)), g_h(t); H)
\]
and
\[
\left(\int_0^t \|(u - u_h)(s)\|^2_V \, ds\right)^{1/2} \leq \|u_0 - u_{h0}\|_H + \left(\int_0^t \mathcal{E}((u_{h,t}, p_{h,t}), g_{h,t}; V^*)^2 \, ds\right)^{1/2}
\]
\[+ \mathcal{E}((u_h(0), p_h(0)), g_h(0); H) + \left(\int_0^t \mathcal{E}((u_h, p_h), g_h; V)^2 \, ds\right)^{1/2}.
\]
Proof. Using again (1.30), we have
\[ \langle u_{h,t} - \tilde{u}_h, v \rangle = 0 \quad \forall v \in V. \]

According to (1.22) and the above relation, we get
\[ \langle u_{h,t}, v \rangle + a(U, v) + b(v, P) = \langle f, v \rangle \quad \forall v \in V. \]

Let \( e := U - u \) and \( \varepsilon := P - p \). By combining (1.4) and (2.1) and using the definitions of \( u \) and \( U \), we have that \( (e, \varepsilon) \in V \times II \) is the solution of the following nonstationary Stokes equation:
\[ \langle e_t, v \rangle + a(e, v) + b(v, \varepsilon) = \langle (U - u_h)_t, v \rangle \quad \forall v \in V, \]
\[ \langle \text{div } e, q \rangle = 0 \quad \forall q \in II, \]

or \( e \in Z \) is the solution of
\[ \langle e_t, v \rangle + a(e, v) = \langle (U - u_h)_t, v \rangle \quad \forall v \in Z, \]

and then \( \varepsilon \in II \) is the solution of
\[ b(v, \varepsilon) = -\langle e_t, v \rangle - a(e, v) + \langle (U - u_h)_t, v \rangle \quad \forall v \in V. \]

Now, since \( e \in Z \), we can choose \( v = e \) in (2.3) to get
\[ \| (u - U)(t) \|_H^2 + \int_0^t \| (u - U)(s) \|_V^2 \, ds \leq \| u(0) - U(0) \|_H^2 + \int_0^t \| (u_{h,t} - U_t)(s) \|_V^2 \, ds. \]

Assumption 1.1 implies that
\[ \| u_{h,t}(t) - U_t(t) \|_{V^*} \leq \mathcal{E}((u_{h,t}(t), p_{h,t}(t)), g_{h,t}(t); V^*) \quad \text{for } 0 \leq t \leq T, \]
which in turn leads to the first assertion of Theorem 2.1. To show the second one, it suffices to note that
Assumption 1.1 yields
\[ \| (u_h - U)(t) \|_H \leq \mathcal{E}((u_h(t), p_h(t)), g_h(t); H) \quad \text{for } 0 \leq t \leq T, \]

which together with
\[ \| u(0) - U(0) \|_H \leq \| u(0) - u_h(0) \|_H + \| u_h(0) - U(0) \|_H \]
\[ \leq \| u_0 - u_0^h \|_H + \mathcal{E}((u_h(0), p_h(0)), g_h(0); H) \]

concludes the proof. □

Next, we establish estimates in \( L^\infty(V) \) for the velocity error.

**Theorem 2.2 (**\( L^\infty(V) \)**-norm error estimates)** Under the assumptions of Theorem 2.1, the following *a posteriori* error bounds hold for \( 0 < t \leq T \):
\[ \int_0^t \| (u - U)_t(s) \|_H^2 \, ds + \| V(u - U)(t) \|_H^2 \]
\[ \leq \| V(u - U)(0) \|_H^2 + \int_0^t \mathcal{E}((u_{h,t}, p_{h,t}), g_{h,t}; H)^2 \, ds \]
and
\[
\| \nabla (u - u_h)(t) \|_H \leq \| u_0 - u_h^0 \|_V + \left( \int_0^t \delta((u_h,t), p_h(t), g_{h,t}; H)^2 \, ds \right)^{1/2} + \delta((u_h(0), p_h(0)), g_h(0); V) + \delta((u_h(t), p_h(t)), g_h(t); V).
\]

**Proof.** The first assertion of the theorem follows by selecting \( v = e_t \) in (2.3). Also, in view of Assumption 1.1, we get
\[
\|(u_h - U)(t)\|_V \leq \delta((u_h(t), p_h(t)), g_h(t); V) \quad \text{for } 0 \leq t \leq T,
\]
which together with
\[
\| \nabla (u - U)(0) \|_H \leq \| u_0 - u_h^0 \|_V + \|(u_h - U)(0)\|_V \\
\leq \| u_0 - u_h^0 \|_V + \delta((u_h(0), p_h(0)), g_h(0); V)
\]
proves the second assertion of this theorem. \( \square \)

**Remark 2.3** Another way to prove the \( H^1 \)-norm error estimate is as follows: Obviously, \( \Delta e \notin Z \) but after integrating by parts in (2.3), one can justify setting \( v = \Delta e \). Then, noting that \( \langle e_t, \Delta e \rangle = \langle e_t, Z \rangle = -\langle \nabla e_t, \Delta e \rangle \) (Heywood & Rannacher, 1982), we find
\[
\max_{0 \leq t \leq T} \| \nabla (u - U)(t) \|_H^2 + \int_0^T \| \Delta (u - U)(s) \|_H^2 \, ds \\
\leq \| \nabla (u - U)(0) \|_H^2 + \int_0^T \delta((u_{h,t}, p_{h,t}), g_{h,t}; H)^2 \, ds.
\]

Estimating the pressure error for the time-dependent problem is a delicate issue. Unlike the stationary problem, where \( \| p - p_h \|_H \) behaves like \( \| u - u_h \|_V \), a corresponding estimate for the pressure error in the time-dependent problem is unclear even in the a priori analysis. In our approach, this problem is reduced to a ‘balanced’ estimate involving \( \| p - P \|_H \) and \( \| u - U \|_V \) in terms of the right-hand side of the PDE (1.32). It turns out that it is unclear how this can be done using simple energy arguments. We refer to the works of Koch and Solonnikov (Solonnikov, 1964; Koch & Solonnikov, 2001, 2002), where stability issues for the time-dependent Stokes problem are addressed. Note also that estimates of the pressure error in \( H^{-1}(L^2) \) can be derived by adopting arguments of Bernardi & Raugel (1985) in estimating (1.32). Below, we estimate \( \max_{t \in [0,T]} \| p(t) - P(t) \|_H \) at the expense of an extra time-derivative in the estimator function; cf. the velocity estimate in Theorem 2.2 for \( \max_{t \in [0,T]} \| u - U \|_V \). Nevertheless, the order of the estimator is the correct one.

