A posteriori error estimates and maximal regularity for approximations of fully nonlinear parabolic problems in Banach spaces

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Abstract A posteriori error estimates are provided for discretizations in time of abstract nonlinear parabolic problems $u' = F(u)$, by the backward Euler method in the maximal regularity framework of Banach spaces. The estimates are of conditional type, i.e., are valid under assumptions on the approximate solution, and the proofs are based on appropriate fixed point arguments.


1 Introduction

In this paper, we derive a posteriori error estimates in the framework of the abstract complex Banach spaces for discretizations in time of fully nonlinear parabolic
problems. We shall consider the initial value problem

\[
\begin{cases}
    u'(t) = F(u(t)), & t > 0, \\
    u(0) = u_0 \in \mathcal{B},
\end{cases}
\]

where \( F : \mathcal{B} \to \mathcal{X} \) under suitable assumptions, \( \mathcal{X} \) and \( \mathcal{B} \) are abstract complex Banach spaces and \( \mathcal{B} \subset \mathcal{B} \) is an open set. This equation is discretized in time by the backward Euler method. Our aim is to show a posteriori estimates for the error between the discrete and the exact solution under verifiable assumptions on the data and the computed approximations.

This is a model problem which serves as a tool in understanding some key aspects of a hopefully successful methodology that can lead to rigorous error control of nonlinear evolution problems. The functional analytic setting that has been chosen is based on maximal (optimal) regularity estimates for a linearized problem. These estimates when applicable are known to handle the corresponding nonlinear problems by using techniques based on appropriate fixed point mappings. At a first stage these techniques imply at the PDE level local existence and regularity results, \([5,10]\). It turns out that this approach when combined with appropriate reconstructions of the approximate solutions can be useful for obtaining a posteriori error estimates. These estimates are valid by imposing restrictions on the time interval of similar nature to the ones which are required in the proofs of local existence results in \([10]\), cf. Remark 3. A further step, will be to show estimates valid in the maximal time interval of existence of the PDE solution. At the PDE level the solution can be extended by a recursive application of similar arguments using as new initial values the terminal values of previous steps. This extension step depends on the properties of the particular underlined PDE and will be studied in forthcoming works.

A posteriori error estimates for certain classes of nonlinear evolution problems have been obtained with different techniques in Hilbert spaces framework in, e.g., \([1,9,11,13]\), and in Banach spaces in \([12,15]\). In \([12]\) problems posed using accretive operators in Banach spaces were considered, while in \([15]\) were derived estimates in \( L^r(0, T; L^\rho(\Omega)) \) for nonlinear parabolic problems. In \([1,15]\) the estimates were valid under certain assumptions of asymptotic nature for the exact solution and the approximations. On the other hand estimates valid solely under assumptions on the approximate solution were derived in \([9,11]\). These assumptions are in principle verifiable and thus the final estimate is usually called conditional (see \([6,9,11]\)). Our paper is an attempt to derive a posteriori error estimates of conditional type for general nonlinear parabolic problems using semigroup–type techniques. It should be noted that the problem considered is a model used to understand the advantages and the limitations of the method. One of the aspects, one should pay attention, is the evaluation of the constants involved. We have made an effort to show that, in the general abstract case considered here, the stability constants provided by the PDE theory are probably involved but in principle computable, see Example 1 and Remark 1. The ideas presented herein might be useful in other applications to evolution PDEs where semigroup and contraction mapping techniques can be used in the analysis. In that case however one has to work with the specific equation taking full advantage of its stability properties.
The paper is organized as follows. In Sect. 2, we introduce the required notation and we describe the functional analytic setting used throughout the paper. We also state precise hypotheses on $F$. These are the main assumptions used to guarantee the local existence and uniqueness of the exact solution in [10]. Section 3 is devoted to describe in detail the continuous reconstruction of the numerical solution given by the backward Euler method which will be crucial to obtain our a posteriori estimates. The main result and relevant discussion are presented in Sect. 4.

2 Analytical framework and notation

In this section, we give the notation and we describe precisely the functional setting used throughout the present paper.

Let $(X, \| \cdot \|)$ and $(B, \| \cdot \|_B)$ be complex Banach spaces such that $B$ is densely embedded in $X$.

We first recall the next definition.

Definition 1 A linear operator, $A : D(A) \subset X \to X$ is called sectorial if there exist $a \in \mathbb{R}, M > 0$ and $\theta \in (0, \pi/2)$ such that the resolvent is analytic outside of the sector

$$a + S_\theta := \{ a + w \in \mathbb{C} : |\arg(-w)| < \theta \},$$

and bounded by

$$\|(z - A)^{-1}\|_{L(X, X)} \leq \frac{M}{|z - a|}, \quad z \notin a + S_\theta.$$

Recall also that a sectorial operator $A$ is an infinitesimal generator of an analytic semigroup $\{e^{tA}\}_{t \geq 0}$.

For a Banach space $(Y, \| \cdot \|_Y)$ and $0 < \alpha < 1$, we will denote $C^\alpha([0, T]; Y)$ the space of bounded and $\alpha$-Hölder continuous functions $g : [0, T] \to Y$, endowed with the norm

$$\|g\|_{C^\alpha([0, T]; Y)} = \sup_{0 \leq t \leq T} \|g(t)\|_Y + \sup_{0 \leq s < t \leq T} \frac{\|g(t) - g(s)\|_Y}{(t - s)^\alpha}.$$

We will also denote $C^\alpha_g((0, T]; Y)$ the set of all bounded functions $g : (0, T] \to Y$, such that $t \to t^\alpha g(t)$ is $\alpha$-Hölder continuous in $(0, T)$, endowed with the norm

$$\|g\|_{C^\alpha_g((0, T]; Y)} = \sup_{0 < t \leq T} \|g(t)\|_Y + \sup_{0 < s < t \leq T} \frac{\|g(t) - g(s)\|_Y}{(t - s)^\alpha} s^\alpha.$$

The corresponding seminorms are denoted by $[g]_{C^\alpha([0, T]; Y)}$ and $[g]_{C^\alpha_g((0, T]; Y)}$.

