Scattering theory for dissipative quantum systems

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The nuclear optical model

The nuclear optical model (I)

Quantum system

- Neutron targeted onto a complex nucleus
- Either the neutron is elastically scattered off the nucleus
- Or it is absorbed by the nucleus $\Longrightarrow$ Formation of a compound nucleus
- Concept of a compound nucleus was introduced by Bohr ('36)

Model

- Feshbach, Porter and Weisskopf ('54) : nuclear optical model describing both elastic scattering and absorption
- "Pseudo-Hamiltonian" on $L^2(\mathbb{R}^3)$

$$H = -\Delta + V(x) - iW(x)$$

  with V and W real-valued, compactly supported, $W \geq 0$
- Empirical model, widely used in Nuclear Physics
- Refined versions include, e.g., spin-orbit interactions

The nuclear optical model (II)

Interpretation

- $-iH$ generates a strongly continuous semigroup of contractions $\{e^{-itH}\}_{t\geq 0}$
- Dynamics described by the Schrödinger equation

$$\begin{cases} i\partial_t u_t = Hu_t \\ u_0 \in \mathcal{D}(H) \end{cases}$$

If the neutron is initially in the normalized state u_0, after a time $t \geq 0$, it is in the unnormalized state $e^{-itH}u_0$

- Probability that the neutron, initially in the normalized state u_0, eventually escapes from the nucleus :

$$p_{\mathrm{scat}} = \lim_{t\to\infty} \left\| e^{-itH} u_0 \right\|^2$$

- Probability of absorption :

$$p_{\mathrm{abs}} = 1 - \lim_{t\to\infty} \left\| e^{-itH} u_0 \right\|^2$$

- If $p_{\mathrm{scat}} > 0$, one expects that there exists an (unnormalized) scattering state u_+ such that $\|u_+\|^2 = p_{\mathrm{scat}}$ and

$$\lim_{t\to\infty} \left\| e^{-itH} u_0 - e^{it\Delta} u_+ \right\| = 0$$

The nuclear optical model (III)

Aim

- Explicit expression of H rests on experimental scattering data
- Need to develop the full scattering theory of a class of models

References : mathematical scattering theory for dissipative operators in Hilbert spaces

- Abstract framework : Lax-Phillips ['73], Martin ['75], Davies ['79,'80], Neidhardt ['85], Exner ['85], Petkov ['89], Kadowaki ['02,'03], Stepin ['04], . . .
- Small perturbations : Kato ['66], Falconi-F-Fröhlich-Schubnel ['17], . . .
- Schrödinger operators : Mochizuki ['68], Simon ['79], Wang-Zhu ['14], . . .

Abstract model

The model

- $\mathcal{H}$ complex Hilbert space
- Pseudo-Hamiltonian

$$H = H_0 + V - iC^*C = H_V - iC^*C,$$

with $H_0 \geq 0$, V symmetric, $C \in \mathcal{L}(\mathcal{H})$ and V, C^*C relatively compact with respect to H_0

- H_V is self-adjoint, H is closed and maximal dissipative, with domains

$$\mathcal{D}(H) = \mathcal{D}(H_V) = \mathcal{D}(H_0)$$

- $-iH$ generates a strongly continuous semigroup of contractions $\{e^{-itH}\}_{t\geq 0}$. More precisely, $-iH$ generates a group $\{e^{-itH}\}_{t\in\mathbb{R}}$ s.t.

$$\left\|e^{-itH}\right\| \leq 1, \ t \geq 0, \qquad \left\|e^{-itH}\right\| \leq e^{\|C^*C\||t|}, \ t \leq 0$$

- $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$ and $\sigma(H) \setminus \sigma_{\mathrm{ess}}(H)$ consists of an at most countable number of eigenvalues of finite algebraic multiplicities that can only accumulate at points of $\sigma_{\mathrm{ess}}(H)$
- Example to keep in mind :

$$\mathcal{H} = L^2(\mathbb{R}^3), \quad H_0 = -\Delta, \quad H_V = -\Delta + V(x) = H_V^*, \quad C = W(x)^{\frac{1}{2}}$$