**Theorem 2.4** (Pressure error estimate) Under the assumptions of Theorem 2.2, we have
\[
\beta^2 \| p - P \|_H^2 \leq C \left[ \| (u - U)(0) \|_H^2 + \| \nabla (u - U)(0) \|_H^2 \\
+ \int_0^t \delta((u_{h,tt}, p_{h,tt}), g_{h,tt}; H)^2 \, ds \\
+ \int_0^t \delta((u_{h,t}, p_{h,t}), g_{h,t}; H)^2 \, ds + \delta((u_{h,t}, p_{h,t}), g_{h,t}; V^*)^2 \right]
\]
and
\[ \beta \| p - p_h \|^2_{H} \leq C \left[ \| (u - U)_t(0) \|^2_{H} + \| \nabla (u - U)(0) \|^2_{H} \right. \]
\[ + \left( \int_{0}^{t} \| (u_{h,t}, p_{h,t}) - (u_{h}, p_{h}) \|^2_{H} \right) \]
\[ + \left( \int_{0}^{t} \| (u_{h,t}, p_{h,t}) - (u_{h}, p_{h}) \|^2_{H} \right)^{1/2} + \| \delta_{h} \|_{H} \]
\[ + \beta \| \delta_{p,h} \|. \]

**Proof.** According to the continuous inf–sup condition (1.14), the error equation (2.2) and Poincaré’s inequality, one finds
\[ \| e \|^2_{H} \leq \sup_{v \in V} \frac{b(v, e)}{\| v \|^2} = \sup_{v \in V} \frac{- (\varepsilon, v) - a(e, v) + (U - u_h)_t, v}{\| v \|^2} \]
\[ \leq c (\| e \|_{H} + \| \nabla e \|_{H} + \| (U - u_h)_t \|_{V'}). \] (2.9)

Differentiating (2.3) with respect to \( t \), we get
\[ \langle e_{tt}, v \rangle + a(e_{t}, v) = \langle (U - u_{h})_{tt}, v \rangle \quad \forall v \in Z. \] (2.10)

Now, setting \( v = e_t \) in the last equation and integrating with respect to \( t \), we obtain
\[ \| (u - U)_t(t) \|^2_{H} + \int_{0}^{t} \| \nabla (u - U)_t(s) \|^2_{H} \, ds \leq \| (u - U)_t(0) \|^2_{H} + \int_{0}^{t} \| (U - u_h)_{tt}(s) \|^2_{H} \, ds. \] (2.11)

According to (2.9), (2.11), Assumption 1.1 and the previous error estimates for the velocity, we obtain the result of this theorem. \( \square \)

### 2.2 Estimates by parabolic duality

Now, we briefly discuss how one can apply our ideas to derive estimates using a parabolic duality argument (Thomée, 1997; Eriksson & Johnson, 1991). Consider the following backward parabolic Stokes problem: fix \( t_s \in (0, T] \) and let \( (z, s) \in Z \times H \) be the solution of the ‘backward’ problem
\[ z_t + \Delta z - \nabla s = 0 \quad \text{in} \quad \Omega \times (0, t_s), \]
\[ \text{div} z = 0 \quad \text{in} \quad \Omega \times (0, t_s), \]
\[ z = 0 \quad \text{on} \quad \partial \Omega \times (0, t_s), \] (2.12)

where \( e = U - u \). Then, for any \( \tau, 0 < \tau < t_s \), we have
\[ \max_{t \in [0, t_s]} \| z(t) \|_{H} \leq \| e(t_s) \|_{H}, \]
\[ \int_{0}^{t_s - \tau} \| z_{t} \|_{H} \, ds \leq \frac{1}{2} L_{\varepsilon} \| e(t_s) \|_{H}, \] (2.13)
where \( L_\tau = (\log(\frac{T}{\tau}))^{1/2} \). This bound follows by entirely similar arguments as in Thomée (1997, Lemma 12.5), working with the single-field version of the Stokes problem

\[
\mathbf{z}_t + \Delta \mathbf{z} = 0 \quad \text{in } \Omega \times (0, t_\ast).
\]  

**Theorem 2.5** Assume that \((\mathbf{u}, p)\) is the solution of the time-dependent Stokes problem (1.1) and \((\mathbf{u}_h, p_h)\) is its finite-element approximation (1.10). Let \((\mathbf{U}, P)\) be the solution of the stationary Stokes problem (1.16). Then, the following \textit{a posteriori} error bound holds: for \( 0 < t_\ast \leq T \) and any \( \tau, 0 < \tau < t_\ast \), we have

\[
\| (\mathbf{u} - \mathbf{u}_h)(t_\ast) \|_H \leq \left( 1 + \frac{1}{2} L_\tau \right) \max_{0 \leq t \leq t_\ast} \| (\mathbf{U} - \mathbf{u}_h)(t) \|_H \\
+ \int_{t_\ast - \tau}^{t_\ast} \| (\mathbf{U} - \mathbf{u}_h) \|_H \, dt + \| (\mathbf{u} - \mathbf{u}_h)(0) \|_H,
\]

where \( L_\tau = (\log(\frac{T}{\tau}))^{1/2} \).

**Proof.** The proof of this estimate is based on the formula obtained by a simple integration by parts

\[
\int_0^{t_\ast} \langle \mathbf{v}_t, \mathbf{z} \rangle + a(\mathbf{v}, \mathbf{z}) + \langle \nabla \mathbf{s}, \mathbf{v} \rangle \, dt + \langle \mathbf{v}(0), \mathbf{z}(0) \rangle = \langle \mathbf{v}(t_\ast), \mathbf{z}(t_\ast) \rangle \quad \text{for } \mathbf{v} \in \mathbf{V}.
\]  

(2.15)

In particular, for \( \mathbf{v} \in \mathbf{Z} \) this relation reduces to

\[
\int_0^{t_\ast} \langle \mathbf{v}_t, \mathbf{z} \rangle + a(\mathbf{v}, \mathbf{z}) \, dt + \langle \mathbf{v}(0), \mathbf{z}(0) \rangle = \langle \mathbf{v}(t_\ast), \mathbf{z}(t_\ast) \rangle.
\]  

(2.16)