We consider the nonlinear initial value problem

$$\begin{cases}
    u'(t) = F(u(t)), & 0 \leq t \leq T, \\
    u(0) = u_0 \in B,
\end{cases} \quad (2)$$
where $B \subset\mathcal{B}$ is an open set and $F : \mathcal{B} \to X$ is continuous and Fréchet differentiable. In addition, we assume that its derivative $F_u$ fulfils the following hypotheses:

[H1]: For each $u^* \in \mathcal{B}$, there exist $R = R(u^*) > 0$ and $L = L(u^*) > 0$ such that

$$
\|F_u(u_2) - F_u(u_1)\|_{L(B, X)} \leq L\|u_2 - u_1\|_B,
$$

for all $u_1, u_2 \in \mathcal{B}$, with $\|u_1 - u^*\|_B \leq R$ and $\|u_2 - u^*\|_B \leq R$.

[H2]: For every $u^* \in \mathcal{B}$, the operator $F_u(u^*) : B \to X$ is $\theta$–sectorial, i.e., there exist $a \in \mathbb{R}$, $M > 0$ and $\theta \in (0, \pi/2)$ such that the resolvent is analytic outside of the sector $a + S_\theta$ and

$$
\|(z - F_u(u^*))^{-1}\|_{L(X, X)} \leq \frac{M}{|z - a|}, \quad z \notin a + S_\theta.
$$

[H3]: For all $u^* \in \mathcal{B}$, the graph-norm of $A = F_u(u^*)$ is equivalent to the norm of $B$. In fact there exists $\gamma = \gamma(u^*) > 0$ such that

$$
\gamma^{-1}\|y\|_B \leq \|y\|_{D(A)} = \|y\| + \|Ay\| \leq \gamma\|y\|_B.
$$

Notice that the constants $M, a$ and $\theta$ do not depend on $u^*$. Moreover, for our purposes, it will be enough that hypotheses [H1], [H2] and [H3] hold for $u^* = u_0$.

Under hypotheses [H1], [H2] and [H3] and appropriate choice of the initial data, the local existence and uniqueness of solution of (2) is guaranteed (see Theorem 8.1.1 in [10]). We therefore assume that there exist $\delta, 0 < \delta \leq T$, and a unique solution of (2), $u \in C([0, \delta]; B) \cap C^1([0, \delta]; X)$, with the additional regularity property $u \in C^\alpha([0, \delta]; B)$.

The proof of local existence and uniqueness is based in linearizing (2) around a state $u^* \in \mathcal{B}$, the optimal (maximal) regularity properties of the linearized problem, and appropriate use of contraction principle. This formally leads to consider the semi–linear problem

$$
\begin{cases}
  u'(t) = Au(t) + f(u(t)), & 0 \leq t \leq T, \\
  u(0) = u_0 \in \mathcal{B},
\end{cases}
$$

where $A = F_u(u^*)$ and $f(u) = F(u) - Au$, for all $u \in \mathcal{B}$. The natural choice $u^* = u_0$ is used in the analysis of [10] and of the present paper. The reiteration process based on (3) with fixed point $u$ gives rise to the linearized problem (see [10, Chap. 8]). The maximal regularity property for the linear problem $u'(t) = Au(t) + f(t)$, with $u(0) = 0$, reads,

$$
\|u'\|_{C^\alpha([0, T]; X)} + \|Au\|_{C^\alpha([0, T]; X)} \leq C\|f\|_{C^\alpha([0, T]; X)},
$$

and is the main tool in the analysis in [10] and also in the present paper.
The ideas in the analysis carried out here for the autonomous case can be extended to the nonautonomous case, i.e., for \( u' = F(t, u) \) instead of \( u' = F(u) \), without relevant additional difficulties.

We next present an example of a problem in one space dimension fitting in the framework considered in this paper. For more examples cf. [10]. This example is useful, since it shows how the constants involved in hypotheses [H1] and [H3] depend on the data of the nonlinear PDE, see also Remark 1. Note however, that in other applications in higher space dimensions the estimation of these constants might be more involved.

**Example 1** Let us consider the boundary value problem

\[
\begin{aligned}
&\partial_t U(t, x) = \partial_x (k(\partial_x U(t, x))\partial_x U(t, x)) + \phi(U(t, x)), \\
&U(0, x) = U_0(x), \quad 0 \leq x \leq 1, \\
&\partial_x U(t, 0) = \partial_x U(t, 1) = 0, \quad t \geq 0,
\end{aligned}
\]

for two given functions \( k \) and \( \phi \), an initial data \( U_0 \) and \((t, x) \in [0, +\infty) \times [0, 1]\). This problem can be seen, e.g., as a model for the evolution of the temperature \( U = U(t, x) \) of a combusting solid fuel at time level \( t \) and at position \( x \in [0, 1] \) (see [3, Sect. 6.7]).

Let us assume that \( k \) and \( \phi \) are sufficiently smooth functions, and that \( k \) satisfies the uniform ellipticity condition

\[
k(y) + yk'(y) \geq \mu > 0, \quad y \in \mathbb{R}.
\]

In addition, let us assume that the initial data \( U_0 \) is twice continuously differentiable, and satisfies the compatibility condition \( U_0(0) = U_0'(1) = 0 \).

Let \( X \) and \( B \) be the function spaces \( C([0, 1]) \) and \( \{v \in C^2([0, 1]) : \partial_x v(0) = \partial_x v(1) = 0\} \) respectively equipped with

\[
\|v\|_X = \sup_{0 \leq t \leq 1} |v(t)|, \quad v \in X,
\]

and

\[
\|v\|_B = \|v\| + \|\partial_x v\| + \|\partial_{xx} v\|, \quad v \in B.
\]

Denoting \( u(t) = U(t, \cdot) \), the boundary value problem (4) can be written in the abstract form (1) as

\[
u' = F(u), \quad u(0) = U_0,
\]

where \( F(v) = \partial_x (k(\partial_x v)\partial_x v) + \phi(v) \).

Under the assumptions on \( k, \phi \) and \( U_0 \) it is straightforward to prove that (5) satisfies the hypotheses [H1], [H2] and [H3] (see [3]). In particular, for \( w \in B \), \( A = F_u(u_0) \) defines the linear operator

\[
Aw = \partial_x (\kappa(\partial_x u_0))\partial_x w + \kappa(\partial_x u_0)\partial_{xx} w + \phi'(u_0)w.
\]
where \( \kappa(v) = k(v) + vk'(v) \). We have,

\[
\|Aw\| \leq \gamma_0 \{\|w\| + \|\partial_x w\| + \|\partial_{xx} w\|\} = \gamma_0 \|w\|_B,
\]

where

\[
\gamma_0 = \max \{\|\kappa(\partial_x u_0)\|, \|\partial_x (\kappa(\partial_x u_0))\|, \|\phi'(u_0)\|\},
\]

which readily leads to \( \gamma = 1 + \gamma_0 \) in [H3].

