Lecture 3.
Continuous systems

Literature

- H. Goldstein, "Classical Mechanics" (Chapter 12.1 - 12.3)
- J.D. Logan, "Applied Mathematics" (Sec. 3.4)
- I.M. Gelfand, S.V. Fomin, "Calculus of Variations" (Sec. 35, 38.1)

Lecture 3a. A many-body problem

Assume particles of mass m on an elastic (massless) rod that can undergo longitudinal displacements and let η_i be the displacement of particle i from its position. Then the kinetic energy of the system is

$$T = \frac{1}{2} \sum_i m \dot{\eta}_i^2.$$

The potential energy is the sum of the potential energies of each elastic part (spring) connecting neighbouring particles

$$V = \frac{1}{2} \sum_i k (\eta_{i+1} - \eta_i)^2.$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2} \sum_i \left[m \dot{\eta}_i^2 - k (\eta_{i+1} - \eta_i)^2 \right]$$

Many close packed particles

The Lagrangian is also written as

$$L = \frac{1}{2} \sum_i \alpha \left[\frac{m}{\alpha} \dot{\eta}_i^2 - k\alpha \left(\frac{\eta_{i+1} - \eta_i}{\alpha} \right)^2 \right]$$

where α is the separation between particles.
Define

- $\mu = m/\alpha$ the mass density,
- $Y = k\alpha$ Young's modulus,
- $\xi_i = (\eta_{i+1} - \eta_i)/\alpha$ the extension per unit length.

In the case that the particles are very close to each other $\alpha \ll 1$ all above quantities are expected to take non-infinite values (for $\alpha \to 0$).

Continuum approximation for the Lagrangian

In the limit $\alpha \to 0$ the discrete position index i becomes a continuous variable x, and $\eta_i \to \eta(x)$. Furthermore,

$$\frac{\eta_{i+1} - \eta_i}{\alpha} \to \frac{\eta(x+\alpha) - \eta(x)}{\alpha} \to \frac{d\eta}{dx}, \qquad \text{for } \alpha \to 0$$

The Lagrangian, that is a sum over the particles, becomes an integral over the variable x with $\alpha = dx$ being the differential.

Lagrangian for a continuous elastic rod

$$L = \frac{1}{2} \int \left[\mu \dot{\eta}^2 - Y \left(\frac{\partial \eta}{\partial x} \right)^2 \right] dx.$$

Continuum approximation for the equation of motion

For the particles we have the Euler-Lagrange equations

$$m\ddot{\eta}_i - k(\eta_{i-1} - 2\eta_i + \eta_{i+1}) = 0.$$

This is also written as

$$\frac{m}{\alpha}\ddot{\eta}_i - k\alpha\frac{\eta_{i-1} - 2\eta_i + \eta_{i+1}}{\alpha^2} = 0.$$

The wave equation

In the limit $\alpha \to 0$ we have

$$\mu\frac{d^2\eta}{dt^2} - Y\frac{d^2\eta}{dx^2} = 0.$$

Equation of motion. Method of small parameter.

Start from the Euler-Lagrange equations for the discrete particles

$$m\ddot{\eta}_i - k(\eta_{i-1} - 2\eta_i + \eta_{i+1}) = 0.$$

Procedure.

- Assume a small parameter ϵ and define the position variable as $x_i = i\,\epsilon$, such that this becomes continuous for $\epsilon \to 0$ ($x_i \to x$).
- Define rescaled parameters

$$m = \epsilon M, \qquad k = \frac{K}{\epsilon^2}.$$

- Assume a continuous field $\eta(x)$ with $\eta_i = \eta(x_i)$. We have the approximations

$$\eta(x_{i\pm1}) \approx \eta(x_i) \pm \epsilon\eta'(x_i) + \frac{\epsilon^2}{2}\eta''(x_i) + O(\epsilon^3).$$

Thus

$$\eta_{i-1} - 2\eta_i + \eta_{i+1} = \epsilon^2\eta'' + O(\epsilon^3).$$

Equation of motion

Substitute the Taylor approximations and the rescaled parameters in the equation,

$$\epsilon \left(M\ddot{\eta} - K\eta'' \right) = 0.$$

The wave equation

In order that the approximation is consistent to order $O(\epsilon)$, the system should satisfy

$$M\frac{d^2\eta}{dt^2} - K\frac{d^2\eta}{dx^2} = 0.$$

(Note. M and μ are identical as are K and Y.)