Therefore, since \( \mathbf{e} \in \mathbf{Z} \), (1.32) yields

\[
\| \mathbf{e}(t_\ast) \|_H^2 = \langle \mathbf{e}(t_\ast), \mathbf{z}(t_\ast) \rangle \\
= \int_0^{t_\ast} \langle \mathbf{e}_t, \mathbf{z} \rangle + a(\mathbf{e}, \mathbf{z}) \, dt + \langle \mathbf{e}(0), \mathbf{z}(0) \rangle \\
= \int_{t_\ast - \tau}^{t_\ast} \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{z} \rangle \, dt + \int_{0}^{t_\ast - \tau} \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{z} \rangle \, dt + \langle \mathbf{e}(0), \mathbf{z}(0) \rangle \\
= \int_{t_\ast - \tau}^{t_\ast} \langle (\mathbf{U} - \mathbf{u}_h)_t, \mathbf{z} \rangle \, dt + \langle (\mathbf{U} - \mathbf{u}_h)(t_\ast - \tau), \mathbf{z}(t_\ast - \tau) \rangle \\
- \langle (\mathbf{U} - \mathbf{u}_h)(0), \mathbf{z}(0) \rangle - \int_{0}^{t_\ast - \tau} \langle (\mathbf{U} - \mathbf{u}_h), \mathbf{z}_t \rangle \, dt + \langle \mathbf{e}(0), \mathbf{z}(0) \rangle.
\]

Using (2.13), we obtain

\[
\| \mathbf{e}(t_\ast) \|_H^2 \leq \left[ \int_{t_\ast - \tau}^{t_\ast} \| (\mathbf{U} - \mathbf{u}_h)_t \|_H \, dt + \left( 1 + \frac{1}{2} \right) L_\tau \max_{0 \leq t \leq t_\ast - \tau} \| (\mathbf{U} - \mathbf{u}_h)(t) \|_H \right] \| \mathbf{e}(t_\ast) \|_H.
\]
It is interesting to note that, in the above estimate, the term
\[ \int_{t_{n-1}}^{t_n} \| (U - u_h)(t) \|_H \, dt \]
can be bounded by \( \max_{t_{n-1} \leq t \leq t_n} \| (U - u_h)(t) \|_H \) in the fully discrete case, provided that \( \tau \) is of the order of the time step. A similar phenomenon appears, of course, in standard proofs by duality in the time-discrete and fully discrete cases (cf. Thomée, 1997; Eriksson & Johnson, 1991). We refer to the forthcoming work of Lakiss & Makridakis (2005), where the fully discrete case for parabolic problems is considered. Thus, assuming for the time being that this term is not present in our case, ignoring the error of the initial condition and recalling Theorem 1.4, we see that the above arguments lead to a bound of the form
\[ \| (u - u_h)(t_n) \|_H \leq \left( 2 + \frac{1}{2} L_{\tau} \right) \max_{0 \leq t \leq t_n} \| \varepsilon \| \left( (u_h, p_h), g_h; H \right). \]

3. Application: Crouzeix–Raviart finite-element discretization

In this section, we will apply the \( a \) posteriori estimates in the case of the classical 2D Crouzeix–Raviart spaces of the lowest order \( V_h \times \Pi_h \) (Crouzeix & Raviart, 1973). We need to extend our notation as follows: For \( D \subset \Omega \) and \( s \geq 0 \), integer, we denote by \( H^s(D) \) the usual Sobolev spaces, and by \( H^s(D) \) their vectorial counterparts. Their norms and seminorms are denoted by \( \| \cdot \|_{s,D} \) and \( | \cdot |_{s,D} \), respectively. We will denote the \( L^2 \)-norm of vector or scalar functions simply by \( \| \cdot \| \). For a piecewise regular vector function \( v \), we define the discrete gradient as the \( L^2 \)-matrix \( \nabla_h v \mid_K = \nabla (v \mid_K), K \in \mathcal{T}_h \).

We consider a family of ‘shape-regular’ triangulations \( \{ \mathcal{T}_h \}_{0 < h < 1} \) of \( \Omega \), i.e. any two triangles in \( \mathcal{T}_h \) share at most a vertex or an edge and there exists a constant \( \theta_0 \) such that \( \theta_K > \theta_0 > 0 \), \( \forall K \), where \( h \) is the maximum diameter of the triangles of \( \mathcal{T}_h \) and \( \theta_K \) is the smallest angle of the triangle \( K \) (Brenner & Scott, 1994; Ciarlet, 1978). By \( E_h(K) \), we denote the set of the edges of \( K \in \mathcal{T}_h \). Also, let \( E_h^n \) be the set of the edges of \( \mathcal{T}_h \) that are not part of \( \partial \Omega \), and define \( E_h^n(K) \) in a similar way. In addition, \( h_K \) denotes the diameter of the triangle \( K \), \( |K| \) its area and \( h_e \) the length of an edge \( e \in E_h(K) \).

Next, let \( V_h \) be the Crouzeix–Raviart nonconforming finite-element space, cf. Crouzeix & Raviart (1973), associated with \( \mathcal{T}_h \), and let \( V_h \) be its vectorial counterpart. Recall that \( V_h \) consists of piecewise linear functions that are continuous at the midpoints of the elements of the triangulation \( \mathcal{T}_h \). The pressure space is just
\[ \Pi_h = \{ \psi \in \Pi = L^2_0 : \psi \mid_K \in P_0 \ \forall \ K \in \mathcal{T}_h \}. \]

The finite-element approximation \( (u_h, p_h) : [0, T] \to V_h \times \Pi_h \) of the semidiscrrete problem is defined by (1.10). It is well known (cf., e.g. Crouzeix & Raviart, 1973, Section 6; Temam, 2001, Proposition 4.13) that the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \), defined in the elementwise sense, satisfy (1.13) and (1.14). The \( a \) priori and \( a \) posteriori error analyses of finite-element approximations to the stationary Stokes equations have been considered in, e.g. Crouzeix & Raviart (1973), Dari et al. (1995), John (1998), Verfürth (1996) and Girault & Raviart (1986).