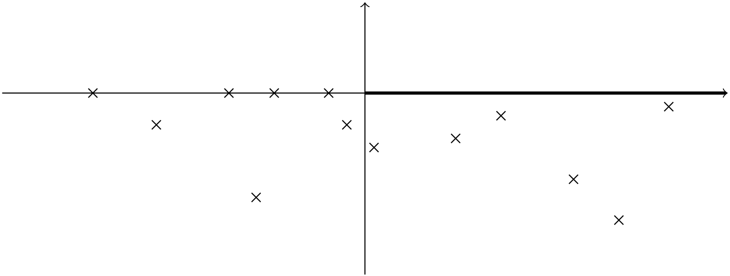

FIGURE: **Form of the spectrum of H.**

Spectral subspaces

Space of bound states

$$\mathcal{H}_{\rm b}(H) = \mathrm{Span}\{u \in \mathcal{D}(H),\ \exists \lambda \in \mathbb{R},\ Hu = \lambda u\}$$

Generalized eigenstates corresponding to non-real eigenvalues

- For $\lambda \in \sigma(H) \setminus \sigma_{\rm ess}(H)$, Riesz projection defined by

$$\Pi_\lambda = \frac{1}{2i\pi} \int_\gamma (z\mathrm{Id} - H)^{-1} dz,$$

  where γ is a circle centered at λ, of sufficiently small radius
- $\mathrm{Ran}(\Pi_\lambda)$ spanned by generalized eigenvectors of H associated to λ, $u \in \mathcal{D}(H^k)$ s.t. $(H-\lambda)^k u = 0$
- Space of generalized eigenstates corresponding to non-real eigenvalues :

$$\mathcal{H}_{\rm p}(H) = \mathrm{Span}\left\{u \in \mathrm{Ran}(\Pi_\lambda),\ \lambda \in \sigma(H),\ \mathrm{Im}\,\lambda < 0\right\}$$

"Dissipative space"

$$\mathcal{H}_{\rm d}(H) = \left\{u \in \mathcal{H},\ \lim_{t\to\infty} \|e^{-itH}u\| = 0\right\} \supset \mathcal{H}_{\rm p}(H)$$

The adjoint operator H^*

Properties of H^*

- $H^* = H_0 + V + iC^*C$
- $\lambda \in \sigma(H^*)$ if and only if $\bar{\lambda} \in \sigma(H)$
- iH^* generates the strongly continuous contraction semigroup $\{e^{itH^*}\}_{t \geq 0}$
- Spectral subspaces

$$\mathcal{H}_{\mathrm{b}}(H^*) = \mathrm{Span}\big\{u \in \mathcal{D}(H),\ \exists \lambda \in \mathbb{R},\ H^*u = \lambda u\big\},$$

$$\mathcal{H}_{\mathrm{p}}(H^*) = \mathrm{Span}\,\big\{u \in \mathrm{Ran}(\Pi^*_\lambda),\ \lambda \in \sigma(H^*),\ \mathrm{Im}\,\lambda > 0\big\},$$

$$\mathcal{H}_{\mathrm{d}}(H^*) = \big\{u \in \mathcal{H},\ \lim_{t \to \infty} \|e^{itH^*}u\| = 0\big\}$$

Hypotheses (I)

(H1) : Spectra of H_0 and H_V

$\sigma(H_0)$ is purely absolutely continuous, $\sigma_{\mathrm{sc}}(H_V) = \emptyset$, H_V has at most finitely many eigenvalues of finite multiplicity, and each eigenvalue of H_V is strictly negative

