Lecture 2.
The canonical form of Euler's equations

- I.M. Gelfand, S.V. Fomin, "Calculus of Variations" (Sec. 16, 17)
- J.D. Logan, "Applied Mathematics" (Sec. 3.5)
- D.W. Jordan, and P. Smith, "Nonlinear Ordinary Differential Equation" (Sec. 8.3, 10.1, 10.2)
- Σ. Κομηνέας και Ε. Χαρμανδάρης, "Μαθηματική Μοντελοποίηση", Κάλλιπος, 2015 (Κεφ. 2.2.8)
- [G.R. Fowles, G. L. Cassidy, "Analytical Mechanics" (Sec. 10.9)]
- [H. Goldstein, "Classical Mechanics" (Section 8.1)]

Lecture 2a. Canonical Momentum

Assume a system with variables y_i, y_i' and a functional

$$J[y_1, \ldots, y_n] = \int_{\alpha}^{b} F(x, y_i, y_i')\, dx.$$

We define the canonical momentum corresponding to y_i as

$$p_i = \frac{\partial F}{\partial y_i'} \qquad (1)$$

Then, the Euler-Lagrange equations read

$$\frac{dp_i}{dx} = \frac{\partial F}{\partial y_i} \quad \text{(system of 2nd order equations).}$$

Energy

The total differential of F is

$$\frac{dF}{dx} = \sum_i \left(\frac{\partial F}{\partial y_i} y_i' + \frac{\partial F}{\partial y_i'} \frac{dy_i'}{dx} \right) + \frac{\partial F}{\partial x}$$

Using the Euler-Lagrange equations, we obtain

$$\frac{d}{dx} \left(\sum_i \frac{\partial F}{\partial y_i'} y_i' - F \right) + \frac{\partial F}{\partial x} = 0.$$

For $\partial F/\partial x = 0$ we have the conserved quantity (Energy)

$$E(y_i, y_i') = \sum_i y_i' \frac{\partial F}{\partial y_i'} - F(y_i, y_i').$$

Example

Let the Lagrangian of a particle moving in one-dimension in a potential $V(x)$

$$L = \frac{1}{2} m\dot{x}^2 - V(x).$$

The energy is

$$E = \dot{x}\frac{dL}{d\dot{x}} - L = \ldots = \frac{1}{2} m\dot{x}^2 + V(x).$$

Example

Let the Lagrangian of a particle moving on the plane, described by polar coordinates (r, θ),

$$L = \frac{1}{2}\, m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta).$$

The energy is

$$E = \frac{1}{2}\, m\, \left(\dot{r}^2 + r^2\, \dot{\theta}^2\right) + V(r, \theta).$$

Hamiltonian

Using Eq. (1) for the definition of the canonical momenta we can write the y_i' as functions of the p_i' (and x, y_i),

assuming that the Jacobian of transformation is nonzero.

Eliminate the y_i' using the p_i in the energy function and write this function (called the Hamiltonian) as

$$H(x, y_i, p_i) = \sum_i y_i' p_i - F(x, y_i, y_i')$$

where the y_i' are regarded as function of y_i, p_i.
Take the differential of H

$$dH = \sum_i p_i dy_i' + \sum_i y_i' dp_i - dF = \ldots = -\frac{\partial F}{\partial x} dx - \sum_i \frac{\partial F}{\partial y_i} dy_i + \sum_i y_i' \, dp_i.$$

This gives the relations

$$\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x}, \qquad \frac{\partial H}{\partial y_i} = -\frac{\partial F}{\partial y_i}, \qquad \frac{\partial H}{\partial p_i} = y_i'.$$

Hamilton's equations

We recall the Euler-Lagrange equations and use them in the last relations.

We obtain the system of $2n$ equations of 1st order

$$\frac{dy_i}{dx} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dx} = -\frac{\partial H}{\partial y_i}.$$

This is also called the canonical system of the Euler equations or Hamilton's equations.

Suppose that H does not depend explicitly on x. Then, using Hamilton's equations we find

$$\frac{dH}{dx} = \sum_i \left(\frac{\partial H}{\partial y_i}\frac{dy_i}{dx} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dx} \right) = \ldots = 0.$$

If F does not depend on x explicitly, the function $H(y_i, p_i)$ is a first integral of Hamilton's equations.

Conservation of momentum

Hamilton's equations show that if H does not depend on some coordinate y_k, then the corresponding momentum p_k remains constant

$$\frac{dp_k}{dx} = 0 \Rightarrow p_k : \text{const.}$$

Exercise

What happens if H depends explicitly on x? What is the derivative of H with x?

Example (Harmonic Oscillator)

Lagrangian and Hamiltonian

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \qquad H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Example (Conserved momentum)

Consider a particle moving on the plane in potential $V(y)$. The Hamiltonian is

$$H(x,y,p_x,p_y) = \frac{p_x^2 + p_y^2}{2m} + V(y).$$

The momentum p_x is conserved

$$\frac{dp_x}{dt} = 0.$$

Example (Lagrangian containing total derivatives)

Assume a particle on the line of on the plane with position $(x(t), y(t))$. Try to find Euler's equations for a hierarchy of Lagrangians.

$$L(x, \dot{x}) = \dot{x}^2 - x^2.$$

$$L(x, \dot{x}) = \dot{x}, \quad L(x, \dot{x}) = \dot{x}^2.$$

$$L(x, y, \dot{x}, \dot{y}) = y\dot{x}.$$

Lecture 2b. Dynamics for a complex variable

We consider a complex variable Ψ

Its dynamical equation is given by

$$i\dot{\Psi} = -\omega_0 \Psi, \qquad \omega_0 : \text{constant}.$$

This linear equation has solutions $\Psi(t) = \Psi_0 e^{i\omega_0 t}$. The Lagrangian that gives the above model is

$$L = i\dot{\Psi}^* \Psi - \omega_0 \Psi \Psi^*$$

where Ψ^* is the complex conjugate of Ψ.