We will now show how the abstract results of the previous sections can be applied when we consider residual-type estimators for the Crouzeix–Raviart spatial discretization. We will adapt the estimators of Dari et al. (1995) and note that other alternatives are possible (Verfürth, 1996; Ainsworth & Oden, 1997). To this end, we let \( \sigma = U - u_h \), \( \zeta = P - p_h \) and recall that Lemma 1.3 implies that \( (u_h, p_h) \)
is the stationary Stokes approximation of a problem with exact solution \((U, P)\). Thus, introducing the classical conforming space

\[
X_h = V_h \cap H^1_0(\Omega),
\]

we have the orthogonality relation

\[
a(\sigma, \varphi) + b(\varphi, \zeta) = 0 \quad \forall \varphi \in X_h.
\]

Then, as in Dari et al. (1995), for \(\text{div}(V_h \sigma) \in H^{-1}(\Omega)\), we introduce \(r \in H^1_0(\Omega)\) and \(q \in L^2(\Omega)\) as the solution of the stationary Stokes problem

\[-\Delta r + \nabla q = -\text{div}(V_h \sigma)\]
\[
\text{div} r = 0.
\]

Note that the first equation can be written as

\[
\text{div}(\nabla r - qI - \nabla h \sigma) = 0,
\]

where \(I\) is the identity matrix. Therefore, there exists \(s \in H^1(\Omega)\) such that the error can be decomposed as \(\nabla h \sigma = \nabla r - qI + \text{curl} s\). Then, \(|s|_1 \leq C(\|\nabla h \sigma\| + |r|_1 + \|q\|)\). In addition, regularity results for the Stokes problem (Dautray & Lions, 1993) imply

\[|r|_1 + \|q\| \leq C\|\nabla h \sigma\|.
\]

Thus, we conclude

\[|r|_1 + |s|_1 \leq C\|\nabla h \sigma\|.\] (3.3)

Next, for two matrices \(B\) and \(D\), we denote their inner product by \(B : D = \sum_{i,j=1}^2 B_{ij}D_{ij}\). Then, since \(\int_\Omega \nabla h \sigma_1 \cdot \text{curl} \psi = 0, \forall \psi \in X_h\), we finally get (Dari et al., 1995)

\[
\|\nabla h \sigma\|^2 = \int_\Omega (\nabla h \sigma - \zeta I) : \nabla(r - \chi)
\]
\[
+ \int_\Omega \nabla h \sigma : \text{curl}(s - \psi), \quad \forall \chi, \psi \in X_h.
\]

We thus have

\[
\|\nabla h \sigma\|^2 = \sum_K \int_K g_h \cdot (r - \chi)
\]
\[
- \int_{\partial K} (\nabla h u_h - p_h I)n \cdot (r - \chi) - \int_{\partial K} \nabla h u_h \tau \cdot (s - \psi),\] (3.4)

for any \(\chi, \psi \in X_h\). Here, \(\tau = (-n_2, n_1)\) is the tangent vector to \(\partial K\). In addition, note the elementwise relation (1.23), whereby

\[
g_h|_K = -\hat{A}_h u_h - f_h + f = -u_{h,t} + f
\]
\[
= -(u_{h,t} - \Delta u_h + \nabla p_h - f) =: -R_K.
\] (3.5)

Obviously, for every element we have that \(\Delta u_h = 0\) and \(\nabla p_h = 0\). Here, \(u_{h,t} - \Delta u_h + \nabla p_h - f = R_K\) denotes the inner residual. Note that the argument above works with higher polynomial degrees as well.
In that case, the inner residual will appear in the estimator since in (3.4) above, instead of \( g_h \) we will have \( g_h + \Delta u_h - \nabla p_h \). Still, we have \( g_h + \Delta u_h - \nabla p_h = -R_K \). We proceed now to define the estimators. To this end, we first use the standard notation to define

\[
J_e = \| (\nabla u_h - p_h I) n_e \|_e = (\nabla u_h |_{K^-} - p_h |_{K^-} I) n_e - (\nabla u_h |_{K^+} - p_h |_{K^+} I) n_e
\]

and

\[
\| \nabla u_h r_e \|_e = (\nabla u_h |_{K^+} r_e) - (\nabla u_h |_{K^+} I) r_e.
\]

For all edges \( e \in E_h \), we let

\[
J_{e,n} = \begin{cases}
(\| (\nabla u_h - p_h I) n_e \|_e, & \text{if } e \in E_h^in, \\
0, & \text{otherwise,}
\end{cases}
J_{e,r} = \begin{cases}
\| \nabla u_h r_e \|_e, & \text{if } e \in E_h^in, \\
2\nabla u_h r_e, & \text{otherwise.}
\end{cases}
\]

We thus define the local error estimators \( \eta_{1,K}(u_h), K \in \mathcal{T}_h \), by

\[
\eta_{1,K}(u_h)^2 = h^2_K \| R_K \|_{0,K}^2 + \frac{1}{2} \sum_{e \in E_h(K)} h_e (\| J_{e,n} \|_e^2 + \| J_{e,r} \|_e^2), \quad (3.6)
\]

where the inner residual \( R_K \) is given by \( R_K = u_h, - \Delta u_h + \nabla p_h - f \). At this point, it is useful to note that the term involving \( J_{e,r} \) can be replaced by the jump of \( u_h \). Indeed, an elementary calculation shows that (John, 1998)

\[
\| \| u_h \|_e \|_0,e^2 = \frac{h^2_e}{12} \| J_{e,r} \|_0,e^2. \quad (3.7)
\]

For the rest of the paper, for clarity of our comparisons, we will use estimators of the form (3.6). The estimator \( \eta_1(u_h) \) is defined by assembling the local estimators:

\[
\eta_1(u_h) = \left( \sum_K \eta_{1,K}(u_h)^2 \right)^{1/2}. \quad (3.8)
\]

The proof is completed by using standard arguments: Let \( I_h \) be a Clement-type interpolant onto \( X_h \), which is locally quasi-stable in \( H^1 \) (Scott & Zhang, 1990; Brenner & Scott, 1994). We choose \( \chi = I_h r \) and \( \psi = I_h s \) in (3.4) and use the approximation properties of the interpolant to conclude the next result.

**Lemma 3.1** The following estimate holds:

\[
\delta'( (u_h(t), p_h(t)), g_h(t); V ) \leq C \eta_1 (u_h(t)). \quad (3.9)
\]

Next, we need an estimate for \( \delta'( (u_{h,t}(t), p_{h,t}(t)), g_{h,t}(t); H) \).