(H2) : Eigenvalues of H

The number of non-real eigenvalues of H is finite

(H3) : Wave operators for H_V and H_0

The wave operators

$$W_\pm(H_V, H_0) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_V} e^{-itH_0}, \quad W_\pm(H_0, H_V) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_0} e^{-itH_V} \Pi_{\mathrm{ac}}(H_V)$$

exist and are asymptotically complete, i.e.,

$$\mathrm{Ran}(W_\pm(H_V, H_0)) = \mathcal{H}_{\mathrm{ac}}(H_V) = \mathcal{H}_{\mathrm{pp}}(H_V)^\perp,$$
$$\mathrm{Ran}(W_\pm(H_0, H_V)) = \mathcal{H}$$

Hypotheses (II)

(H4) : Relative smoothness of C with respect to H_V

There exists a constant $\mathrm{c}_V > 0$, such that

$$\int_{\mathbb{R}} \left\| C e^{-itH_V} \Pi_{\mathrm{ac}}(H_V) u \right\|^2 dt \leq \mathrm{c}_V^2 \|\Pi_{\mathrm{ac}}(H_V) u\|^2,$$

for all $u \in \mathcal{H}$

Remarks

- Estimates of this form considered in [Kato '66]
- (H4) is equivalent to

$$\int_{\mathbb{R}} \Big(\left\| C (H_V - (\lambda + i0^+))^{-1} u \right\|^2 + \left\| C (H_V - (\lambda - i0^+))^{-1} u \right\|^2 \Big) d\lambda \leq 2\pi \mathrm{c}_V^2 \|u\|^2,$$

for all $u \in \mathrm{Ran}(\Pi_{\mathrm{ac}}(H_V))$

- The following estimate is satisfied

$$\int_0^\infty \left\| C e^{-itH} u \right\|^2 dt \leq \|u\|^2$$

Hypotheses (III)

Definition : spectral singularity

$\lambda \in [0, \infty)$ is a regular spectral point of H if there exists a compact interval $K_\lambda \subset \mathbb{R}$ whose interior contains λ and such that the limit

$$C\big(H - (\mu - i0^+)\big)^{-1} C^* = \lim_{\varepsilon \downarrow 0} C\big(H - (\mu - i\varepsilon)\big)^{-1} C^*$$

exists uniformly in $\mu \in K_\lambda$ in the norm topology of $\mathcal{L}(\mathcal{H})$. If λ is not a regular spectral point of H, we say that λ is a "*spectral singularity*" of H

(H5) : Spectral singularities of H

H has a finite number of spectral singularities $\{\lambda_1, \dots, \lambda_n\} \subset [0, \infty)$ and, for each spectral singularity $\lambda_j \in [0, \infty)$, there exist an integer $\nu_j > 0$ and a compact interval K_{λ_j}, whose interior contains λ_j, such that the limit

$$\lim_{\varepsilon \downarrow 0} (\mu - \lambda_j)^{\nu_j} C\big(H - (\mu - i\varepsilon)\big)^{-1} C^*$$

exists uniformly in $\mu \in K_{\lambda_j}$ in the norm topology of $\mathcal{L}(\mathcal{H})$. Moreover there exists $m > 0$ such that

$$\sup_{\mu \geq m,\, \varepsilon > 0} \left\| C\big(H - (\mu - i\varepsilon)\big)^{-1} C^* \right\| < \infty$$

Main results

Dissipative space

Theorem

Suppose that Hypotheses (H1)–(H5) hold. Then

$$\mathcal{H}_{\mathrm{d}}(H) = \mathcal{H}_{\mathrm{p}}(H)$$

Remarks

- Finding conditions implying this result quoted as open in [Davies '80]
- For small perturbations, the theorem follows from similarity of H and H_0 [Kato '66], implying that $\mathcal{H}_{\mathrm{d}}(H) = \mathcal{H}_{\mathrm{p}}(H) = \{0\}$
- Interpretation for the nuclear optical model : unless the initial state is a linear combination of generalized eigenstates corresponding to non-real eigenvalues of H, the probability that the neutron eventually escapes from the nucleus is always strictly positive

