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Calculus of Variations

Liouville type properties for a class of weighted anisotropic elliptic equations

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Abstract

We establish Liouville type results for weighted anisotropic elliptic equations in divergence form in the strip $\mathbb{R}^{N-1} \times (-1, 1)$, $N \geq 2$. The weights depend on one variable and they include the case where they are powers of the distance functions to the boundary of the strip.

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1 Introduction and main results

In this work our interest is to prove Liouville type results for the *anisotropic* elliptic operator

$$\mathcal{L}u = w_1 \Delta_{x'} u + \partial_\lambda (w_1 w_2 \partial_\lambda u), \tag{1.1}$$

where

$$x = (x', \lambda) \in S := \mathbb{R}^{N-1} \times (-1, 1), \quad N \geq 2,$$

and $w_i(\lambda) = w_i(|\lambda|)$ for $i = 1, 2$, are locally positive and bounded weight functions. That is, we look for conditions on $w_1,\ w_2$ under which the only bounded weak solutions of $\mathcal{L}u = 0$ are the constant solutions.

Let us recall the uniformly elliptic case

$$\sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) = 0,$$

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with

$$c_1|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq c_2|\xi|^2, \qquad \forall\xi \in \mathbb{R}^N, \quad c_1, c_2 > 0.$$

The pioneering work of De Giorgi and Moser [6, 20, 21], see also [13], played a crucial role in establishing many properties of weak solutions such as Harnack inequality, Liouville type results, Holder continuity etc. Several extensions of these results were made by various authors in a number of directions, see e.g., [7, 11, 12].

To discuss the nonuniformly elliptic case we denote by $a(x)$ the matrix with entries $a_{ij}(x)$ and set

$$\kappa(x) := \inf_{\xi\in\mathbb{R}^N} \frac{\xi \cdot a(x)\xi}{|\xi|^2}, \qquad\qquad \mu(x) := \sup_{\xi\in\mathbb{R}^N} \frac{|a(x)\xi|^2}{\xi \cdot a(x)\xi}.$$

Assume that for $p, q \in (1, +\infty]$, $\mu \in L^p_{loc}(\mathbb{R}^N)$, $\kappa^{-1} \in L^q_{loc}(\mathbb{R}^N)$, and

$$\limsup_{R\to\infty} |B_R|^{-\left(\frac{1}{p}+\frac{1}{q}\right)} \|\mu\|_{L^p(B_R)} \|\kappa^{-1}\|_{L^q(B_R)} < \infty.$$

Under essentially these assumptions, and provided that

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{N},$$

Trudinger [24], established Harnack inequality and Hölder continuity for nonnegative weak solutions, see also [22]. Quite recently the same results have been proved by Bella and Schäffner [2] under the weaker condition

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{N-1}.$$

As a consequence, every bounded weak solution is constant, cf Corollary 4.4 of [2] for the precise result and the definition of weak solutions.

There is a recent interest in the study of anisotropic operators see e.g. [5, 14, 16, 17]. Our motivation for studying (1.1) comes from the work of Caffarelli and Cordoba [3] in phase transition analysis and is a continuation of [9] and [19]. In [9] the aim was to establish various Sobolev type inequalities for anisotropic weighted operators whereas in [19], Liouville type Theorems for (1.1) are presented, for particular choices of the weights.

We first consider the model *anisotropic* elliptic operator

$$\mathcal{L}_{\alpha,\nu}u = (1-|\lambda|)^{\alpha}\Delta_{x'}u + \partial_\lambda((1-|\lambda|)^{\alpha+\nu}\partial_\lambda u) \tag{1.2}$$

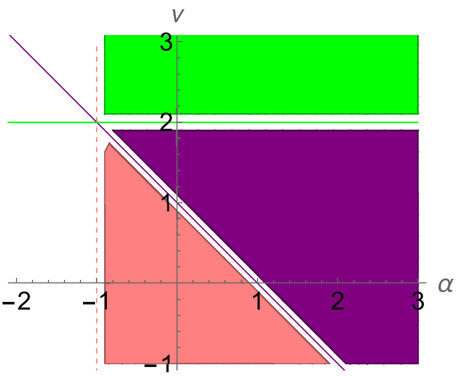
for $(x', \lambda) \in S := \mathbb{R}^{N-1} \times (-1, 1)$. We focus our attention only in the case $\alpha > -1$ and we state the results in three cases, the subcritical one, that is $\nu < 2$ and the critical or supercritical case corresponding to $\nu = 2$ and $\nu > 2$ respectively (Fig. 1). Then, our first result reads