Now, let us fix \( R > 0 \), select \( v_1, v_2, w \in B \) such that \( \|v_1 - u_0\|_B < R \) and \( \|v_2 - u_0\|_B < R \). Then we have

\[
\|(F_u(v_2) - F_u(v_1))w\| \leq \|\kappa(\partial_x v_2) - \kappa(\partial_x v_1)\| \|\partial_{xx} w\| + \|\partial_x (\kappa(\partial_x v_1))\| \|w\| + \|\partial_x (\kappa(\partial_x v_2) - \kappa(\partial_x v_1))\| \|w\|
\]

\[
\leq \max\{K_1, K_2, L_0\} \|v_2 - v_1\|_B \|w\|_B,
\]

where

\[
K_1 = \max_{\min \partial_x u_0 - R \leq \xi \leq \max \partial_x u_0 + R} |k'\!(\xi)|,
\]

\[
K_2 = \max_{\min \partial_x u_0 - R \leq \xi \leq \max \partial_x u_0 + R} |k''(\xi)|,
\]

\[
L_0 = \max_{\min u_0 - R \leq \xi \leq \max u_0 + R} |\phi''(\xi)|.
\]

Therefore [H1] follows with \( L = \max\{K_1, K_2, L_0\} \).

The framework described above will be used in this paper where the nonlinear equation is discretized with the backward Euler method. Despite the fact in this paper, we discretize the nonlinear equation by means of the backward Euler method, in principle the general approach is applicable to other more general methods. The main idea of deriving a posteriori estimates is to construct a new function defined through the nodal values of the backward Euler approximation, call it \( U \). The reconstruction \( U \) satisfies then a perturbation of the original equation (11). Since \( U \) is known the a posteriori estimate relies on estimating \( u - U \). This is achieved by adopting the strategy of optimal regularity of Lunardi [10] to derive appropriate perturbation estimates that will lead to a posteriori error control.

### 3 The backward Euler method and its regular reconstruction

Let \( 0 = t_0 < t_1 < t_2 < \cdots < t_N = \delta \) be a partition of \([0, \delta]\). Let us denote \( I_n = (t_n, t_{n+1}] \) and \( k_n = t_{n+1} - t_n \) the \( n \)-th time step. The backward Euler method applied to (2) reads

\[
\frac{U^{n+1} - U^n}{k_n} = F(U^{n+1}), \quad n = 0, 1, \ldots, N - 1,
\]
where \( U^0 = u_0 \) and \( U^n \), which stands for the approximation to \( u(t_n) \), is assumed to belong in \( \mathcal{B} \), in fact,

\[
\{ U^0, U^1, U^2, \ldots, U^N \} \subset \mathcal{B}.
\] (7)

Notice that it is very natural to select \( U^0 = u_0 \). Otherwise, if \( U^0 \neq u_0 \), then an additional term must be added in the final estimate but we shall not consider this case in the present paper.

The existence of the discrete solution (7), i.e., the solvability of (6), as well as the convergence via a priori estimates, was studied in detail in [7]. In fact, in our paper, we will assume that the solution of (7) exists as long as the solution of (2) exists.

We denote by \( U = U(t) \) the piecewise linear interpolant of the values \( \{ U^1, U^2, \ldots, U^N \} \), i.e., \( U \in \mathbb{P}_1(I_n; \mathcal{B}) \) such that \( U(t_n) = U^n \). This function was used for deriving a posteriori error estimates for the backward Euler method in several function space settings (see, e.g., [12, 13]). In our case however this choice is not sufficient due to the lack of time regularity. Thus, we introduce a continuously differentiable in time reconstruction of the numerical solution denoted by \( U \), compare with [4], defined as follows:

\[
\begin{align*}
U|_{I_n} & \in \mathbb{P}_3(I_n; \mathcal{B}), \\
U(t_n) &= U^n, \quad n = 0, 1, \ldots, N, \\
U'(t_n) &= F(U^n), \quad n = 1, \ldots, N.
\end{align*}
\] (8)

Note also that \( U \) is locally defined in each \( I_n \) as a linear combination of \( U^{n-1}, U^n \) and \( U^{n+1} \). To be more precise, \( U \) is expressed as

\[
U|_{I_n}(t) = U^n + \frac{(t - t_n)}{k_{n-1}}(U^n - U^{n-1}) + 2 \frac{(t - t_n)^2}{k_n^2} V^n - \frac{(t - t_n)^3}{k_n^3} V^n,
\] (9)

where \( V^n = U^{n+1} - U^n - \frac{k_n}{k_{n-1}}(U^n - U^{n-1}) \). In addition, if \( U^0 \in \mathcal{B} \), then since \( U^n \in \mathcal{B} \), for all \( n \), it follows that \( U(t) \in \mathcal{B}, t \in I_n \), i.e., \( U \) is well defined. In fact by the definition of the scheme \( F(U^n) \) is a linear combination of \( U^{n-1} \) and \( U^n \), therefore \( F(U^n) \) belongs to \( \mathcal{B} \). In this case, clearly \( U \in C^1((0, \delta]; \mathcal{B}) \), and since \( U|_{I_n} \in \mathbb{P}_3(I_n; \mathcal{B}) \), for each \( I_n \), it follows that \( U' \in C_\alpha^0((0, \delta]; X) \), for any \( 0 < \alpha < 1 \).

Define now the computable residual \( r : [0, \delta] \to X \) by

\[
r(t) := U'(t) - F(U(t)).
\] (10)

Let us point out that \( F(U) \) is well defined because in each interval \( F \) is evaluated in a linear combination of \( U^{n-1}, U^n \) and \( U^{n+1} \) which belong to \( \mathcal{B} \). Moreover, since

\[
F(U(t)) - F(U(s)) = \int_0^1 F_u(\tau U(t) + (1 - \tau) U(s)) \, d\tau (U(t) - U(s)),
\]

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for $0 < s \leq t \leq \delta$, $F(U) \in C^\alpha_\alpha([0, \delta]; X)$, for any $0 < \alpha < 1$, therefore $r$ also belongs to $C^\alpha_\alpha((0, \delta]; X)$. This conclusion requires $\|F(t\mu(t) + (1-t)\mu(s))\|_{L^B(X)}$ to be bounded, which follows by [H1] in view of the forthcoming hypothesis [H4].