Uniform boundedness of the solution to the Schrödinger equation

Theorem

Suppose that Hypotheses (H1)–(H5) hold and that H has no spectral singularities in $[0,\infty)$. Then there exist $m_1>0$ and $m_2>0$ such that, for all $u\in\mathcal{H}_{\rm p}(H^*)^\perp$,

$$m_1\|u\| \le \left\|e^{-itH}u\right\| \le m_2\|u\| \ , \quad t\in\mathbb{R}$$

Remarks

- The second inequality shows that the solution to the Schrödinger equation

$$\begin{cases} i\partial_t u_t = Hu_t \\ u_0\in\mathcal{H}_{\rm p}(H^*)^\perp \end{cases}$$

  cannot blow up, as $t\to-\infty$
- For Schrödinger operators with a complex potential, related result in [Goldberg '10]

Asymptotic completeness (I)

Definition : wave operator

$$W_-(H,H_0) = \operatorname*{s-lim}_{t\to\infty} e^{-itH} e^{itH_0}$$

Theorem

Suppose that Hypotheses (H1)–(H5) hold and that H has no spectral singularities in $[0,\infty)$. Then $W_-(H,H_0)$ exists and is asymptotically complete, in the sense that

$$\mathrm{Ran}(W_-(H,H_0)) = \big(\mathcal{H}_{\mathrm{b}}(H^*) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\big)^{\perp}.$$

In particular,

$$\mathcal{H} = \mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H) \oplus \big(\mathcal{H}_{\mathrm{b}}(H^*) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\big)^{\perp}$$

and the restriction of H to $(\mathcal{H}_{\mathrm{b}}(H^*) \oplus \mathcal{H}_{\mathrm{p}}(H^*))^{\perp}$ is similar to H_0

Remarks

- Existence of $W_-(H,H_0)$ follows from standard arguments
- Not difficult to verify that $\overline{\mathrm{Ran}(W_-(H,H_0))} = \big(\mathcal{H}_{\mathrm{b}}(H^*) \oplus \mathcal{H}_{\mathrm{d}}(H^*)\big)^{\perp}$
- Main issues : $\mathcal{H}_{\mathrm{d}}(H^*) = \mathcal{H}_{\mathrm{p}}(H^*)$ and $\mathrm{Ran}(W_-(H,H_0))$ is closed

Asymptotic completeness (II)

Theorem

Suppose that Hypotheses (H1)–(H5) hold. Assume that there exist an interval $J \subset [0, \infty)$ and a vector $u \in \mathcal{H}$ such that

$$\lim_{\varepsilon \downarrow 0} \int_J \left\| C(H - (\lambda - i\varepsilon))^{-1} C^* u \right\|^2 d\lambda = \infty \qquad (*)$$

Then $W_-(H, H_0)$ is *not* asymptotically complete :

$$\mathrm{Ran}(W_-(H, H_0)) \subsetneq \left(\mathcal{H}_{\mathrm{b}}(H^*) \oplus \mathcal{H}_{\mathrm{p}}(H^*)\right)^\perp$$

Remark

For Schrödinger operators with bounded, compactly supported V and C, a spectral singularity corresponds to a real resonance. In particular condition $(*)$ is always satisfied for any such singularity

The wave operator $W_+(H_0, H)$ and the scattering operator

Definition

- Let $\Pi_{\mathrm{b}}(H)$ denotes the orthogonal projection onto $\mathcal{H}_{\mathrm{b}}(H)$ and
$$W_+(H_0,H) = \underset{t\to\infty}{\text{s-lim}}\, e^{itH_0} e^{-itH} \Pi_{\mathrm{b}}(H)^{\perp} = W_+(H^*, H_0)^*$$
- Scattering operator : $S(H, H_0) = W_+(H_0, H) W_-(H, H_0)$

Theorem

Suppose that Hypotheses (H1)–(H5) hold and that H has no spectral singularities in $[0, \infty)$. Then $W_+(H_0, H) : \mathcal{H} \to \mathcal{H}$ is surjective and
$$\mathrm{Ker}(W_+(H_0, H)) = \mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H).$$
Moreover, $S(H, H_0) : \mathcal{H} \to \mathcal{H}$ is bijective