Theorem 1.1 (Subcritical case) *Let* $\alpha > -1$.
(a) *If* $\nu < 1 - \alpha$ *then the function*

$$u(\lambda) = \int_{-1}^{\lambda} (1-|t|)^{-\alpha-\nu}dt,$$

is a nonnegative (and bounded) weak solution of $\mathcal{L}_{\alpha,\nu}u = 0$ *in* S.
(b) *If* $1 - \alpha \leq \nu < 2$ *then any nonnegative weak solution of* $\mathcal{L}_{\alpha,\nu}u = 0$ *in* S *is constant.*

Fig. 1 For $\alpha > -1$, the lines $\nu = 1 - \alpha$ and $\nu = 2$ define three regions in the plane α–ν. In the pink region (subcritical) there exist nonnegative non constant solutions. In the purple region (also subcritical) all nonnegative solutions are constants. Finally in the green region (supercritical) as well as in the case $\nu = 2$ (critical) all bounded solutions are constants



When $\nu \geq 2$ our result reads

Theorem 1.2 (Critical and supercritical cases) *Let $\alpha > -1$ and $\nu \geq 2$. Every bounded weak solutions of $\mathcal{L}_{\alpha,\nu} u = 0$ in S is constant.*

The critical case $\nu = 2$ in the case $\alpha = 1$ was already treated in [19]; in such a case the validity of a Liouville type result entails a positive answer to De Giorgi conjecture under the additional assumption that level sets are Lipschitz graphs, see also [1], [3].

An operator like $\mathcal{L}_{\alpha,\nu}$ when $\nu = 2\alpha$ and $0 < \alpha \leq 1$ (which corresponds to the subcritical and critical case in the present terminology) is naturally related to the phase transition analysis in [3].

When $1 - \alpha \leq \nu < 2$ our result is stronger than establishing that the only bounded weak solutions are the constant ones and is proved by means of an elliptic Harnack inequality.

We note that our results are outside the range of applicability of the ones by Bella and Schäffner [2] mentioned above.

We next consider the more general elliptic operator (1.1). We assume that $w_i(\lambda) = w_i(|\lambda|)$, $i = 1, 2$, $-1 < \lambda < 1$, and $w_i \in L^\infty_{loc}(-1, 1)$. We only consider the case $w_1 \in L^1(0, 1)$ and we state the results in two cases, the subcritical one, which corresponds to the case $w_2^{-\frac{1}{2}} \in L^1(0, 1)$ and the critical or supercritical case which corresponds to the case $w_2^{-\frac{1}{2}} \notin L^1(0, 1)$. Before stating the results we introduce the following two assumptions.

(H1) Suppose $w_1 \in L^1(0, 1)$, $(w_2)^{-\frac{1}{2}} \in L^1(0, 1)$, $(w_1 w_2)^{-1} \notin L^1(0, 1)$ and there exists $\theta \geq 1$ and constants $c_1, c_2 > 0$ such that for any $\lambda \in (-1, 1)$ there holds

$$c_1 \left(\int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt \right)^\theta \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \leq c_2 \left(\int_{|\lambda|}^1 w_2^{-\frac{1}{2}}(t)dt \right)^\theta .$$

(H2) Suppose $w_1 \in L^1(0, 1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0, 1)$ and define

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t)dt. \tag{1.3}$$

We assume that there exists $m > 2$ such that $\varphi^{-\frac{1}{m}} w_2^{-\frac{1}{2}} \in L^1(0, 1)$ and $\theta > 0$ such that for some constants $c_1, c_2 > 0$ and any $\lambda \in (-1, 1)$ there holds

$$c_1 \left(\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt \right)^\theta \leq w_1(|\lambda|) w_2^{\frac{1}{2}}(|\lambda|) \varphi^{\frac{1}{m}}(|\lambda|) \leq c_2 \left(\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t)dt \right)^\theta .$$