On the other hand, an important fact is that by the definition (10) the function $U$ is the solution of the initial value problem

\[
\begin{align*}
    U'(t) &= F(U(t)) + r(t), \quad 0 \leq t \leq \delta, \\
    U(0) &= U^0 \in \mathcal{B}.
\end{align*}
\]

(11)

3.1 The estimator in the linear case

Before going to the nonlinear case, it will be instructive to see what is the form of estimator in the linear case. In fact we consider the linear parabolic problem

\[
\begin{align*}
    u'(t) &= Au(t) + f(t), \quad 0 \leq t \leq \delta, \\
    u(0) &= u_0 \in \mathcal{B}.
\end{align*}
\]

(12)

where $A$ stands for a linear and bounded operator of sectorial type and $f$ is regular enough, e.g., $f \in C^\alpha_\alpha((0, \delta]; X)$. The existence, uniqueness and optimal regularity properties in Banach spaces of this problem have been studied in, e.g., [10, Chap. 4]. We therefore assume that $A$ is sectorial and (12) enjoys optimal regularity properties.

Assume that \{\text{Un}_n\}^{N}_{n=0}$ and $U_n$ are respectively the numerical solution of (12) by the backward Euler method and its continuous reconstruction defined previously, adapted in the linear case. Notice that the stability of the backward Euler method for linear problems in the framework of the analytic semigroups has been studied, e.g., in [8,14].

Then, the residual $r$ for the initial value problem (12), defined as in (10) by

\[
r(t) := U'(t) - AU(t) - f(t), \quad 0 \leq t \leq \delta,
\]

can be explicitly written as

\[
r|_{t_n}(t) = P_n(t) - f(t), \quad t_{n-1} \leq t \leq t_n,
\]

where, in view of (9) with the notation used there, $P_n$ is the computable polynomial

\[
P_n(t) = f(t_n) + (t - t_n) \left( \frac{4}{k^2_n} V^n - \frac{1}{k^2_{n-1}} A(U^n - U^{n-1}) \right) \\
+ (t - t_n)^2 \left( -\frac{3}{k^3_{n}} V^n - \frac{2}{k^3_{n}} A V^n \right) + \frac{(t - t_n)^3}{k^3_{n}} A V^n.
\]

Moreover, by construction, $r(t_n) = 0$, for $0 \leq n \leq N$, and the piecewise polynomial $P$ defined by $P|_{t_n} = P_n$ is continuously differentiable in the time interval $[0, \delta]$.

Next, to estimate the error $e = U - u$, we notice that it satisfies,

\[
\begin{align*}
    e'(t) &= Ae(t) + r(t), \quad 0 \leq t \leq \delta, \\
    e(0) &= 0.
\end{align*}
\]
The maximal regularity of this problem in the space \( C^\alpha_{\alpha}((0, \delta]; X) \), for \( 0 < \alpha < 1 \), implies
\[
\|e\|_{C^\alpha_{\alpha}((0,\delta]; D(A))} \leq \tilde{C} \|r\|_{C^\alpha_{\alpha}((0,\delta]; X)},
\]
where \( \tilde{C} \) is the constant of Lemma 2 in the next section. Thus, the a posteriori estimate follows in the linear case. The nonlinear case is considered in the next section.

4 A posteriori analysis for the nonlinear problem

Let us consider the problem (1) under hypotheses [H1], [H2] and [H3]. Thus, Theorem 1 below provides an a posteriori error estimate for the numerical solution (7). The theorem is valid solely under assumptions on the approximate solution (via its regular reconstruction). These assumptions are in principle verifiable and thus the final estimate is \( \text{conditional} \) in the spirit of [6,9,11].

We start with a preliminary lemma that will be used in the main theorem. Lemma 2 is taken from Lunardi [10] but for completeness we include the main idea of the proof. It provides the maximal regularity in the \( C^\alpha_{\alpha} \) framework for the linearized problem and it is the main ingredient in the proof of the final estimate. Note that we need this regularity result only for the simplified problem with initial data \( u_0 = 0 \), and thus technical issues related to the regularity of the initial data are avoided. The proof of our main result (Theorem 1) is based on an appropriate use of a fixed point mapping for a linearized error equation written using the regular reconstruction \( \mathcal{U} \). Note that all assumptions are of \( \text{a posteriori} \) type and thus the result is \( \text{conditional} \).

To fix notation, let \( g : [0, \delta] \times \mathcal{B} \to X \) be the function
\[
g(t, v) = F(\mathcal{U}(t)) - F(\mathcal{U}(t) - v) - Av, \tag{13}
\]
where \( A = F_u(u_0) \) is the operator defined in (3) and \( \mathcal{U} \) the continuous reconstruction (8). Besides, let \( \rho > 0 \) be a constant such that
\[
\rho \leq \frac{1}{2} R(u_0), \tag{14}
\]
where \( R(u_0) \) is the constant given by hypothesis [H1], a fixed \( 0 < \alpha < 1 \), and \( Y_\rho \) the metric space to be used also there defined by
\[
Y_\rho := \{ w \in C^\alpha_{\alpha}((0, \delta]; B) \cap C([0, \delta]; X) : w(0) = 0, \|w\|_{C^\alpha_{\alpha}((0,\delta]; B)} \leq \rho \}. \tag{15}
\]

The next additional hypothesis, is our first \( \text{a posteriori} \), and thus verifiable, assumption on the continuous reconstruction (8):
[H4]: The continuous reconstruction $\mathcal{U}$ fulfils

$$\|\mathcal{U}(\cdot) - u_0\|_{C^\alpha_g((0,\delta]; B)} < \rho.$$  

The radius $\rho$ will be restricted below in Theorem 1, cf. (25). Notice that we implicitly require that the exact solution will belong to this ball, which is possible by appropriately selecting the time interval $[0, \delta]$, see Remark 3 and Lunardi [10, Theorem 8.1.1]. Once $[0, \delta]$ is fixed, the hypothesis [H4] merely stands for an assumption to be satisfied by the continuous reconstruction in the proof of Theorem 1. Otherwise the estimate in Theorem 1 is not longer valid and a more accurate numerical solution should be computed.