Remarks

- [Davies '80] : Closedness of $\mathrm{Ran}(W_-(H, H_0))$ implies bijectivity of $S(H, H_0)$
- Interpretation for the nuclear optical model : for any initial state $u_0 \neq 0$ orthogonal to all the generalized eigenstates of H, there exists a scattering state $u_+ \neq 0$ s.t. $\| e^{-itH} u_0 - e^{-itH_0} u_+ \| \to 0$, as $t \to \infty$

Application : the nuclear optical model (I)

Theorem

Let $H = -\Delta + V(x) - iW(x)$ on $L^2(\mathbb{R}^3)$ with $W \geq 0$, $W(x) > 0$ on some non-trivial open set and $V, W \in L^\infty_c(\mathbb{R}^3; \mathbb{R})$. Suppose that 0 is neither an eigenvalue nor a resonance of $H_V = -\Delta + V(x)$. Then

$$\mathcal{H}_{\mathrm{d}}(H) = \mathcal{H}_{\mathrm{p}}(H).$$

Moreover, the wave operator $W_-(H, H_0) = \text{s-lim}_{t\to\infty} e^{-itH} e^{itH_0}$, with $H_0 = -\Delta$, is asymptotically complete in the sense that

$$\mathrm{Ran}(W_-(H, H_0)) = \mathcal{H}_{\mathrm{p}}(H^*)^\perp$$

if and only if H does not have real resonances. In this case, the restriction of H to $\mathcal{H}_{\mathrm{p}}(H^*)^\perp$ is similar to H_0 and there exist $m_1 > 0$ and $m_2 > 0$ such that, for all $u \in \mathcal{H}_{\mathrm{p}}(H^*)^\perp$,

$$m_1 \|u\| \leq \left\| e^{-itH} u \right\| \leq m_2 \|u\|, \quad t \in \mathbb{R}$$

Application : the nuclear optical model (II)

Verification of the abstract assumptions

- Spectra of H_0 and H_V : well-known
- H has a finite number of eigenvalues : well-known for compactly supported potentials, see e.g. [Frank,Laptev,Safronov '16] for more general conditions
- Wave operators for H_0 and H_V : well-known
- Relative smoothness of C w.r.t. H_V : [Ben-Artzi,Klainerman '92] for all $\varepsilon > 0$

$$\int_{\mathbb{R}} \left\| (1+x^2)^{-\frac{1+\varepsilon}{2}} e^{-itH_V} \Pi_{\mathrm{ac}}(H_V) u \right\|^2 dt \le \mathrm{c}_\varepsilon^2 \| \Pi_{\mathrm{ac}}(H_V) u \|^2$$

- Spectral singularities : resonances theory

Remarks

- H does not have real eigenvalues
- [Wang '11] : 0 cannot be a resonance of $H = -\Delta + V(x) - iW(x)$
- [Wang '12] : For any $\lambda > 0$, one can construct smooth compactly supported potentials V and W such that λ is a spectral singularity of H

Application : scattering for Lindblad master equations (I)

Lindbladian and quantum dynamical semigroup

- If one considers a quantum particle interacting with a dynamical target, takes the trace over the degrees of freedom of the target and studies the reduced effective evolution of the particle, then, in the kinetic limit, the dynamics of the particle is given by a quantum dynamical semigroup $\{e^{-it\mathcal{L}}\}_{t\geq 0}$ generated by a Lindbladian $\mathcal{L}$
- On $\mathcal{J}_1(\mathcal{H})$ (space of trace-class operators), $\mathcal{L}$ is given by

$$\mathcal{L}(\rho) = H\rho - \rho H^* + i\sum_{j\in\mathbb{N}} W_j \rho W_j^*, \quad H = H_V - \frac{i}{2}\sum_{j\in\mathbb{N}} W_j^* W_j,$$