Notice that if $w_1 \in L^1(0,1)$ and $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$ then necessarily $(w_1 w_2)^{-1} \notin L^1(0,1)$, as it follows easily from the decomposition $w_2^{-\frac{1}{2}} = w_1^{\frac{1}{2}}(w_1 w_2)^{-\frac{1}{2}}$, whence

$$\varphi(\lambda) \to +\infty \quad \text{as} \quad |\lambda| \to 1.$$

The results then are the following

Theorem 1.3 (Subcritical case) **(a)** *If* $w_1 \in L^1(0,1)$, $(w_2)^{-\frac{1}{2}} \in L^1(0,1)$ *and* $(w_1 w_2)^{-1} \in L^1(0,1)$ *then the function*

$$u(\lambda) = \int_{-1}^{\lambda} (w_1 w_2)^{-1}(t)dt,$$

is a nonnegative (and bounded) weak solution of $\mathcal{L}u = 0$ *in* S *.*
(b) *If* w_1, w_2 *satisfy (H1) for some* $\theta \geq 1$, *then any nonnegative weak solution of* $\mathcal{L}u = 0$ *in* S *is constant.*

Also,

Theorem 1.4 (Critical and supercritical cases) *If* w_1, w_2 *satisfy (H2) for some* $\theta > 0$ *and* $m > 2$, *then any bounded weak solution of* $\mathcal{L}u = 0$ *in* S *is constant.*

The result of Theorem 1.4 is weaker than the one in Theorem 1.3(b). Nevertheless, the result of Theorem 1.4 is optimal and one can not have a Liouville result similar to Theorem 1.3 for nonnegative weak solutions. Indeed, if w_1, w_2 are as in Theorem 1.4 then the function

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t)dt,$$

is a nonnegative weak solution of $\mathcal{L}u = 0$ in S, which is actually unbounded. This function is in the proper energy space, see Sect. 3. Hence the requirement of boundedness of weak solutions in Theorem 1.4 cannot be replaced by the nonnegativity of weak solutions.

To prove Theorem 1.3(b) we establish a Harnack inequality for nonnegative weak solutions $u(x)$ of

$$\mathcal{L}u = 0, \qquad \text{in } C_R := \{|x'| < R, \ \ |\lambda| < 1\}. \tag{1.4}$$

The Harnack inequality follows once one establishes Poincaré and Sobolev inequalities as well as a doubling volume growth condition as is shown in [4, 7]. See also [12, 23] for extensions on complete Riemannian manifolds. In the present work we follow an adaptation made in [8], cf Theorem 2.11 there. In particular the proper energy space is now given by the following norm

$$||u||^2_{H^1_{w_1,w_2}(C_R)} := \int_{C_R} \left(u^2 + |\nabla_{x'} u|^2 + w_2(\partial_\lambda u)^2\right) w_1 dx' d\lambda.$$

To prove Theorem 1.4 we make use of the oscillation decrease method, cf Sect. 4.3 of [13], as adapted in Theorem 1.4 of [19] to the anisotropic setting. This is done in Sect. 3.

In Sect. 4 we give the proofs of Theorems 1.1 and 1.2. We also discuss various extensions of our results.

2 Subcritical case: Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3(b). This will be done by means of an elliptic Harnack inequality, using the Moser iteration scheme, as adapted to isotropic degenerate elliptic operators on bounded domains in [8]. There is a difference in the cut off functions used here as compared to the ones used in [8]. In this work our cut off functions take into account the geometry of the cylinder and they depend only on x'. We combine this with a density argument similar to [8] that takes care of the λ direction.

The three ingredients needed for the scheme to work are the doubling volume-growth condition, a local weighted Sobolev inequality as well as a local weighted Poincaré inequality.

The doubling property follows easily from the fact that

$$V(C_R) = \int_{C_R} w_1 dx' d\lambda = \left(\int_{B'_R} dx' \right) \int_{-1}^{1} w_1 d\lambda = C R^{N-1}, \tag{2.1}$$

for some uniform constant C (independent from R) and any $R > 0$. Here we denote with B'_R the Euclidean ball of radius R in $\mathbb{R}^{N-1}$. We also denote the half cylinder,

$$C_R^+ = C_R \cap \{\lambda > 0\} = \{|x'| < R,\ 0 < \lambda < 1\}.$$

For the Moser iteration scheme to work, we will also need the analogue of Theorem 2.11 of [8]. We first introduce the following norm

