Lecture 1.
Calculus of variations

Literature

- I.M. Gelfand, S.V. Fomin, "Calculus of Variations" (Sec. 1-5, 7)
- J.D. Logan, "Applied Mathematics" (Sec. 3.1 - 3.5)
- H. Goldstein, "Classical Mechanics" (Chapter 2)

Functionals

A functional assigns a number to each function in a class. Thus, it is a function where the independent variable is itself a function.

Examples

1. Associate its length to each rectifiable curve.
2. For a continuously differentiable function $y(x)$ define the number

$$J[y] = \int_{\alpha}^{b} y'^2(x)\, dx.$$

3. More generally, for $y(x)$ and a function $F(x, y, y')$, define

$$J[y] = \int_{\alpha}^{b} F(x, y, y')\, dx.$$

Problems with functionals

Examples (Shortest curve)

1. Find the shortest plane curve joining two points A and B, i.e., find the curve $y = y(x)$ for which the following functional achieves its minimum,

$$\int_{\alpha}^{b} \sqrt{1 + y'^2}\, dx.$$

Examples (Brachistochrone)

Find the minimum time it takes for a particle to move from point A to B under the influence of gravity. The time it takes for the particle to move depends on the path it follows and it is, therefore, a functional.

The above problems involve functionals of the form

$$\int_{\alpha}^{b} F(x, y, y')\, dx.$$

Classical Calculus and functionals

Assume an interval $[\alpha, b]$ and divide it in $n+1$ equal parts

$$x_0 = \alpha,\ x_1, \ldots,\ x_n,\ x_{n+1} = b, \qquad h = x_{i+1} - x_i$$

A curve $y = y(x)$ from point $y(x_0) = y_\alpha$ to $y(x_{n+1}) = y_b$ is represented by the points (x_i, y_i) where $y_i = y(x_i)$. We consider a function on this curve of the form.

$$J(y_1, \ldots, y_n) = \sum_{i=1}^{n} F(x_i, y_i) h$$

If we allow $n \to \infty$, $h \to 0$ then the function J bocomes a functional

$$J[y] = \int_\alpha^b F(x, y)\, dx, \quad y(\alpha) = y_\alpha,\ y(b) = y_b.$$

Functionals may be regarded as functions of infinitely many variables

Function spaces

Spaces whose elements are functions are called function spaces

Functionals are defined on function spaces. We have to choose the functions that are useful for each particular problem.

Example

For a functional of the form

$$\int_{\alpha}^{b} F(x, y, y')\, dx$$

we consider functions $y(x)$ with a continuous first derivative.

Definition. Linear space

(1) $x + y = y + x$, (2) $(x + y) + z = x + (y + z)$
(3) $x + 0 = x$, (4) $x + (-x) = 0$ (5) $1 \cdot x = x$
(6) $\alpha(\beta x) = (\alpha\beta)x$, (7) $(\alpha + \beta)x = \alpha x + \beta x$
(8) $\alpha(x + y) = \alpha x + \alpha y$.

Normed function spaces

Normed space $\mathcal{R}$. For each element x in $\mathcal{R}$

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha| \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$.

Example

1. For a function $y(x)$ in the space of cont functions $\mathcal{C}(\alpha, b)$, we define

$$\|y\|_0 = \max_{\alpha \leq x \leq b} |y(x)|.$$

2. For $y(x), y'(x)$ continuous, we define

$$\|y\|_1 = \max_{\alpha \leq x \leq b} |y(x)| + \max_{\alpha \leq x \leq b} |y'(x)|.$$

Exercise (Use the norm in order to)

1. Define the distance of $y(x)$ from α standard function $\hat{y}(x)$.

2. Define continuity for functionals.

Definition (Variation, or differential, of a functional)

Let a functional $J[y]$ defined in some normed space and let

$$\Delta J[h] = J[y+h] - J[y]$$

be its increment corresponding to the increment $h(x)$ of the argument $y(x)$. Then, $\Delta J[h]$ is a functional of h. Suppose

$$\Delta J[h] = \phi[h] + \epsilon ||h||,$$

where $\phi[h]$ is a linear functional and $\epsilon \to 0$ as $||h|| \to 0$. Then $J[y]$ is *differentiable* and the *principal linear part* of $\Delta J[h]$, i.e., $\phi[h]$ is called the *variation* (or differential) of $J[y]$. We write

$$\delta J[h] = \phi[h].$$

Remark. A *linear functional* is defined similarly to a linear function. E.g., $\phi[\alpha h] = \alpha \phi[h]$.

Calculus for functionals

Definition (Extrema of functionals)

A functional $J[y]$ has a minimum at $y = \hat{y}$ if

$$\Delta J = J[y] - J[\hat{y}] > 0$$

in a neighbourhood of $\hat{y}(x)$.

A similar definition holds for a maximum.

Theorem

A necessary condition for a differentiable functional $J[y]$ to have an extremum at $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$,

$$\delta J[h] = 0$$

for $y = \hat{y}$ and all admissible h.