**Lemma 1** If $\mathcal{U}$ fulfils the hypothesis [H4] and $\rho$ is the constant considered in (14), then $g(\cdot, w(\cdot)) \in C^\alpha_g((0, \delta]; X)$, for every $w \in Y_\rho$. Moreover,

$$\|g(\cdot, w(\cdot))\|_{C^\alpha_g((0, \delta]; X)} \leq A_1 \|w\|_{C^\alpha_g((0, \delta]; B)},$$

where

$$A_1 = \frac{9L\rho}{2},$$

and $L = L(u_0)$ is the constant in the hypothesis [H1].

**Proof** Let $w$ be a function belonging to $Y_\rho$. By the definition (13) we have

$$g(t, w(t)) = \int_0^1 \{ F_u (\tau \mathcal{U}(t) + (1 - \tau)(\mathcal{U}(t) - w(t))) - A \} \, d\tau \, w(t).$$

Therefore, by hypothesis [H1] we have

$$\|g(t, w(t))\| \leq L \int_0^1 \|\mathcal{U}(t) + (1 - \tau)(\mathcal{U}(t) - w(t)) - u_0\|_B \, d\tau \|w(t)\|_B$$

$$\leq L \sup_{0 \leq t \leq \delta} \|w(t)\|_B \int_0^1 \|\mathcal{U}(t) - u_0 - (1 - \tau)w(t)\|_B \, d\tau$$

$$\leq \frac{3}{2} \rho L \sup_{0 \leq t \leq \delta} \|w(t)\|_B.$$
On the other hand, for $0 < \varepsilon \leq s \leq t \leq \delta$, we have

$$g(t, w(t)) - g(s, w(s)) = \int_0^1 (M(\tau, t) - M(\tau, s)) \, d\tau \, w(t) + \int_0^1 (M(\tau, s) - A)(w(t) - w(s)) \, d\tau,$$

where $M(\tau, s) : B \to X$ is a family of linear operators defined by

$$M(\tau, s) = F_u(\tau \mathcal{U}(s) + (1 - \tau)(\mathcal{U}(s) - w(s)),$$

for $(\tau, s) \in [0, 1] \times [0, \delta]$. We now proceed to bound the first term of (16) for which we have

$$\left\| \int_0^1 (M(\tau, t) - M(\tau, s))w(t) \, d\tau \right\| \leq L_1 \int_0^1 \|\tau(\mathcal{U}(t) - \mathcal{U}(s)) + (1 - \tau)(\mathcal{U}(t) - \mathcal{U}(s) + w(s))\|_B \, d\tau \|w(t)\|_B \leq L \left\{ \|\mathcal{U}(t) - \mathcal{U}(s)\|_B \cdot \|w(t)\|_B + \int_0^1 (1 - \tau)\|w(t) - w(s)\|_B \, d\tau \|w(t)\|_B \right\}$$

$$\leq L \varepsilon^{-\alpha}(t - s)^\alpha \sup_{0 \leq t \leq \delta} \|w(t)\|_B + \left[ w \right]_{C^\alpha((0, \delta] ; B)} \left[ \sup_{0 < \varepsilon \leq s \leq t \leq \delta} \left\{ \varepsilon^\alpha \|\mathcal{U}(t) - u_0 - \mathcal{U}(s) + u_0\|_B \right\} + \frac{1}{2} \rho \right]$$

$$\leq L \varepsilon^{-\alpha}(t - s)^\alpha \|w\|_{C^\alpha((0, \delta] ; B)} \left\{ \|\mathcal{U}(\cdot) - u_0\|_{C^\alpha((0, \delta] ; B)} + \frac{1}{2} \rho \right\}. \quad (17)$$

Moreover, the second term of (16) is bounded as follows

$$\left\| \int_0^1 (M(\tau, s) - A)(w(s) - w(t)) \, d\tau \right\| \leq L \int_0^1 \|\tau \mathcal{U}(s) + (1 - \tau)(\mathcal{U}(s) - w(s)) - u_0\|_B \, d\tau \varepsilon^{-\alpha}(t - s)^\alpha \|w\|_{C^\alpha((0, \delta] ; B)} \leq \frac{3}{2} L \rho \varepsilon^{-\alpha}(t - s)^\alpha \|w\|_{C^\alpha((0, \delta] ; B)}. \quad (18)$$
Therefore, by (17) and (18) we have
\[ \| g(t, w(t)) - g(s, w(s)) \| \leq L e^{-\alpha(t-s)} \| w \|_{C^2_0((0, \delta]; B)} \left\{ \| U(\cdot) - u_0 \|_{C^2_0((0, \delta]; B)} + 2\rho \right\}. \]

Combining the above bounds we conclude
\[ \| g(\cdot, w(\cdot)) \|_{C^2_0((0, \delta]; X)} \leq L \| w \|_{C^2_0((0, \delta]; B)} \left\{ \| U(\cdot) - u_0 \|_{C^2_0((0, \delta]; B)} + \frac{7}{2}\rho \right\}, \]
and the proof then follows. \(\square\)

The next lemma establishes the maximal regularity bound for the linear problem
\[ \begin{cases} v'(t) = Av(t) + f(t), & 0 \leq t \leq \delta, \\ v(0) = 0. \end{cases} \]

Its proof is part of the proof of Theorem 4.3.5 in Lunardi [10], however the sketch we show in the present paper intends to clarify that all constants involved are in principle computable. They depend on, cf. proof of Lemma 2,
\[ M_k = \sup_{0 \leq t \leq \delta+1} \| t^k A^k e^{tA} \|_{L(X,X)}, \quad k = 0, 1, 2, \]
and
\[ M_{k,\alpha} = \sup_{0 \leq t \leq \delta+1} \| t^{k-\alpha} A^k e^{tA} \|_{L(D_A(\alpha, \infty), X)}, \quad k = 1, \]
for \(0 < \alpha < 1\), where \(D_A(\alpha, \infty)\) stands for a real interpolation space between \(X\) and the domain of \(A\) in \(L^\infty\). In addition, notice that one can compute upper bounds for \(M_k, M_{k,\alpha}\) in terms of the sectorial constants for \(A\), i.e., the angle \(\theta\), \(a\) and \(M\). To be more precise, following Sect. 2.1 of [10] one can verify
\[ M_0 \leq \frac{Me^{a(\delta+1)}}{2\pi} \left( 2 \int_{1}^{+\infty} \int_{-\nu}^{\nu} \rho^{-1} e^{-\rho \cos(v)} d\rho \cos(v) d\omega \right), \]
\[ M_1 \leq \frac{Me^{a(\delta+1)}}{2\pi} \left( 2 \int_{1}^{+\infty} \int_{-\nu}^{\nu} e^{\rho \cos(v)} d\rho \cos(v) d\omega \right), \]
\[ M_2 \leq 2(M_1 e)^2 \]
\[ M_{1,\alpha} \leq \frac{Me^{a(\delta+1)}}{2\pi} \left( 2 \int_{1}^{+\infty} \int_{-\nu}^{\nu} \rho^{-\alpha} e^{-\rho \cos(v)} d\rho \cos(v) d\omega \right), \]
for every \(\nu \in (\theta, \pi/2)\).
Lemma 2 Assume that $f \in C^\alpha_\alpha((0, \delta]; X)$, and $A$ is a linear and sectorial operator. Then, $v \in C^\alpha_\alpha((0, \delta]; D(A))$ and there exists a constant $\tilde{C} = \tilde{C}(M_0, M_1, M_2, M_1, \alpha) > 0$ such that