$$||u||^2_{H^1_{w_1,w_2}(C_R)} := \int_{C_R} \left(u^2 + |\nabla_{x'} u|^2 + w_2 (\partial_\lambda u)^2\right) w_1 dx' d\lambda,$$

and we denote by $H^1_{0,w_1,w_2}(C_R)$ the completion of $C_0^\infty(\{|x'| < R\})$ under the above norm, whereas $H^1_{00,w_1,w_2}(C_R)$ is the completion under the same norm, of functions that in addition have compact support in $\lambda \in (-1, 1)$, that is

$$H^1_{00,w_1,w_2}(C_R) = \overline{C_0^\infty(C_R)}^{\,||\cdot||_{H^1_{w_1,w_2}(C_R)}}.$$

We then have

Proposition 2.1 *(Density) Suppose w_1, w_2 satisfy (H1). Then*

$$H^1_{0,w_1,w_2}(C_R) = H^1_{00,w_1,w_2}(C_R),$$

that is, we are free from boundary conditions at $|\lambda| = 1$.

Proof We change variables by defining

$$s = s(\lambda) = \left(\int_\lambda^1 w_2^{-\frac{1}{2}}(t) dt \right) \left(\int_0^1 w_2^{-\frac{1}{2}}(t) dt \right)^{-1}, \qquad g(x', s) = f(x', \lambda), \tag{2.2}$$

In the new variables, domain C_R^+ becomes (using calligraphic C)

$$\mathcal{C}_R^+ = \{|x'| < R,\ 0 < s < 1\}.$$

We recall that $\lambda = 1$ corresponds to $s = 0$. The norm now takes the form

$$||u||^2_{H^1(\mathcal{C}_R^+, s^\theta dx' ds)} := \int_{\mathcal{C}_R^+} \left(u^2 + |\nabla_{x'} u|^2 + (\partial_s u)^2\right) s^\theta dx' ds,$$

and $H^1(\mathcal{C}_R^+, s^\theta dx'ds)$ is the corresponding function space. In addition we denote by $H_0^1(\mathcal{C}_R^+, s^\theta dx'ds)$ the completion of $C_0^\infty(|x'| < R,\ s > 0)$ under the above norm We need to prove that any function in $H^1(\mathcal{C}_R^+, s^\theta dx'ds)$ can be approximated by functions in $H_0^1(\mathcal{C}_R^+, s^\theta dx'ds)$.

By Theorem 7.2 of [15] it is known that the set $C^\infty(\overline{\mathcal{C}_R^+})$ is dense in $H^1(\mathcal{C}_R^+, s^\theta dx'ds)$. Hence for any $v \in H^1(\mathcal{C}_R^+, s^\theta dx'ds)$ and any $\varepsilon > 0$, there exists $w \in C^\infty(\overline{\mathcal{C}_R^+})$ such that $||v - w||_{H^1} \leq \epsilon$. We then define the function

$$\varphi_k(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{k^2}\ , \\ 1 + \frac{\ln(ks)}{\ln(k)} & \text{if } \frac{1}{k^2} < s < \frac{1}{k}\ , \\ 1 & \text{if } s \geq \frac{1}{k} \end{cases}$$

and set

$$w_k := w\varphi_k \in C_0^{0,1}(\{s > 0\})\Big|_{\mathcal{C}_R^+}.$$

Then,

$$\begin{aligned} ||w - w_k||_{H^1}^2 &= ||w(1-\varphi_k)||_{H^1}^2 \\ &\leq 2\int_{\mathcal{C}_R^+} (w^2 + |\nabla_{x'} w|^2 + (\partial_s w)^2)(1-\varphi_k)^2 s^\theta dx'ds + 2\int_{\mathcal{C}_R^+} w^2(\partial_s \varphi_k)^2 s^\theta dx'ds \\ &\leq 2\int_{\{|x'|<R,\ 0<s<\frac{1}{k}\}} (w^2 + |\nabla_{x'} w|^2 + (\partial_s w)^2)\ s^\theta dx'ds \\ &\qquad + C R^{N-1} ||w||_{L^\infty(\mathcal{C}_R^+)}^2 \int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s^2(\ln(k))^2}\ s^\theta ds. \end{aligned}$$