It is standard to consider the dual problem: find \( (z, s) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1(\Omega) \cap L^2(\Omega)) \) such that

\[
-\Delta z - \nabla s = \sigma \quad \text{in } \Omega, \\
\div z = 0 \quad \text{in } \Omega, \\
z = 0 \quad \text{on } \partial \Omega.
\]

\[
\sigma = \nabla \cdot (u - g_{h,t} - p_{h,t} I), \quad (3.10)
\]
Under known conditions on $\Omega$, the solution $(z, s)$ satisfies the elliptic regularity estimate

$$\|z\|_2 + \|s\|_1 \leq C \|\sigma_t\|.$$  (3.11)

Multiplying the first equation in (3.10) by $\sigma_t$ and using integration by parts and the second equation in (3.10), we obtain

$$\|\sigma_t\|^2 = \sum_K \left\{ \int_K (\nabla z : \nabla (U_t - u_{h,t}) + s \operatorname{div}(U_t - u_{h,t}) - (P_t - p_{h,t}) \operatorname{div} z) \right. \right.$$

$$+ \int_{\partial K} \nabla z \cdot u_{h,t} + \int_{\partial K} s n \cdot u_{h,t} \left. \right\}. \quad (3.12)$$

Also, differentiating (3.2) with respect to $t$, we have

$$a(\sigma_t, \phi) + b(\phi, \xi_t) = 0 \quad \forall \phi \in X_h.$$  

In addition, $b(U_t - u_{h,t}, \psi) = 0, \psi \in \Pi_h$. Therefore,

$$\|\sigma_t\|^2 = \sum_K \left\{ \int_K (-\Delta \sigma_t + \nabla \xi_t) \cdot (z - \chi) \right. \right.$$

$$+ \int_K (s - \psi) \operatorname{div}(\sigma_t) + \int_{\partial K} (\nabla (\sigma_t) n - \xi_t n) \cdot (z - \chi) \right. \right.$$

$$+ \int_{\partial K} (\nabla z n \cdot u_{h,t} + sn \cdot u_{h,t}) \left. \right\}. \quad (3.13)$$

Since in each element $K$ we have $g_{h,t} = -\partial_t R_K = -R_{K,t}$, we obtain

$$\|\sigma_t\|^2 = \sum_K \left\{ \int_K (-R_{K,t} \cdot (z - \chi) - (s - \psi) \operatorname{div} u_{h,t}) \right. \right.$$

$$+ \int_{\partial K} (\nabla u_{h,t} n - p_{h,t} n) \cdot (z - \chi) + \int_{\partial K} (\nabla z n + sn) \cdot u_{h,t} \left. \right\}, \quad (3.13)$$

for any $\chi \in X_h$ and $\psi \in \Pi_h$.

Now, define the local error estimators $\eta_{0,K}$ by

$$\eta_{0,K}(u_{h,t})^2 = h_K^2 \|R_{K,t}\|^2_{0,K} + h_K^2 \|\operatorname{div} u_{h,t}\|^2_{0,K}$$

$$+ \frac{1}{2} \sum_{e \in E_h^t(K)} (h_e^3 \|\partial_t J_e.n\|^2_{0,e} + h_e^3 \|\partial_t J_e.r\|^2_{0,e}).$$

The $L^2$-spatial estimator is given by

$$\eta_0(u_{h,t}) = \left( \sum_K \eta_{0,K}(u_{h,t})^2 \right)^{1/2}. \quad (3.14)$$
To conclude the estimate, one can show as in John (1998) that

\[ \sum_{K} \sum_{e \in E(K)} \int_{e} (\nabla zn + s n) \cdot u_{h,t} \leq C \left( \sum_{e \in E_{h}^{n2}} h_{e} \left\| u_{h,t} \right\|_{0,e} \right)^{1/2} \left\| \sigma \right\| \]

\[ \leq C \left( \sum_{e \in E_{h}^{n2}} h_{e}^{2} \left\| \partial_{t} J_{e}, \tau \right\|_{0,e} \right)^{1/2} \left\| \sigma \right\|. \quad (3.15) \]

Here, we utilized the fact that \( \int_{e} \left[ \left[ u_{h,t} \right]_{e} \right]_{h} = 0 \), the approximation properties of the \( L^{2}(e) \)-projection onto \( P_{0}(e) \) and (3.11).

Then, choosing \( \chi \) as the standard nodal interpolant of \( z \) and \( \psi = I_{h}s \), we finally conclude the following.

**Lemma 3.2** The following estimate holds:

\[ \mathcal{E}((u_{h,t}(t), p_{h,t}(t)), g_{h,t}(t); H) \leq C \eta_{1}(u_{h,t}(t)). \quad (3.16) \]

Further, assuming that (3.11) holds, we have

\[ \mathcal{E}((u_{h,t}(t), p_{h,t}(t)), g_{h,t}(t); H) \leq C \eta_{0}(u_{h,t}(t)). \quad (3.17) \]

We can now apply Theorems 2.1 and 2.2 to obtain \( H^{1} \) estimates in our case.

**Theorem 3.3** (Residual-based \( L^{2}(H^{1}) \)- and \( L^{\infty}(H^{1}) \)-norm error estimates) Assume that \( (u, p) \) is the solution of the time-dependent Stokes problem (1.1) and \( (u_{h}, p_{h}) \) is its Crouzeix–Raviart finite-element approximation (1.10). Then, the following \textit{a posteriori} bounds hold for \( 0 < t \leq T \):

\[ \left( \int_{0}^{t} \left\| u - u_{h}(s) \right\|_{V}^{2} ds \right)^{1/2} \leq \left\| u_{0} - u_{h}^{0} \right\|_{H} + C \left( \int_{0}^{t} \eta_{1}(u_{h,t}(s))^{2} ds \right)^{1/2} \]

\[ + C \eta_{1}(u_{h}(0)) + C \eta_{0}(u_{h,t}(t)). \]

\[ \| \nabla (u - u_{h})(t) \|_{H} \leq \| u_{0} - u_{h}^{0} \|_{V} + C \left( \int_{0}^{t} \eta_{1}(u_{h,t}(s))^{2} ds \right)^{1/2} \]

\[ + C \eta_{1}(u_{h}(0)) + C \eta_{1}(u_{h}(t)). \]

**Remark 3.4** In the above estimates, we did not assume the elliptic regularity estimate (3.11) and we have used the (crude) first bound of Lemma 3.2. Still the estimates in Theorem 3.3 are of optimal order. In the case of, e.g. convex polygonal domains, where (3.11) holds, the estimator \( \eta_{1}(u_{h,t}(s)) \) in these bounds should be replaced by \( \eta_{0}(u_{h,t}(s)) \).
For the \( L^\infty(L^2) \)-estimate, note that due to the fact that we use the Crouzeix–Raviart elements of lowest order, it is true that the term \( \mathcal{E}((u_{h,t}, p_{h,t}), g_{h,t}; V^*) \) in Theorem 2.1 is simply bounded by \( \mathcal{E}((u_{h,t}, p_{h,t}), g_{h,t}; H) \).