Exercise (* Prove the theorem.)

Lecture 1b.
The simplest variational problem

Let $F(x,y,z)$ a function with continuous derivatives (in all its arguments) and consider the functional

$$J[y] = \int_{\alpha}^{b} F(x,y,y')\,dx.$$

We want to find $y(x)$ for which $J[y]$ has an extremum.

We will search among $y(x)$ that are continuously differentiable for $x \in [\alpha, b]$ and satisfy the boundary conditions

$$y(\alpha) = y_{\alpha}, \quad y(b) = y_b.$$

The solution of the problem is a curve joining two points, $(\alpha, y_{\alpha}), (b, y_b)$.

Variation and extremum

The increment of $J[y]$ is

$$\Delta J = J[y+h] - J[y] = \int_\alpha^b [F(x, y+h, y'+h') - F(x, y, y')]\, dx$$

and by Taylor's formula

$$\Delta J = \int_\alpha^b [F_y(x, y, y')h + F_{y'}(x, y, y')h']\, dx + \dots$$

The rhs gives the principal linear part of ΔJ.

Theorem

A necessary condition for an extremum is that the variation vanish

$$\delta J = \int_\alpha^b [F_y(x, y, y')h + F_{y'}(x, y, y')h']\, dx = 0$$

for a definite $y(x)$ and all admissible functions $h(x)$.

Lemmata of the calculus of variations

Lemma (Fundamental lemma of the calculus of variations)

Let $\alpha(x)$ *continuous in* $[\alpha, b]$. *If*

$$\int_{\alpha}^{b} \alpha(x) h(x)\, dx = 0$$

for every $h(x) \in \mathcal{C}(\alpha, b)$ *such that* $h(\alpha) = h(b) = 0$, *then* $\alpha(x) = 0$ *for every* $x \in [\alpha, b]$.

Lemma

Let $\alpha(x)$ *continuous in* $[\alpha, b]$. *If*

$$\int_{\alpha}^{b} \beta(x) h'(x)\, dx = 0$$

for every $h(x) \in \mathcal{D}_1(\alpha, b)$ *such that* $h(\alpha) = h(b) = 0$, *then* $\beta(x) = c$ *for every* $x \in [\alpha, b]$, *where* c *is a constant.*

Lemma

Let $\alpha(x), \beta(x)$ *continuous in* $[\alpha, b]$*. If*

$$\int_{\alpha}^{b} [\alpha(x)h(x) + \beta(x)h'(x)\, dx = 0$$

for every $h(x) \in \mathcal{D}_1(\alpha, b)$ *such that* $h(\alpha) = h(b) = 0$*, then, for every* $x \in [\alpha, b]$*,*

$$\beta'(x) = \alpha(x).$$

Apply a partial integration in order to understand the plausibility of the result.

A differential equation

Theorem (Euler-Lagrange equation)

A necessary condition for an extremum of $J[y]$ is that $y(x)$ satisfy the Euler-Lagrange equation

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

For the proof we use the lemmas in the preceeding pages.

A more intuitive derivation of Euler's equation

Take $y(x) \to y(x) + \epsilon h(x)$ and define the function

$$\mathcal{J}(\epsilon) = J[y + \epsilon h].$$

We may write

$$\mathcal{J}(\epsilon) = \mathcal{J}(0) + \mathcal{J}'(0)\epsilon + \dots$$

An extremum (for $\mathcal{J}$ and for the corresponding functional) requires that $\frac{d}{d\epsilon}\mathcal{J}(\epsilon{=}0) = 0$. That is

$$\frac{d}{d\epsilon}\mathcal{J}(\epsilon)\Big|_{\epsilon=0} = \int_\alpha^b (F_y h + F_{y'} h')\, dx = \int_\alpha^b \left(F_y - \frac{d}{dx}F_{y'}\right) h\, dx$$

By the fundamental lemma

$$F_y - \frac{d}{dx}F_{y'} = 0.$$

Example (arc length)

The arc length $J[y]$ between two points on the plane has an extremum when y is a straight line.

Exercise (Fermat's principle in geometric optics)

Find the extremum of

$$T[y] = \int_{x_1}^{x_2} n(x,y)\sqrt{1+y'^2}\,dx$$

Exercise

Find the general solution of Euler's equation corresponding to the functional

$$J[y] = \int_{\alpha}^{b} f(x)\sqrt{1+y'^2}\,dx = 0$$

and investigate the special cases $f(x) = \sqrt{x}$ *and* $f(x) = x$.

Hamilton's principle in mechanics

Hamilton's principle

A particle $x(t)$ moving from time t_1 until time t_2 between two points $x(t_1), x(t_2)$ follows a trajectory that minimizes a functional

$$I = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

called the *action*.

Lagrangian

The function L is called the Lagrangian and it is derived from the energy of the system

$$L = T - V$$

where T is its kinetic energy and V its potential energy.