For $\theta > 1$ there holds

$$\int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s^2(\ln(k))^2}\ s^\theta ds \leq \frac{1}{\theta - 1}\left(\frac{1}{k}\right)^{\theta-1} \frac{1}{(\ln(k))^2}$$

whereas for $\theta = 1$,

$$\int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s^2(\ln(k))^2}\ s^\theta ds = \frac{1}{(\ln(k))^2}\int_{\frac{1}{k^2}<s<\frac{1}{k}} \frac{1}{s} ds \leq \frac{1}{\ln(k)}.$$

Thus, for $\theta \geq 1$ and k large enough we have $||v - w_k||_{H^1} \leq 2\epsilon$ and the result follows. □

Concerning the local weighted Sobolev inequality we have

Lemma 2.2 *(local weighted Sobolev) Suppose w_1, w_2 satisfy (H1). Then, for $q = \frac{2(N+\theta)}{N-2+\theta}$ there exists a positive constant C_S such that for any $R \geq 1$ and for all $f \in C_0^\infty(\{|x'| < R\})$ there holds*

$$\left(\int_{C_R} |f|^q w_1 dx'd\lambda\right)^{\frac{2}{q}} \leq C_S R^2 (V(C_R))^{\frac{2}{q}-1} \int_{C_R} \left(|\nabla_{x'} f|^2 + w_2(\partial_\lambda f)^2\right) w_1 dx'd\lambda. \tag{2.3}$$

Proof It is clear that it is enough to prove the inequality in the half cylinder,

$$C_R^+ = C_R \cap \{\lambda > 0\} = \{|x'| < R,\ 0 < \lambda < 1\}.$$

Thus, we will prove that for any $f \in C_0^\infty(\{|x'| < R\})$

$$\left(\int_{C_R^+} |f|^q w_1 dx' d\lambda\right)^{\frac{2}{q}} \le C_S R^2 (V(C_R^+))^{\frac{2}{q}-1} \int_{C_R^+} \left(|\nabla_{x'} f|^2 + w_2(\partial_\lambda f)^2\right) w_1 dx' d\lambda. \tag{2.4}$$

We next change variables by (2.2). We recall that domain C_R^+ becomes

$$\mathcal{C}_R^+ = \{|x'| < R,\ 0 < s < 1\},$$

with $V(C_R^+) = V(\mathcal{C}_R^+) = c_N R^{N-1}$. Taking into account (H1), inequality (2.4) takes the following equivalent form

$$\left(\int_{\mathcal{C}_R^+} |g|^q s^\theta dx' ds\right)^{\frac{2}{q}} \le C_S\, R^2 (V(\mathcal{C}_R^+))^{\frac{2}{q}-1} \int_{\mathcal{C}_R^+} \left(|\nabla_{x'} g|^2 + (\partial_s g)^2\right) s^\theta dx' ds. \tag{2.5}$$

For $R = 1$ the above inequality is written

$$\left(\int_{\mathcal{C}_1^+} |g|^q s^\theta dx' ds\right)^{\frac{2}{q}} \le C_S\, (V(\mathcal{C}_1^+))^{\frac{2}{q}-1} \int_{\mathcal{C}_1^+} \left(|\nabla_{x'} g|^2 + (\partial_s g)^2\right) s^\theta dx' ds. \tag{2.6}$$

This is true by Proposition 2.1 of [9] with $QB = 2A = \theta$ there. As a consequence $q = \frac{2(N+\theta)}{N-2+\theta}$.

To establish (2.5), after a rescaling in the x' variables the inequality takes the form

$$\left(\int_{\mathcal{C}_1^+} |g|^q s^\theta dx' ds\right)^{\frac{2}{q}} \le C_S\, (V(\mathcal{C}_1^+))^{\frac{2}{q}-1} \int_{\mathcal{C}_1^+} \left(|\nabla_{x'} g|^2 + R^2\, (\partial_s g)^2\right) s^\theta dx' ds.$$

This is true by (2.6) and the fact that $R \ge 1$. This completes the proof. □

We next consider the local weighted Poincare iequality. If

$$\bar{f} := \frac{1}{V(C_R)} \int_{C_R} f(x', \lambda)\, w_1 dx' d\lambda$$

we have

Lemma 2.3 *(local weighted Poincare) Suppose w_1, w_2 satisfy (H1), then there exist positive constant C_P such that for any $R \ge 1$ and for all $f \in C^1(\overline{C_R})$ there holds*

$$\int_{C_R} |f - \bar{f}|^2 w_1 dx' d\lambda \le C_P R^2 \int_{C_R} \left(|\nabla_{x'} f|^2 + w_2(\partial_\lambda f)^2\right) w_1 dx' d\lambda, \tag{2.7}$$