In the example with the rod, we notice that

- The unknown is a function of two variables $\eta = \eta(x,t)$.
- The Lagrangian in an integral over a Lagrangian density

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t\right).$$

- Hamilton's principle must be formulated over an action given by a double integral

$$I = \int_1^2 \int \mathcal{L}\, dx\, dt.$$

Functionals of several variables

Let a functional J that depends on a function $u = u(x,y)$

$$J[u] = \iint_R F(x,y,u,u_x,u_y)\,dx\,dy.$$

Assume $u \in A$ and the boundary condition on the boundary C of A

$$u(x,y) = f(x,y).$$

The action for a continuous system is a multiple integral, as in the above form.

Extrema for functionals of several variables

For an increment $\epsilon h(x,y)$ in u we have the necessary condition for an extremum

$$\frac{d}{d\epsilon} J[u+\epsilon h]_{\epsilon=0} = 0.$$

Using Green's theorem we obtain

$$\iint_R \left[F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} \right] h\, dxdy + \int_C (-hF_{u_y} dx + hF_{u_x} dy) = 0.$$

The integral on C is zero because $h=0$ on the boundary.
Using an extension of the fundamental lemma of the calculus of variations we have the following.

Euler-Lagrange equations for a functional of two variables

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0.$$

Example

Find Euler's equation for the functional

$$J[u] = \iint \left[\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \rho(x,y)u\right] dx\, dy = 0.$$

Solution. This is

$$u_{xx} + u_{yy} = \rho(x,y).$$

It is called the Poisson equation.

Action and Euler-Lagrange equations for a continuous system

For a Lagrangian density $\mathcal{L}$, defined for a function $\eta(x,t)$, the action is

$$I = \iint \mathcal{L}\, dx\, dt.$$

Vanishing of the first variation gives the following.

Euler-Lagrange equation for a continuous system

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \eta)}\right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0.$$

This gives a partial differential equation for $\eta = \eta(x,t)$.

Example

We have seen the Lagrangian for an elastic rod. Derive the Euler-Lagrange equation.

Conservation laws

We consider a field of many variables x_i and the time, $u(x_i, t)$, and a Lagrangian density $\mathcal{L}(u, \dot{u}, u_i)$, where $u_i \equiv \partial u/\partial x_i$.

We may derive a conserved quantity as follows (we sum over ν).

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial u}\dot{u} + \frac{\partial \mathcal{L}}{\partial \dot{u}}\ddot{u} + \sum_\nu \frac{\partial \mathcal{L}}{\partial u_\nu}\dot{u}_\nu = \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}} + \frac{d}{dx_\nu}\frac{\partial \mathcal{L}}{\partial u_\nu}\right)\dot{u} + \frac{\partial \mathcal{L}}{\partial u_\nu}\dot{u}_\nu + \frac{\partial \mathcal{L}}{\partial \dot{u}}\ddot{u}$$

where the Euler-Lagrange equation was used in the last step. We have

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\dot{u}\right) + \frac{d}{dx_\nu}\left(\frac{\partial \mathcal{L}}{\partial u_\nu}\dot{u}\right)$$

or

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\dot{u} - \mathcal{L}\right) + \frac{d}{dx_\nu}\left(\frac{\partial \mathcal{L}}{\partial u_\nu}\dot{u}\right) = 0.$$

Energy

We integrate over all space and we may apply the divergence theorem in the second term. For fields that vanish at spatial infinity we obtain that the energy is conserved

$$\frac{dE}{dt} = 0, \qquad E = \int \left(\frac{\partial \mathcal{L}}{\partial \dot{u}} \dot{u} - \mathcal{L} \right) dx.$$

Example

Energy for the eave equation The Lagrangian density for the eave equation is

$$\mathcal{L} = \frac{1}{2} \left(\mu \dot{u}^2 - Y u_x^2 \right), \quad u_x = \frac{\partial u}{\partial x}.$$

The corresponding energy is

$$E = \frac{1}{2} \int \left(\mu \dot{u}^2 + Y u_x^2 \right) dx$$