In the estimate of the next theorem, as in the elliptic case, to gain a power of \( h \) in the order, we have to use the elliptic regularity bound (3.11). Due to the abstract form of our estimators in Theorem 2.1, other, refined, choices can be made when (3.11) is not valid. To address this issue, detailed work related to the specific form of possible singularities of the exact solution is required. This case will not be considered in this paper.

We therefore have the following result.

**Theorem 3.5** (Residual-based \( L^\infty(L^2) \)-norm error estimates.) Assume that \( (u, p) \) is the solution of the time-dependent Stokes problem (1.1) and \( (u_h, p_h) \) is its Crouzeix–Raviart finite-element approximation (1.10). Assume further that (3.11) holds. Then, the following \textit{a posteriori} bound holds for 0 < \( t \leq T \):

\[
\| (u - u_h)(t) \|_H \leq \| u_0 - u_0^h \|_H + C \left( \int_0^t \eta_0(u_{h,t}(s))^2 \, ds \right)^{1/2} + C \eta_0(u_h(0)) + C \eta_0(u_h(t)).
\]

**4. Application: a finite-volume scheme**

Finite-volume methods rely on local conservation properties of the differential equation. Thus, integrating (1.1) over a region \( b \subset \Omega \) and using Green’s formula, we obtain

\[
\int_b u_t - \int_{\partial b} \nabla u n + \int_{\partial b} p n = \int_b f.
\] (4.1)

In addition, (1.1) gives \( \int_{\partial b} \text{div} \, u = 0 \), for appropriate domains \( \partial b \). The finite-volume scheme approximations in the Crouzeix–Raviart couple \( V_h \times \Pi_h \), used in Section 3, satisfying the local conservation property (4.1) over the ‘control volumes’. The number of these control volumes is equal to the dimension of \( V_h \). To fix the notation, we consider the following construction: Let \( z_K \) be an inner point of \( K \in T_h \). We connect \( z_K \) with line segments to the vertices of \( K \), thus partitioning \( K \) into three subtriangles \( K_e, e \in \mathcal{E}_h \). Then, with each side \( e \in \mathcal{E}_h \), we associate a quadrilateral \( b_e \), which consists of the union of the subregions \( K_e \).

The corresponding finite-volume method for the time-dependent Stokes problem is: seek \( (u_h, p_h): [0, T] \to V_h \times \Pi_h \) satisfying

\[
\int_{b_e} u_{h,t} \, \n - \int_{\partial b_e} \nabla u_h n + \int_{\partial b_e} p_h n = \int_{b_e} f \quad \forall e \in \mathcal{E}_h, \tag{4.2}
\]

\[
\int_K \text{div} u_h = 0 \quad \forall K \in \mathcal{T}_h. \tag{4.3}
\]

We refer to Chou (1997), Chou & Kwak (1997), and Chatzipantelidis et al. (2003), respectively, for \textit{a priori} and \textit{a posteriori} estimates for the finite-volume methods for the stationary Stokes problem.

For the rest of this paper, it is important to note that the finite-volume scheme admits a variational formulation similar to the one in the finite-element case (Chatzipantelidis et al., 2003, 2004; Chatzipantelidis, 2002).
There exists a unique solution \((u_h, p_h) \in V_h \times \Pi_h\) of the finite-volume method (4.2)–(4.3), which satisfies
\[
\langle u_{h,t}, A_h \varphi \rangle + a(u_h, \varphi) + b(\varphi, p_h) = \langle f, A_h \varphi \rangle \quad \forall \varphi \in V_h,
\]
\[
b(u_h, \psi) = 0 \quad \forall \psi \in \Pi_h,
\]
where the operator \(A_h : C(\Omega)^2 + V_h \rightarrow V_h\) is defined by
\[
A_h \varphi = \sum_{e \in E_h^{in}} v(m_e) \chi_b e,
\]
\(\chi_b e\) being the characteristic function of \(b_e\) and \(m_e\) the midpoint of the edge \(e\).

For future reference, we list the main properties of the operator \(A_h\) (Chatzipantelidis et al., 2003; Chatzipantelidis, 2002):
\[
\| \varphi - A_h \varphi \|_{0,K}^2 = \sum_{e \in E_h(K)} \| \varphi - A_h \varphi \|_{0,K_e}^2 \lesssim C h_K^2 |\varphi|_{1,K}^2 \quad \forall \varphi \in V_h,
\]
\[
\int_e A_h \varphi = \int_e \varphi \quad \forall \varphi \in V_h, \ \forall e \in E_h.
\]

We now introduce two operators that will be useful in the sequel. Let \(\mathcal{L}_h : L^2(\Omega)^2 \rightarrow V_h\) and \(\mathcal{L}_h : L^2(\Omega)^2 \rightarrow Z_h\) be defined by
\[
\langle \mathcal{L}_h v, \varphi \rangle = \langle v, A_h \varphi \rangle \quad \forall \varphi \in V_h,
\]
\[
\langle \mathcal{L}_h v, \omega \rangle = \langle v, A_h \omega \rangle \quad \forall \omega \in Z_h.
\]

Obviously, \(\mathcal{L}_h\) and \(\mathcal{L}_h\) are well defined. In view of the above definitions and Lemma 4.1, we conclude that the solution of the finite-volume scheme satisfies
\[
\mathcal{L}_h u_{h,t} = \tilde{A}_h u_h - \tilde{\mathcal{L}}_h f = 0.
\]

The derivation of the \textit{a posteriori} estimates follows the lines of the abstract analysis that was presented in Section 1, but certain modifications are still required. We start by redefining the Stokes reconstruction. Let \((U, P) \in V \times \Pi\) be the solution of the stationary Stokes problem
\[
a(U, v) + b(v, P) = \langle g_h, v \rangle \quad \forall v \in V,
\]
\[
b(U, q) = 0 \quad \forall q \in \Pi,
\]
where
\[
g_h := -\tilde{A}_h u_h - \tilde{\mathcal{L}}_h f + f + \mathcal{L}_h u_{h,t} - u_{h,t}.
\]