$$
\|v\|_{C^\alpha_\alpha((0, \delta]; D(A))} \leq \tilde{C} \|f\|_{C^\alpha_\alpha((0, \delta]; X)}.
$$

Proof Using the expression $v(t) = \int_0^t e^{(t-s)A} f(s) \, ds$, we immediately see that

$$
\|v(t)\| \leq M_0 \delta \|f\|_{L^\infty((0, \delta]; X)}, \quad 0 < t \leq \delta.
$$

On the other hand the splitting

$$
v(t) = \int_0^t e^{(t-s)A} (f(s) - f(t)) \, ds + \int_0^t e^{\sigma A} f(t) \, d\sigma,
$$

implies

$$
\|Av(t)\| \leq M_1 \int_0^t (t-s)^{\alpha-1} s^{-\alpha} \, ds \|f\|_{C^\alpha_\alpha((0, \delta]; X)} + \|e^{tA} - I\| \|f(t)\|_{L^\infty((0, \delta]; X)}
$$

$$
\leq M_1 \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} \, ds \|f\|_{C^\alpha_\alpha((0, \delta]; X)} + (M_0+1) \|f\|_{L^\infty((0, \delta]; X)}.
$$

Therefore, in view of (19) and (20)

$$
\|v(t)\|_{D(A)} \leq \tilde{C}_1 \|f\|_{C^\alpha_\alpha((0, \delta]; X)}, \quad 0 \leq t \leq \delta.
$$

where $\tilde{C}_1 = \max\{M_0 \delta, M_1 \beta(\alpha, 1 - \alpha), M_0 + 1\}$ and $\beta$ stands for the beta function.

Further, from the equality

$$
v'(t) = Av(t) + f(t) = \int_0^t A e^{(t-s)A} (f(t) - f(s)) \, ds + e^{tA} f(t),
$$

one can prove that $t^\alpha v'(t)$ is bounded in $(0, \delta]$ with values in $D_A(\alpha, \infty)$, in particular

$$
\|v'(t)\|_{D_A(\alpha, \infty)} \leq \frac{\tilde{C}_2}{t^{\alpha}} \|f\|_{C^\alpha_\alpha((0, \delta]; X)},
$$

for each $0 < t \leq \delta$, where

$$
\tilde{C}_2 = \max \left\{ 2^{\alpha+1} M_2 + M_1, 2^\alpha M_2 \int_0^{\delta/2} \sigma^\alpha (\sigma + 1)^{-2} \, d\sigma \right\}.
$$
To prove that $v \in C^\alpha_{\alpha}((0, \delta]; D(A))$ we notice that, for every $\varepsilon \in (0, \delta)$,

$$v(t) = e^{(t-\varepsilon)}v(\varepsilon) + \int_{\varepsilon}^{t} e^{(t-s)}A f(s) \, ds, \quad \varepsilon \leq t \leq \delta.$$ 

Since $f \in C^\alpha([\varepsilon, \delta]; X)$ and $v(\varepsilon) \in D(A)$, $A v(\varepsilon) + f(\varepsilon) = v'(\varepsilon) \in D(A)$, then from Theorem 4.3.1(iii) in [10] applied in the interval $[\varepsilon, \delta)$ instead of $[0, \delta)$ and from (22), if follows that $v \in C^\alpha((\varepsilon, \delta]; D(A)) \cap C^{\alpha+1}((\varepsilon, \delta]; X)$ and that

$$\|Av\|_{C^\alpha([\varepsilon, \delta]; X)} \leq \frac{M_{1, \alpha}}{\alpha} \|Av(\varepsilon) + f(\varepsilon)\|_{D_A(\alpha, \infty)} + \left(\frac{M_2}{\alpha(1-\alpha)} + 3M_0 + \frac{2M_1}{\alpha} + 1\right) \|f\|_{C^\alpha((\varepsilon, \delta]; X)}$$

$$\leq \frac{M_{1, \alpha}}{\alpha} \tilde{C}_2 \|f\|_{C^\alpha((0, \delta]; X)} + \left(\frac{M_2}{\alpha(1-\alpha)} + 3M_0 + \frac{2M_1}{\alpha} + 1\right) \|f\|_{C^\alpha((\varepsilon, \delta]; X)}$$

$$\leq \frac{\tilde{C}_3}{\varepsilon^{\alpha}} \|f\|_{C^\alpha((0, \delta]; X)}, \quad (23)$$

where

$$\tilde{C}_3 = \left(\frac{M_2}{\alpha(1-\alpha)} + 3M_0 + \frac{2M_1}{\alpha} + 1\right) \delta^{\alpha} + \frac{M_{1, \alpha}}{\alpha} \tilde{C}_2.$$

In view of (21) and (23), the proof of the lemma now follows. \qed

It is clear from this proof that the splitting (19) is one of the key ingredients in the maximal regularity of the $C^\alpha_{\alpha}$ framework, cf. [10] for details.

**Remark 1** (Evaluation of constants) By the arguments presented above it follows that the maximal regularity constant $\tilde{C}$ in Lemma 2 is computable, but its estimate is rather involved. Here, we assume that the basic sectoriality constant $M$ is at our disposal (Further evaluation of $M$ in the general case depends on the norm in $X$, it is in principle possible, see, e.g., [10, Sect. 3.1], but goes beyond the scope of this paper). It is expected that for specific simpler PDEs $\tilde{C}$ could be computed in a more straightforward manner.