Exercise

Show that the same model is given by the real-valued Lagrangian

$$L = \frac{i}{2}(\dot{\Psi}^* \Psi - \dot{\Psi} \Psi^*) - \omega_0 \Psi \Psi^*$$

Canonical momenta

Let us consider Ψ^* as the variable of the problem. The canonical momentum is

$$p_{\Psi^*} = \frac{\partial L}{\partial \dot{\Psi}^*} = i\Psi.$$

That is, Ψ^* and $i\Psi$ are canonically conjugate.

Hamiltonian

Considering the Hamiltonian $H = \omega_0 \Psi \Psi^*$, we have the equations of motion

$$i\dot{\Psi}^* = \frac{\partial H}{\partial p_\Psi^*} \Rightarrow i\Psi^* = \omega_0 \Psi^*.$$

Models with damping

Newton's law for a harmonic oscillator

$$m\ddot{x} = -kx.$$

In a Hamiltonian form, this is

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \quad p = m\dot{x}, \qquad H = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

Let us add an extra term on the rhs (a force)

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} - \frac{\lambda}{m}p \quad \text{or} \quad \ddot{x} = -kx - \lambda\dot{x}$$

We show that the energy is not constant under the modified equation

$$\frac{dH}{dt} = \ldots = -\lambda\dot{x}^2 < 0.$$

A damped system

Let us consider a Hamiltonian $H(x,p)$ and the system of equations

$$\dot{x} = -\frac{\partial H}{\partial x}, \qquad \dot{p} = -\frac{\partial H}{\partial p}.$$

Decreasing energy

We can show that the Hamiltonian function of the system along a solution path is a decreasing function of time

$$\frac{dH}{dt} = \ldots = -\left(\frac{\partial H}{\partial x}\right)^2 - \left(\frac{\partial H}{\partial p}\right)^2 < 0.$$

A Hamiltonian system including damping

Let us consider a Hamiltonian $H(x,p)$ and the system of equations

$$\dot{x} = \frac{\partial H}{\partial p} - \frac{\partial H}{\partial x}, \qquad \dot{p} = -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial p}.$$

Exercise (Decreasing energy)

Show that the energy of the system along a solution path is a decreasing function of time

$$\frac{dH}{dt} < 0.$$

Non-conservative models

In general, we could add any type of term on the right-hand-side of Hamilton's equations, modifying the conservative system

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x} - f(x,p).$$

Damping in a system with a complex field

Let us consider a Hamiltonian $H(\Psi, \Psi^*)$ and the system of equations

$$\dot{\Psi} = -\frac{\partial H}{\partial \Psi^*}, \qquad \dot{\Psi}^* = -\frac{\partial H}{\partial \Psi}.$$

They are equivalent if $H \in \mathbb{R}$.

Decreasing energy

We can show that the energy of the system along a solution path is a decreasing function of time

$$\frac{dH}{dt} = \ldots = -2\frac{\partial H}{\partial \Psi}\frac{\partial H}{\partial \Psi^*} < 0.$$

Note that $\frac{dH}{dt} \in \mathbb{R}$.

Exercise

Write explicitly the system of equations shown in the beginning of this page.

Stability of paths

Definition

A function $V(\boldsymbol{x})$ is called positive definite in a neighbourhood $\mathcal{N}$ of $\boldsymbol{x} = \mathbf{0}$ if $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq 0$ in $\mathcal{N}$ and $V(\mathbf{0}) = 0$

An example is given by the Hamiltonian for a harmonic oscillator.

Definition (asymptotic stability)

Let $\boldsymbol{x}^*$ be a solution of the system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$. We say that $\boldsymbol{x}^*$ is asymptotically stable if there exists δ and a time t_0 such that

$$||\boldsymbol{x}(t_0) - \boldsymbol{x}^*|| < \delta \Rightarrow \lim_{t\to\infty} ||\boldsymbol{x}(t) - \boldsymbol{x}^*|| \to 0.$$

The definition can be generalised for a non-constant solution $\boldsymbol{x}^*(t)$.

Lianunov functions

Theorem

Let $\boldsymbol{x}^*(t) = \mathbf{0}$ be a solution of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$$

where $\boldsymbol{F}(\mathbf{0}) = \mathbf{0}$. Then, $\boldsymbol{x}^*$ is *asymptotically stable* if there exist $V(\boldsymbol{x})$ with the following properties in some neighbourhood of $\boldsymbol{x} = \mathbf{0}$.
(i) $V(\boldsymbol{x})$ and its partial derivatives are continuous, (ii) $V(\boldsymbol{x})$ is positive definite, (iii) $\dot{V}(\boldsymbol{x})$ is negative definite.

Such a function V is called a *Liapunov function* for the system.

Example

Find a Liapunov function for the system

$$\dot{x} = -x - 2y^2, \qquad \dot{y} = xy - y^3.$$

Solution. Try the function $V(x,y) = x^2 + \alpha y^2$. Then

$$\dot{V} = -2x^2 + 2(\alpha - 2)xy^2 - 2\alpha y^4.$$

If we choose $\alpha = 2$, then $V(x,y)$ has the properties of a Lianunov function.

This shows that the solution of the system $(0,0)$ is asymptotically stable. That is, every trajectory starting in the neighbourhood of this solution will approach it as $t \to \infty$.

Exercise

Give an example of a Lianunov function and a corresponding dynamical system based on the Hamiltonians that we saw earlier.