Examples

Example (Extremum of the energy for a free particle)

Assume the energy of a free particle and find its equation of motion,

$$L = T = \frac{1}{2} m\dot{x}^2$$

Example (Extremum of the energy for a particle)

Assume a particle in a potential $V(x)$. Write its equation of motion.

Newton's equation is obtained as a problem of the calculus of variations from a principle of least action.

Case of many functions

Assume a functional of two functions $y_1(x), y_2(x)$

$$J[y_1, y_2] = \int_{\alpha}^{b} F(x, y_1, y_2, y_1', y_2')\, dx$$

with boundary conditions

$$y_1(\alpha) = A_1, \quad y_1(b) = B_1$$
$$y_2(\alpha) = A_2, \quad y_2(b) = B_2.$$

Theorem (Euler-Lagrange equations for a functional of n functions y_i)

A necessary condition for an extremum of $J[y_i]$ is that the $y_i(x)$ satisfy the set of equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0, \qquad i = 1, 2, \ldots, n$$

Particle motion in two dimensions

Example (Equations for a particle in 2D)

Find the equations for a free particle moving in a two-dimensional space.

Example (Equations in polar coordinator)

Assume a particle in a potential $V(r)$ where (r, θ) are polar coordinates. Write its equation of motion in polar coordinates.

Integrals for the Euler-Lagrange equation

Case 1

Suppose F does not depend on y, i.e., we have a functional

$$\int_{\alpha}^{b} F(x, y')dx.$$

Euler's equation is

$$\frac{d}{dx}F_{y'} = 0 \Rightarrow F_{y'} = C.$$

Exercise

Give an example.

Case 2

Suppose F does not depend on x, i.e., we have the functional

$$\int_{\alpha}^{b} F(y, y')dx.$$

Euler's equation is

$$F_y - \frac{d}{dx}F_{y'} = 0 \Rightarrow F_y - F_{y'y}y' - F_{y'y'}y'' = 0.$$

Exercise

Give an example.

We can find an integral for the equation

Multiply by y'

$$F_y y' - F_{y'y}y'^2 - F_{y'y'}y'y'' = 0 \Rightarrow \frac{d}{dx}(F - y'F_{y'}) = 0 \Rightarrow F - y'F_{y'} = C.$$

Case 3

Suppose F does not depend on y', i.e., we have the functional

$$\int_{\alpha}^{b} F(x,y)dx.$$

Euler's equation is

$$F_y = 0$$

and it is an equation for a curve $y = y(x)$.

Examples of first Integrals

Exercise (Arc length)

Write Euler's equation for the extremum of the arc length

$$J[y] = \int_{\alpha}^{b} \sqrt{1+y'^2}\, dx.$$

Exercise (Energy for a particle in a potential)

Find an integral of the motion for a particle in a potential $V = V(x)$ *with a Lagrangian of the form*

$$L = L(x, \dot{x}).$$

Exercise (A particle in a central potential)

Write the equations of motion for a particle in a central potential

$$L = L(r, \dot{r}, \dot{\theta}).$$

A second look into The variational (or functional) derivative

[I.M. Gelfand, S.V. Fomin, "Calculus of Variations", Section 7.]

Assume a functional

$$J[y] = \int_{\alpha}^{b} F(x, y, y')\, dx, \qquad y(\alpha) = y_\alpha,\ y(b) = y_b$$

and devide $[\alpha, b]$ in $n+1$ equal subintervals at points

$$x_0 = \alpha, x_1, \ldots, x_{n+1} = b, \quad (x_{i+1} - x_i = \Delta x).$$

Let $y(x_i) = y_i$. We have the approximation

$$J(y_1, \ldots, y_n) = \sum_{i=1}^{n} F\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \Delta x.$$

Functional derivative and Euler's equation

We calculate $\partial J/\partial y_k$ and find

$$\frac{\partial J}{\partial y_k \Delta x} = F_y\left(x_k, y_k, \frac{y_{k+1}-y_k}{\Delta x}\right) - \frac{1}{\Delta x}\left[F_{y'}\left(x_k, y_k, \frac{y_{k+1}-y_k}{\Delta x}\right) - F_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k-y_{k-1}}{\Delta x}\right)\right].$$

The quantity $\partial y_k \Delta x$ gives the area between the increment h of y and the x-axis in an interval Δx.

Definition of functional derivative

As $\Delta x \to 0$ we obtain the following expression

$$\frac{\delta J}{\delta y} = F_y - \frac{d}{dx} F_{y'}$$

Euler's equation is expressed as

$$\frac{\delta J}{\delta y} = 0.$$

Functional derivative for functions of many variables

In analogy to the previous derivation we have the following for $u = u(x, y)$.

Definition of functional derivative

$$\frac{\delta J}{\delta u} = F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y}$$

Euler's equation is expressed as

$$\frac{\delta J}{\delta u} = 0.$$

Exercise (Variational problems with constraints)

Read: I.M. Gelfand, S.V. Fomin, "Calculus of Variations", Section 12.