**Remark 2** (The parameter $\alpha$) In [10], $\alpha$ is associated with the Hölder regularity of $F$ seen as a function of time. However in the present paper, we consider the autonomous case and consequently all results will be valid for any $0 < \alpha < 1$, in particular, the estimate in Theorem 1. In that case the estimate depends on $\alpha$ through the constant $\tilde{C}$ in Lemma 2.
Remark 3 (Choice of the time interval \([0, \delta]\)) In the local existence result \([10, \text{Theorem } 8.1.1.}\), the exact solution is proved to belong in the ball of \(C^\alpha_{\alpha}((0, \delta]; B)\) with center \(u_0\) and radius \(\rho\) at most

\[
\rho \leq \min \left\{ R, \frac{1}{8L\tilde{C}\gamma} \right\},
\]

see (8.1.11) in [10]; compare to the assumption \([H4]\). To be consistent with the assumption \([H4]\) and (25) we shall require that the maximal time interval \([0, \delta]\) is chosen such that the exact solution also belongs to the ball with radius \(\rho\) bounded by (25) instead of (24). This is possible according the proof of Theorem 8.1.1. in [10].

Remark 4 (Existence of the discrete solution) The existence of the discrete solution (7), was proved under certain assumptions in [7]. In the present paper, we will assume that the solution of (7) exists as long as the solution of (2) exists without requiring necessarily that the hypotheses in [7] are valid.

Remark 5 (Local inversion theorem) The main tool in the proof of our next Theorem 1 is an appropriate use of a fixed point argument. This result in an abstract setting follows also by an appropriate use of the Local Inversion Theorem in Banach spaces, cf. e.g., Theorem 1.2 in Ambrosetti and Prodi [2]. Note however, that an essential feature of our approach is the explicit form of the conditional assumptions and their influence to the final result. This is not a straightforward task and should be performed no matter which route is taken to the final result.

We are ready now to prove the main result in this paper. Notice however that our second a posteriori assumption is (26).

**Theorem 1** Let \(u, U : [0, \delta] \to B\) be the solution of (2) under the hypotheses \([H1], [H2] and [H3]\), and the continuous reconstruction (8) of the Backward Euler approximations \(\{U^n\}\) (6) with \(U^0 = u^0\) respectively. We select \(\rho\) such that

\[
\rho \leq \min \left\{ \frac{1}{2} R, \frac{1}{9L\tilde{C}\gamma} \right\},
\]

where \(\tilde{C}\) is the constant in Lemma 2, \(L = L(u_0), R = R(u_0)\) and \(\gamma = \gamma(u_0)\) the constants in hypotheses \([H1]\) and \([H3]\). Let us assume in addition to the hypothesis \([H4]\) that

\[
\|r\|_{C^\alpha_{\alpha}((0,\delta]; X)} \leq \frac{\rho}{2\tilde{C}\gamma}.
\]

Then,

\[
\|U - u\|_{C^\alpha_{\alpha}((0,\delta]; B)} \leq 2\tilde{C}\gamma\|r\|_{C^\alpha_{\alpha}((0,\delta]; X)},
\]

where \(r\) is the residual defined in (10).
Proof Recall that by definition (10), $U$ is a solution of the nonlinear initial value problem

$$
\begin{cases}
U'(t) = F(U(t)) + r(t), & 0 \leq t \leq \delta, \\
U(0) = U^0 \in B.
\end{cases}
$$

Besides, if we define $e : [0, \delta] \to B$ as $e := U - u$, which stands for the error of the backward Euler method, then $e$ is a solution of the initial value problem

$$
\begin{cases}
e'(t) = G(e(t)) + r(t), & 0 \leq t \leq \delta, \\
e(0) = 0,
\end{cases}
$$

where $G : [0, \delta] \times B \to X$ is defined by $G(t, v) = F(U(t)) - F(U(t) - v)$. Notice that $v$ will stand for the error, therefore the assumption that $v$ will be small enough as $U(t) - v \in B$, for $0 \leq t \leq \delta$, is reasonable.

The proof of the statement of the theorem is based in the existence and uniqueness of solution of (27). To this end, we linearize (27) around $u^* = u_0$ and consider the formal semilinear initial value problem

$$
\begin{cases}
e'(t) = Ae(t) + g(t, e(t)) + r(t), & 0 \leq t \leq \delta, \\
e(0) = 0,
\end{cases}
$$

where $A$ is the operator defined in (3) and $g$ is the function defined in (13) with $u^* = u_0$. As usual in the framework of the semilinear problems, the proof of existence and uniqueness of solution of (28) is based on the existence of a unique fixed point for a nonlinear operator, in our case the operator $\Gamma : Y_\rho \to Y_\rho$, defined by $\Gamma(w) = v$, where $Y_\rho$ is the metric space defined in (15) and $v$ is the solution of the initial value problem

$$
\begin{cases}
v'(t) = Av(t) + g(t, w(t)) + r(t), & 0 \leq t \leq \delta, \\
v(0) = 0.
\end{cases}
$$

The fixed point of $\Gamma$ must be the solution of (28) $e$, and we will have as a consequence that

$$e \in C_\alpha^\alpha((0, \delta]; B) \quad \text{and} \quad \|e\|_{C_\alpha^\alpha((0,\delta]; B)} < \rho.\]

Recall that, by the regularity of $U$ and $F(U)$ we have $r \in C_\alpha^\alpha((0, \delta]; X)$. Besides, by Lemma 1, for every $w \in Y_\rho$, we have that $g(\cdot, w(\cdot)) \in C_\alpha^\alpha((0, \delta]; X)$. Therefore, by Lemma 2 (see also Corollary 4.3.6 (ii) in [10]) the solution of (29), $\Gamma(w)$, belongs to $C_\alpha^\alpha((0, \delta]; D(A))$ and, by hypothesis [H3], $\Gamma(w) \in C_\alpha^\alpha((0, \delta]; B)$.