According to the definition of \(g_h\) and (4.9), we get
\[
g_h = f - u_{h,t}.
\]
One of the reasons motivating the definition of $g_h$ is that in view of (4.12), the error equation for $e = U - u$ and $\varepsilon = P - p$ is the one in Theorem 1.4:

$$e_t - \Delta e + \nabla \varepsilon = (U - u_h)_t$$

$$\text{div } e = 0. \tag{4.13}$$

Thus, the estimation of $e = U - u$ and $\varepsilon = P - p$ is done in a similar fashion as in Section 2, provided that we know how to handle $U - u_h$ and $(U - u_h)_t$. In the remaining part of this section, we show that although Lemma 1.3 is no longer valid, $U - u_h$ satisfies the necessary orthogonality relations needed to estimate $U - u_h$ and $(U - u_h)_t$ by applying the stationary a posteriori theory for the finite-volume scheme (Chatzipantelidis et al., 2003). In fact, it is interesting that $(u_h, p_h)$ is the stationary finite-volume solution to Problem (4.10). This is shown in Lemma 4.2.

**Lemma 4.2** Assume that $(U, P)$ is the unique solution of the stationary Stokes problem (4.10) and $(U_h, P_h)$ is its finite-volume solution, namely,

$$a(U_h, \varphi) + b(\varphi, P_h) = \langle g_h, A_h \varphi \rangle \quad \forall \varphi \in V_h,$$

$$b(U_h, q) = 0 \quad \forall q \in \Pi_h. \tag{4.14}$$

Then,

$$U_h(t) = u_h(t) \quad \text{and} \quad P_h(t) = p_h(t), \tag{4.15}$$

where $(u_h, p_h)$ is the solution of (4.4).

**Proof.** Let $\varphi \in Z_h$, then $b(\varphi, P_h) = 0$ and $a(U_h, \varphi) = \langle g_h, A_h \varphi \rangle$. Now, in view of (4.4) and (4.12), we have

$$a(U_h, \varphi) = \langle g_h, A_h \varphi \rangle = \langle f - u_{h,t}, A_h \varphi \rangle = a(u_h, \varphi), \tag{4.16}$$

i.e.

$$a(U_h - u_h, \varphi) = 0 \quad \forall \varphi \in Z_h. \tag{4.17}$$

Since $U_h, u_h \in Z_h$, we get $U_h(t) = u_h(t)$. Subtracting (4.14) from (4.4),

$$b(\varphi, p_h) - b(\varphi, P_h) = a(u_h, \varphi) - \langle u_{h,t}, A_h \varphi \rangle + \langle f, A_h \varphi \rangle$$

$$- [a(U_h, \varphi) + \langle g_h, A_h \varphi \rangle] \tag{4.18}$$

for all $\varphi \in V_h$. According to (4.12) and the fact that $U_h = u_h$, we get

$$b(\varphi, p_h - P_h) = 0 \quad \forall \varphi \in V_h.$$ 

Due to the discrete inf–sup assumption (1.14), we obtain $p_h = P_h$. Therefore, $(u_h, p_h) \in Z_h \times \Pi_h$ is the finite-volume solution of the stationary Stokes equation whose exact solution is $(U, P)$. \qed

We now turn to the estimate of $U - u_h$. As in Section 3, $\sigma = U - u_h$ and $\xi = P - p_h$. Note that Lemma 4.1 implies that $(u_h, p_h)$ satisfies

$$a(u_h, \varphi) + b(\varphi, p_h) = \langle L_h f, \varphi \rangle - \langle L_h u_{h,t}, \varphi \rangle \quad \forall \varphi \in V_h. \tag{4.19}$$
Therefore, in view of the definition of \((\mathbf{U}, P)\) and \((4.12)\), we have the following orthogonality relation on the conforming space \(X_h\):

\[
a(\sigma, \varphi) + b(\varphi, \xi) = \langle f - \mathcal{L}f, \varphi \rangle - \langle u_{h,t} - \mathcal{L}u_{h,t}, \varphi \rangle \quad \forall \varphi \in X_h.
\]

(4.20)

The proof rests on applying again the argument in Dari et al. (1995) as in Section 3 and taking into account the additional error term resulting from the finite-volume discretization. Thus, with the same notation as in Section 3, we use the decomposition \(\nabla_h \sigma = \nabla r - qI + \text{curl} s\), where \(q \in L^2(\Omega)\) and \(r \in H^1_0(\Omega)\) with \(\text{div} r = 0\) and \(s \in H^1(\Omega)\). Thus, cf. Section 3, using \((4.20)\) and the definition of \(\mathcal{L}_h\), we finally get

\[
\|\nabla_h \sigma\|^2 = \sum_K \int_K g_h \cdot (r - \chi) + \int_K (f - u_{h,t}) \cdot (\chi - A_h \chi)
\]

\[-\int_{\partial K} (\nabla_h u_h - p_h I) n \cdot (r - \chi) - \int_{\partial K} \nabla_h u_h \tau \cdot (s - \psi),\]

(4.21)

for any \(\chi, \psi \in X_h\). Still we denote by \(R_K\), the inner elementwise residual \(R_K = u_{h,t} - \Delta u_h + \nabla p_h - f\). Then, \(g_h|_K = -u_{h,t} + f = -R_K\). Consider now the local error estimators \(\eta_{1,K}(u_h)\) defined in \((3.6)\) and the global estimator \(\eta_1(u_h)\) defined by \((3.8)\). Then, we choose \(\chi = I_h r\) and \(\psi = I_h s\) in \((4.21)\), \(I_h\) being a Clement-type interpolant onto \(X_h\). Using the approximation properties of the interpolant and the operator \(A_h\), \((4.6)\), we conclude as in Chatzipantelidis et al. (2003) the following result.