  where, for all $j \in \mathbb{N}$, $W_j \in \mathcal{L}(\mathcal{H})$, and $\sum_{j\in\mathbb{N}} W_j^* W_j \in \mathcal{L}(\mathcal{H})$
- H is a dissipative operator on $\mathcal{H}$
- On a suitable domain, $\mathcal{L}$ is the generator of a quantum dynamical semigroup $\{e^{-it\mathcal{L}}\}_{t\geq 0}$ (strongly continuous semigroup on $\mathcal{J}_1(\mathcal{H})$ such that, for all $t \geq 0$, $e^{-it\mathcal{L}}$ preserves the trace and is a completely positive operator)
- Free dynamics : group of isometries $\{e^{-it\mathcal{L}_0}\}_{t\in\mathbb{R}}$,

$$\mathcal{L}_0(\rho) = H_0\rho - \rho H_0$$

Application : scattering for Lindblad master equations (II)

Modified wave operator ([Davies '80], [Alicki '81])

- $\Pi_{\mathrm{pp}}^{\perp} : \mathcal{H} \to \mathcal{H}$: orthogonal projection onto $(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H))^{\perp}$
- Modified wave operator :

$$\tilde{\Omega}_{+}(\mathcal{L}_0, \mathcal{L}) := \underset{t \to +\infty}{\text{s-lim}} \; e^{it\mathcal{L}_0} \left(\Pi_{\mathrm{pp}}^{\perp} e^{-it\mathcal{L}} (\cdot) \Pi_{\mathrm{pp}}^{\perp} \right)$$

Theorem

Suppose that Hypotheses (H1)–(H5) hold and that H has no spectral singularities in $[0, \infty)$. Then $\tilde{\Omega}_{+}(\mathcal{L}_0, \mathcal{L})$ exists on $\mathcal{J}_1(\mathcal{H})$

Interpretation

For all $\rho \in \mathcal{J}_1(\mathcal{H})$ with $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$, the number $\mathrm{tr}(\tilde{\Omega}_{+}(\mathcal{L}_0, \mathcal{L})\rho) \in [0, 1]$ is interpreted as the probability that the particle, initially in the state ρ, eventually escapes from the target

Ideas of the proofs

Step 1 : the range of the wave operator and the inverse semigroup

Lemma

Suppose that Hypotheses (H1), (H3), (H4) hold. Then

$$\mathrm{Ran}(W_-(H,H_0)) = S(H) \cap \mathcal{H}_{\mathrm{b}}(H)^\perp$$

where

$$S(H) = \Big\{ u \in \mathcal{H},\ \sup_{t\geq 0} \big\| e^{itH} u \big\| < \infty \Big\}$$

Elements of the proof

- [Davies '80] : $\mathcal{H}_{\mathrm{b}}(H)^\perp = \mathcal{H}_{\mathrm{ac}}(H)$, where $\mathcal{H}_{\mathrm{ac}}(H)$ is the closure of

$$M(H) = \Big\{ u \in \mathcal{H}, \exists \mathrm{c}_u > 0, \forall v \in \mathcal{H}, \int_0^\infty \big| \langle e^{-itH} u, v \rangle \big|^2 dt \leq \mathrm{c}_u \|v\|^2 \Big\}$$

- Easy to verify that $\mathrm{Ran}(W_-(H,H_0)) \subset S(H) \cap \mathcal{H}_{\mathrm{b}}(H)^\perp$
- Converse inclusion : uses that for all $u \in S(H)$ and $v \in \mathcal{H}_{\mathrm{ac}}(H)$,

$$\big\langle v, e^{itH} u \big\rangle \to 0, \quad t \to \infty$$

Step 2 : spectral projections

Spectral projections for non-self-adjoint operators

- Let $I \subset [0, \infty)$ be a closed interval and

$$E_H(I) = \underset{\varepsilon \downarrow 0}{\text{w-lim}} \frac{1}{2i\pi} \int_I \left((H - (\lambda + i\varepsilon))^{-1} - (H - (\lambda - i\varepsilon))^{-1} \right) d\lambda$$

  Then $E_H(I)$ is a well-defined projection if H does not have spectral singularities in I, $E_H(I)E_H(J) = E_H(I \cap J)$