Moreover, also by Lemma 2, there exists a constant $\tilde{C} > 0$ such that

$$
\|\Gamma(w)\|_{C_\alpha^\alpha((0,\delta]; D(A))} \leq \tilde{C} \|g(\cdot, w(\cdot)) + r\|_{C_\alpha^\alpha((0,\delta]; X)},
$$

\[\square\] Springer
and, if $\gamma > 0$ is the constant given by hypothesis [H3], then

$$
\| \Gamma( w ) \|_{C^\alpha((0,\delta]; B)} \leq \tilde{C} \gamma \| g(\cdot, w(\cdot)) + r \|_{C^\alpha((0,\delta]; X)}.
$$

(30)

Now, in view of Lemma 1, (26) and the hypothesis [H4] we have

$$
\| \Gamma( w ) \|_{C^\alpha((0,\delta]; B)} \leq \tilde{C} \gamma \left( \frac{9L\rho}{2} \| w \|_{C^\alpha((0,\delta]; B)} + \| r \|_{C^\alpha((0,\delta]; X)} \right)
\leq \tilde{C} \gamma \rho \left( \frac{9L\rho}{2} + \frac{1}{2\tilde{C} \gamma} \right),
$$

for every $w \in C^\alpha((0, \delta]; B)$. Moreover, (25) and (26) lead up to

$$
\| \Gamma( w ) \|_{C^\alpha((0,\delta]; B)} \leq \rho.
$$

Therefore, $\Gamma$ maps $Y_\rho$ into $Y_\rho$.

On the other hand we will show that $\Gamma$ is a contraction in the space $C^\alpha((0, \delta]; B)$. We have, for every $v_1, v_2 \in Y_\rho$,

$$
\| \Gamma(v_2(t)) - \Gamma(v_1(t)) \|_B \leq \tilde{C} \gamma \| F(U(t) - v_2(t)) - F(U(t) - v_1(t)) - A(v_2(t) - v_1(t)) \|
\leq \tilde{C} \gamma \| \int_0^1 \{ F_u(\tau(U(t) - v_2(t)) + (1 - \tau)(U(t) - v_1(t))) - A \} d\tau \|_{L(B, X)}
\times \| v_2(t) - v_1(t) \|_B
\leq \tilde{C} \gamma L \int_0^1 \| U(t) - u_0 - \tau v_2(t) - (1 - \tau)v_1(t) \|_B d\tau \| v_2 - v_1 \|_{C((0,\delta]; B)}
\leq 2\tilde{C} \gamma L\rho \| v_2 - v_1 \|_{C((0,\delta]; B)}.
$$

(31)

Moreover, for every $0 < \varepsilon \leq s \leq t \leq \delta$,

$$
\| \Gamma(v_2(t)) - \Gamma(v_1(t)) - \Gamma(v_2(s)) + \Gamma(v_1(s)) \|_B
\leq \tilde{C} \gamma \| F(U(t) - v_2(t)) - F(U(t) - v_1(t)) - F(U(s) - v_2(s))
+ F(U(s) - v_1(s)) - A(v_2(t) - v_1(t) - v_2(s) + v_1(s)) \|.
$$
\begin{align*}
&\leq \tilde{C}\gamma \left[ \int_0^1 \{ F_u(\tau(\mathcal{U}(t) - v_2(t)) + (1 - \tau)(\mathcal{U}(t) - v_1(t)))ight. \\
&\quad - F_u(\tau(\mathcal{U}(s) - v_2(s)) + (1 - \tau)(\mathcal{U}(s) - v_1(s))) \} \, d\tau (v_2(t) - v_1(t)) \\
&\quad + \int_0^1 \{ F_u(\tau(\mathcal{U}(s) - v_2(s)) + (1 - \tau)(\mathcal{U}(s) - v_1(s))) - A \} \, d\tau \\
&\quad \times (v_2(t) - v_1(t) - v_2(s) + v_1(s)) \\
&\leq \tilde{C}\gamma L \int_0^1 \{ \| \mathcal{U}(t) - \mathcal{U}(s) - \tau(v_2(t) - v_2(s)) - (1 - \tau)(v_1(t) - v_1(s)) \|_B \\
&\quad \times \| v_2(t) - v_1(t) \|_B + \| \mathcal{U}(t) - u_0 - \tau v_2(s) + (1 - \tau)v_1(s) \|_B \\
&\quad \times \| v_2(t) - v_1(t) - v_2(s) + v_1(s) \|_B \} \, d\tau \\
&\leq 2\tilde{C}\gamma L \rho e^{-\alpha}(t - s)^{\alpha} \left\{ \sup_{0 < t \leq \delta} \| v_2(t) - v_1(t) \|_B + \left[ v_2 - v_1 \right]_{C^g_{\alpha}((0, \delta]; B)} \right\} \\
&\leq 2\tilde{C}\gamma L \rho e^{-\alpha}(t - s)^{\alpha} \| v_2 - v_1 \|_{C^g_{\alpha}((0, \delta]; B)}. \tag{32}
\end{align*}

Here, we have used that

\[ \| \mathcal{U}(t) - \mathcal{U}(s) - \tau(v_2(t) - v_2(s)) - (1 - \tau)(v_1(t) - v_1(s)) \|_B \leq 2\rho e^{-\alpha}(t - s)^{\alpha} \]

since \[ \| \mathcal{U} - u_0 \|_{C^g_{\alpha}((0, \delta]; B)}, \| v_1 \|_{C^g_{\alpha}((0, \delta]; B)}, \| v_2 \|_{C^g_{\alpha}((0, \delta]; B)} \leq \rho, \] and in particular

\[ [\mathcal{U}]_{C^g_{\alpha}((0, \delta]; B)} = [\mathcal{U} - u_0]_{C^g_{\alpha}((0, \delta]; B)} \leq \rho. \]

Thus, by (31) and (32), we have

\[ \| \Gamma(v_2) - \Gamma(v_1) \|_{C^g_{\alpha}((0, \delta]; B)} \leq 4\tilde{C}\gamma L \rho \| v_2 - v_1 \|_{C^g_{\alpha}((0, \delta]; B)}. \]

Besides, by (25)

\[ \sigma := 4\tilde{C}\gamma L \rho \leq \frac{4}{9} < 1, \]

and consequently \( \Gamma \) is a \( \sigma \)-contraction. Therefore, \( \Gamma \) has a unique fixed point which is at the same time the solution of (28). In addition we have that \( e \in C^g_{\alpha}((0, \delta]; B) \), and

\[ \| e \|_{C^g_{\alpha}((0, \delta]; B)} \leq \rho. \]

Moreover, by Lemma 1, (25) and (30)

\[ \| e \|_{C^g_{\alpha}((0, \delta]; B)} \leq \frac{1}{2} \| e \|_{C^g_{\alpha}((0, \delta]; B)} + \tilde{C}\gamma \| r \|_{C^g_{\alpha}((0, \delta]; X)}, \]

and the statement of the theorem follows. \( \square \)
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