Lemma 4.3 The following estimate holds:

\[
\|\mathbf{U}(t) - \mathbf{u}_h(t)\|_V \leq C \eta_1(u_h(t)), t \in [0, T].
\]

(4.22)

Next, we will provide an \textit{a posteriori} estimator for the \(L^2\)-norm error of the velocity. From the \textit{a priori} analysis of finite-volume methods, it is known (Chatzipantelidis, 2002) that in order to get \(O(h^2)\) convergence in \(L^2\), \(z_K\) has to be chosen as the barycentre of \(K\). Therefore, in the sequel, we assume that in the construction of the control volumes \(b_c\), the point \(z_K\) is chosen to be the barycentre of \(K\). In this case, we will have

\[
\int_K (\varphi - A_h \varphi) = 0 \quad \forall \varphi \in V_h.
\]

(4.23)

We consider again the dual problem \((3.10)\). Then \((3.12)\) is still valid. In addition, since all operators commute with time differentiation, if we differentiate \((4.20)\) with respect to \(t\), we obtain

\[
a(\sigma_t, \varphi) + b(\varphi, \xi_t) = \langle f_t - \mathcal{L}f_t, \varphi \rangle - \langle u_{h,t,t} - \mathcal{L}u_{h,t}, \varphi \rangle \quad \forall \varphi \in X_h.
\]

In the next equation, we also use the fact that \(b(U_t - u_{h,t}, \psi) = 0, \psi \in \Pi_h\). Therefore,

\[
\|\sigma_t\|^2 = \sum_K \left\{ \int_K (-\Delta \sigma_t + \nabla \xi_t) \cdot (z - \chi)
\]

\[+ \int_K (s - \psi) \text{div} \sigma_t + \int_{\partial K} (\nabla (\sigma_t) n - \xi_t n) \cdot (z - \chi)
\]

\[+ \int_K (f_t - u_{h,t,t}) \cdot (\chi - A_h \varphi) + \int_{\partial K} (\nabla z n \cdot u_{h,t} + s n \cdot u_{h,t}) \right\}.
\]
Using the fact that $g_{h,t} = -\partial_t R_K = -R_{K,t}$ and (4.23), we obtain
\[
\|\sigma_t\|^2 = \sum_K \left\{ \int_K (-R_{K,t} \cdot (z - \chi) - (s - \psi) \text{div} \, u_{h,t}) \right.
\]
\[
- \int_K (R_{K,t} - \overline{R_{K,t}}) \cdot (\chi - A_h \chi)
\]
\[
+ \int_{\partial K} (\nabla u_{h,t} n - p_{h,t} n) \cdot (z - \chi) + \int_{\partial K} (\nabla u n + s n) \cdot u_{h,t} \right\}, \tag{4.24}
\]
for any $\chi \in X_h$ and $\psi \in \Pi_h$. Here, $\overline{R_{K,t}}$ denotes the average of $R_{K,t}$ over $K$. The local error estimators $\tilde{\eta}_{0,K}$ in the finite-volume case are slightly different than the corresponding ones in the finite-element case:
\[
\tilde{\eta}_{0,K}(u_{h,t})^2 = h_K^4 \|R_{K,t}\|_{0,K}^2 + h_K^2 \|\text{div} \, u_{h,t}\|_{0,K}^2
\]
\[
+ \frac{1}{2} \sum_{e \in E_h^p(K)} \left( h_e^3 \|\partial_t J_{e,n}\|_{0,e}^2 + h_e^3 \|\partial_t J_{e,r}\|_{0,e}^2 \right) .
\]

The $L^2$-estimator is defined by
\[
\tilde{\eta}_0(u_{h,t}) = \left( \sum_K \tilde{\eta}_{0,K}(u_{h,t})^2 \right)^{1/2}. \tag{4.25}
\]

Then, we choose $\chi$ as the standard nodal interpolant of $z$, denoted by $I_{h,N} z$ and $\psi = I_h s$. Following Chatzipantelidis et al. (2003), we observe
\[
\int_K (R_{K,t} - \overline{R_{K,t}}) \cdot (I_{h,N} z - A_h I_{h,N} z) \leq \|R_{K,t} - \overline{R_{K,t}}\|_{0,K} \|I_{h,N} z - A_h I_{h,N} z\|_{0,K}
\]
\[
\leq C |K|^{1/2} \|R_{K,t} - \overline{R_{K,t}}\|_{0,K} \|I_{h,N} z\|_{1,K}
\]
\[
\leq C |K|^{1/2} \|R_{K,t} - \overline{R_{K,t}}\|_{0,K} \|z\|_{2,K}.
\]

The proof is thus complete.

Therefore, we have proved the following result.

**Lemma 4.4** The following estimate holds:
\[
\|u_{h,t} - U_t\|_H \leq C \eta_1(u_{h,t}(t)). \tag{4.26}
\]

Further, assuming that (3.11) and (4.23) hold, we have
\[
\|u_{h,t} - U_t\|_H \leq C \tilde{\eta}_0(u_{h,t}(t)). \tag{4.27}
\]

We can now apply Theorem 2.2 to obtain $H^1$ estimates in our case.
THEOREM 4.5 (Residual-based $L^2(H^1)$- and $L^\infty(H^1)$-norm error estimates) Assume that $(u, p)$ is the solution of the time-dependent Stokes problem (1.1) and $(u_h, p_h)$ is its finite-volume approximation (4.2) and (4.3). Then, the following a posteriori bounds hold for $0 < t \leq T$:

$$
\| \nabla (u - u_h)(t) \|_H \leq \| u_0 - u_h^0 \|_V + C \left( \int_0^t \eta_1(u_h, s)^2 \, ds \right)^{1/2} + C \eta_1(u_h(0)) + C \eta_1(u_h(t))
$$

and

$$
\left( \int_0^t \| (u - u_h)(s) \|_V^2 \, ds \right)^{1/2} \leq \| u_0 - u_h^0 \|_H + C \left( \int_0^t \eta_1(u_h, s)^2 \, ds \right)^{1/2} + C \eta_1(u_h(0)) + C \eta_1(u_h(t)).
$$

REMARK 4.6 In the case where (3.11) and (4.23) hold, the estimator $\eta_1(u_h, s)$ in these bounds should be replaced by $\tilde{\eta}_0(u_h, s)$.

Further, we prove the following $L^\infty(L^2)$ a posteriori estimate. The remark preceding Theorem 3.5 applies also here regarding the assumption of the elliptic regularity bound (3.11).

THEOREM 4.7 (Residual-based $L^2$-norm error estimates) Assume that $(u, p)$ is the solution of the time-dependent Stokes problem (1.1) and $(u_h, p_h)$ is its finite-volume approximation (4.2) and (4.3). Assume further that (3.11) and (4.23) hold. Then, the following a posteriori bound holds for $0 < t \leq T$:

$$
\| (u - u_h)(t) \|_H \leq \| u_0 - u_h^0 \|_H + C \left( \int_0^t \tilde{\eta}_0(u_h, s)^2 \, ds \right)^{1/2} + C \tilde{\eta}_0(u_h(0)) + C \tilde{\eta}_0(u_h(t)).
$$

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