- Considered in [Dunford '52, '58], [J. Schwartz '60]
- Studied in relation with stationary scattering theory : [Mochizuki '67,'68], [Goldstein '70,'71], [Huige '71]

Lemma

Suppose that Hypotheses (H1), (H3) and (H4) hold. Let $I \subset [0, \infty)$ be a closed interval containing no spectral singularities of H. Then

$$\mathrm{Ran}(E_H(I)) \subset \mathrm{Ran}(W_-(H, H_0))$$

Element of the proof

Use that $\mathrm{Ran}(W_-(H, H_0)) = S(H) \cap \mathcal{H}_{\mathrm{b}}(H)^{\perp}$

Step 3 : proof that $\mathcal{H}_{\mathrm{d}}(H) = \mathcal{H}_{\mathrm{p}}(H)$ (I)

Use of spectral projections

- $\mathrm{Ran}(E_{H^*}(I)) \subset \mathrm{Ran}(W_+(H^*, H_0))$
- Taking orthogonal complements
$$\mathrm{Ran}(W_+(H^*, H_0))^{\perp} \subset \bigcap_{I \subset [0,\infty)} \mathrm{Ran}(E_{H^*}(I))^{\perp},$$
i.e.
$$\mathrm{Ker}(W_+(H_0, H)) \subset \bigcap_{I \subset [0,\infty)} \mathrm{Ker}(E_H(I)),$$
where the intersection runs over all closed intervals $I \subset [0, \infty)$ with the property that I does not contain any spectral singularities of H
- Since
$$\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H) = \mathrm{Ker}(W_+(H_0, H)),$$
it suffices to prove that
$$\mathcal{K} := \bigcap_{I \subset [0,\infty)} \mathrm{Ker}(E_H(I)) \subset \mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H)$$

Step 3 : proof that $\mathcal{H}_{\rm d}(H) = \mathcal{H}_{\rm p}(H)$ (II)

Spectral mapping theorem and Riesz projections

- Let $R = (H-i)^{-1}$. Then

$$\mathrm{Id} = \Pi_{\rm pp}(R) + \frac{1}{2i\pi}\oint_{\Gamma_\varepsilon}(\mu - R)^{-1}d\mu, \qquad (**)$$

  where $\Pi_{\rm pp}(R) =$ sum of Riesz projections corresponding to isolated eigenvalues of R, and $\Gamma_\varepsilon = \Gamma_{1,\varepsilon}\cup\Gamma_{2,\varepsilon}\cup\Gamma_{3,\varepsilon}\cup\Gamma_{4,\varepsilon}$
- $\mathrm{Ran}(\Pi_{\rm pp}(R)) = \mathcal{H}_{\rm b}(H)\oplus\mathcal{H}_{\rm p}(H)$
- If H does not have spectral singularities, taking the weak limit $\varepsilon\downarrow 0$ gives

$$\mathrm{Id} = \Pi_{\rm pp}(R) + \operatorname*{w-lim}_{\varepsilon\downarrow 0}\frac{1}{2i\pi}\int_0^\infty\{(H-(\lambda+i\varepsilon))^{-1} - (H-(\lambda-i\varepsilon))^{-1}\}d\lambda$$

  and therefore

$$\mathcal{K} = \bigcap_{I\subset[0,\infty)}\mathrm{Ker}(E_H(I)) = \mathrm{Ker}(E_H([0,\infty))) \subset \mathcal{H}_{\rm b}(H)\oplus\mathcal{H}_{\rm p}(H)$$

