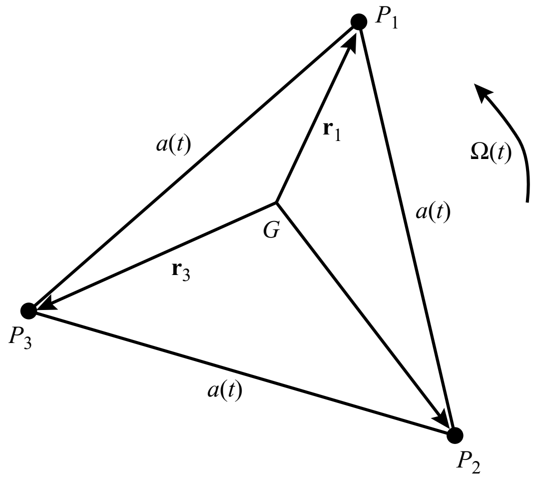

Figure 2.15 Lagrange equilateral configuration for a three-body problem with $P_1P_2 = P_2P_3 = P_3P_1 = a(t)$.

If $|\mathbf{r}_i| = r_i$, deduce the polar equations

$$\ddot{r}_i - r_i\Omega^2 = -\frac{\mu_1 + \mu_2 + \mu_3}{a^3} r_i, \quad r_i^2\Omega = \text{constant} \qquad (i = 1, 2, 3).$$

Explain why a satisfies

$$\ddot{a} - a\Omega^2 = -\frac{\mu_1 + \mu_2 + \mu_3}{a^2}, \; a^2\Omega = \text{constant} = K,$$

say, and that solutions of these equations completely determine the position vectors. Express the equation in non-dimensionless form by the substitutions $a = K^2/(\mu_1 + \mu_2 + \mu_3)$, $t = K^3\tau/(\mu_1+\mu_2+\mu_3)^2$, sketch the phase diagram for the equation in μ obtained by eliminating Ω, and discuss possible motions of this Lagrange configuration.

2.22 A disc of radius a is freely pivoted at its centre A so that it can turn in a vertical plane. A spring, of natural length $2a$ and stiffness λ connects a point B on the circumference of the disc to a fixed point O, distance $2a$ above A. Show that θ satisfies

$$I\ddot{\theta} = -Ta\sin\phi, \qquad T = \lambda a[(5 - 4\cos\theta)^{1/2} - 2],$$

where T is the tension in the spring, I is the moment of inertia of the disc about A, $\widehat{OAB} = \theta$ and $\widehat{ABO} = \phi$. Find the equilibrium states of the disc and their stability.

2.23 A man rows a boat across a river of width a occupying the strip $0 \le x \le a$ in the x, y plane, always rowing towards a fixed point on one bank, say (0, 0). He rows at a constant speed u relative to the water, and the river flows at a constant speed v. Show that

$$\dot{x} = -ux/\sqrt{(x^2 + y^2)}, \quad \dot{y} = v - uy/\sqrt{(x^2 + y^2)},$$

where (x, y) are the coordinates of the boat. Show that the phase paths are given by $y + \sqrt{(x^2 + y^2)} = Ax^{1-\alpha}$, where $\alpha = v/u$. Sketch the phase diagram for $\alpha < 1$ and interpret it. What kind of point is the origin? What happens to the boat if $\alpha > 1$?

2.24 In a simple model of a national economy, $\dot{I} = I - \alpha C$, $\dot{C} = \beta(I - C - G)$, where I is the national income, C is the rate of consumer spending, and G the rate of government expenditure; the constants α and β satisfy $1 < \alpha < \infty$, $1 \le \beta < \infty$. Show that if the rate of government expenditure G is constant G_0 there is an equilibrium state. Classify the equilibrium state and show that the economy oscillates when $\beta = 1$.

Consider the situation when government expenditure is related to the national income by the rule $G = G_0 + kI$, where $k > 0$. Show that there is no equilibrium state if $k \geq (\alpha - 1)/\alpha$. How does the economy then behave?

Discuss an economy in which $G = G_0 + kI^2$, and show that there are two equilibrium states if $G_0 < (\alpha - 1)^2/(4k\alpha^2)$.

2.25 Let $f(x)$ and $g(y)$ have local minima at $x = a$ and $y = b$ respectively. Show that $f(x) + g(y)$ has a minimum at (a, b). Deduce that there exists a neighbourhood of (a, b) in which all solutions of the family of equations

$$f(x) + g(y) = \text{constant}$$

represent closed curves surrounding (a, b).

Show that (0, 0) is a centre for the system $\dot{x} = y^5, \dot{y} = -x^3$, and that all paths are closed curves.

2.26 For the predator–prey problem in Section 2.2, show by using Problem 2.25 that all solutions in $y > 0$, $x > 0$ are periodic.

2.27 Show that the phase paths of the Hamiltonian system $\dot{x} = -\partial H/\partial y$, $\dot{y} = \partial H/\partial x$ are given by $H(x, y) = constant$. Equilibrium points occur at the stationary points of $H(x, y)$. If (x_0, y_0) is an equilibrium point, show that (x_0, y_0) is stable according to the linear approximation if $H(x, y)$ has a maximum or a minimum at the point. (Assume that all the second derivatives of H are nonzero at x_0, y_0.)

2.28 The equilibrium points of the nonlinear parameter-dependent system $\dot{x} = y$, $\dot{y} = f(x, y, \lambda)$ lie on the curve $f(x, 0, \lambda) = 0$ in the x, λ plane. Show that an equilibrium point (x_1, λ_1) is stable and that all neighbouring solutions tend to this point (according to the linear approximation) if $f_x(x_1, 0, \lambda_1) < 0$ and $f_y(x_1, 0, \lambda_1) < 0$.

Investigate the stability of $\dot{x} = y$, $\dot{y} = -y + x^2 - \lambda x$.

2.29 Find the equations for the phase paths for the general epidemic described (Section 2.2) by the system

$$\dot{x} = -\beta xy, \qquad \dot{y} = \beta xy - \gamma y, \qquad \dot{z} = \gamma y.$$

Sketch the phase diagram in the x, y plane. Confirm that the number of infectives reaches its maximum when $x = \gamma/\beta$.

2.30 Two species x and y are competing for a common food supply. Their growth equations are

$$\dot{x} = x(1 - x - y), \quad \dot{y} = y(3 - x - \tfrac{3}{2}y), \qquad (x, y > 0).$$

Classify the equilibrium points using linear approximations. Draw a sketch indicating the slopes of the phase paths in $x \geq 0$, $y \geq 0$. If $x = x_0 > 0$, $y = y_0 > 0$ initially, what do you expect the long term outcome of the species to be? Confirm your conclusions numerically by computing phase paths.

2.31 Sketch the phase diagram for the competing species x and y for which

$$\dot{x} = (1 - x^2 - y^2)x, \qquad \dot{y} = (\tfrac{5}{4} - x - y)y.$$

2.32 A space satellite is in free flight on the line joining, and between, a planet (mass m_1) and its moon (mass m_2), which are at a fixed distance a apart. Show that

$$-\frac{\gamma m_1}{x^2} + \frac{\gamma m_2}{(a - x)^2} = \ddot{x},$$

where x is the distance of the satellite from the planet and γ is the gravitational constant. Show that the equilibrium point is unstable according to the linear approximation.

2.33 The system

$$\dot{V}_1 = -\sigma V_1 + f(E - V_2), \quad \dot{V}_2 = -\sigma V_2 + f(E - V_1), \qquad \sigma > 0, E > 0$$