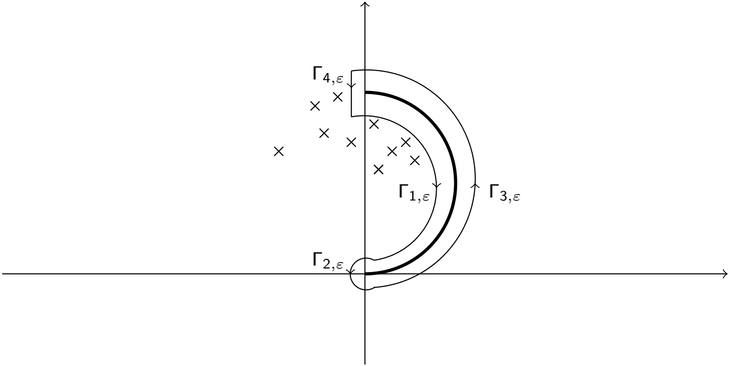

FIGURE: **The spectrum of $R = (H - i)^{-1}$ and the contour Γ_ε.**

Step 3 : proof that $\mathcal{H}_{\mathrm{d}}(H) = \mathcal{H}_{\mathrm{p}}(H)$ (III)

Regularizing the spectral singularities

- If $\{\lambda_1, \dots \lambda_n\} \subset [0, \infty)$ are the spectral singularities of H, let $\mu_j = (\lambda_j - i)^{-1}$, $j = 1, \dots, n$, be the corresponding "spectral singularities" of $R = (H - i)^{-1}$
- Composing $(**)$ by $R^4 \prod_{j=1}^n (R - \mu_j)^{\nu_j}$ gives

$$R^4 \prod_{j=1}^n (R - \mu_j)^{\nu_j} = R^4 \prod_{j=1}^n (R - \mu_j)^{\nu_j} \Pi_{\mathrm{pp}}(R) - \frac{1}{2i\pi} \oint_{\Gamma_\varepsilon} \mu^4 \prod_{j=1}^n (\mu - \mu_j)^{\nu_j} (R - \mu)^{-1} d\mu$$

- Taking the weak limit $\varepsilon \downarrow 0$ gives the modified spectral decomposition formula

$$\prod_{j=1}^n (R - \mu_j)^{\nu_j} = \prod_{j=1}^n (R - \mu_j)^{\nu_j} \Pi_{\mathrm{pp}}(R) + \operatorname*{w-lim}_{\varepsilon \downarrow 0} \frac{1}{2i\pi} \int_0^\infty \prod_{j=1}^n \left((\lambda - i)^{-1} - \mu_j\right)^{\nu_j} \left\{ \big(H - (\lambda + i\varepsilon)\big)^{-1} - \big(H - (\lambda - i\varepsilon)\big)^{-1} \right\} d\lambda$$

- Using in particular Lebesgue's dominated convergence theorem

$$\mathcal{K} = \bigcap_{I \subset [0,\infty)} \mathrm{Ker}(E_H(I)) \subset \mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H)$$

Step 4 : asymptotic completeness

Parseval's theorem

For all $\varepsilon > 0$ and $u \in (\mathcal{H}_{\rm b}(H^*) \oplus \mathcal{H}_{\rm p}(H^*))^\perp$,

$$\int_0^\infty e^{-s\varepsilon} \left\| C e^{isH} u \right\|^2 ds = \frac{1}{2\pi} \int_{\mathbb{R}} \left\| C(H-(\lambda - i\varepsilon))^{-1} u \right\|^2 d\lambda \qquad (***)$$

Asymptotic completeness

- Recall $\mathrm{Ran}(W_-(H,H_0)) = S(H) \cap \mathcal{H}_{\rm b}(H)^\perp$
- Observe that

$$S(H) = \left\{ u \in \mathcal{H}, \int_0^\infty \left\| C e^{isH} u \right\|^2 ds < \infty \right\}$$

- If H does not have spectral singularities, uniform bound in $\varepsilon > 0$ in the right side of $(***)$ implies that $u \in S(H)$ and hence to $\mathrm{Ran}(W_-(H,H_0))$
- If H does have a spectral singularity, one constructs a vector

$$u \in (\mathcal{H}_{\rm b}(H^*) \oplus \mathcal{H}_{\rm p}(H^*))^\perp$$

such that the limit as $\varepsilon \to 0$ in the right side of $(***)$ is infinite and therefore $u \notin \mathrm{Ran}(W_-(H,H_0))$

Thank you !