Proof The result will follow once we establish that for any $f \in C^1(\overline{C_R})$ we have the following inequality in the upper half cylinder C_R^+,

$$\int_{C_R^+} |f - \xi|^2 w_1 dx' d\lambda \le C_P R^2 \int_{C_R^+} \left(|\nabla_{x'} f|^2 + w_2(\partial_\lambda f)^2\right) w_1 dx' d\lambda \tag{2.8}$$

for some positive constant C_P (independent on R), with the choice

$$\xi = \frac{\int_{|x'|<R} f(x', 0) dx'}{\omega_{N-1} R^{N-1}}.$$

A similar inequality will hold in the lower half cylinder C_R^- with the same choice of ξ. Then, since

$$\int_{C_R} |f-\bar{f}|^2 w_1 dx' d\lambda = \min_{\xi\in\mathbb{R}} \int_{C_R} |f-\xi|^2 w_1 dx' d\lambda,$$

the required inequality in C_R will follow.

Making use of the change of variables (2.2) and taking into account (H1) inequality (2.8) takes the following equivalent form (modulo absolute constants)

$$\int_{\{|x'|<R,\ 0<s<1\}} |g-\xi|^2 s^\theta dx' ds \le C_P R^2 \int_{\{|x'|<R,\ 0<s<1\}} \left(|\nabla_{x'} g|^2 + (\partial_s g)^2\right) s^\theta dx' ds. \tag{2.9}$$

We note that

$$\xi = \frac{\int_{\{|x'|<R\}} g(x',1) dx'}{\omega_{N-1} R^{N-1}}.$$

Once again it is enough to establish the result for $R=1$. The general case then follows by scaling in x' and using the fact that $R \ge 1$, as it was done in the proof of (2.5).

For $s \in [0,1]$ we define

$$\bar{g}(s) = \frac{\int_{\{|x'|<1\}} g(x',s) dx'}{\omega_{N-1}},$$

and note that $\xi = \bar{g}(1)$. There holds

$$\begin{aligned} &\int_{\{|x'|<1,\ 0<s<1\}} |g(x',s)-\xi|^2 s^\theta dx' ds \\ &\le 2\int_{\{|x'|<1,\ 0<s<1\}} |g(x',s)-\bar{g}(s)|^2 s^\theta dx' ds + 2\int_{\{|x'|<1,\ 0<s<1\}} |\bar{g}(s)-\bar{g}(1)|^2 s^\theta dx' ds. \end{aligned} \tag{2.10}$$

We next consider the first integral on the right hand side. By Poincaré in the x' variables we have

$$\begin{aligned} \int_{\{|x'|<1,\ 0<s<1\}} |g(x',s)-\bar{g}(s)|^2 s^\theta dx' ds &= \int_0^1 s^\theta \left(\int_{|x'|<1} |g(x',s)-\bar{g}(s)|^2 dx' \right) ds \\ &\le C \int_{\{|x'|<1,\ 0<s<1\}} |\nabla_{x'} g(x',s)|^2 s^\theta \, dx' ds. \end{aligned} \tag{2.11}$$

Concerning the second integral on the right hand side of (2.10) we have the following one dimensional Poincaré

$$\begin{aligned} \int_0^1 |\bar{g}(s)-\bar{g}(1)|^2 s^\theta \, ds &= \int_0^1 |\bar{g}(s)-\bar{g}(1)|^2 \left(\frac{s^{\theta+1}}{\theta+1}\right)' ds \\ &= -\frac{2}{\theta+1}\int_0^1 (\bar{g}(s)-\bar{g}(1))\bar{g}'(s) s^{\theta+1} \, ds \\ &\le \frac{2}{\theta+1} \left(\int_0^1 (\bar{g}(s)-\bar{g}(1))^2 s^{\theta+2} \, ds\right)^{\frac12} \left(\int_0^1 \bar{g}'^2(s) s^\theta \, ds\right)^{\frac12} \end{aligned}$$

$$\leq \frac{2}{\theta+1}\left(\int_0^1 (\bar{g}(s)-\bar{g}(1))^2 s^\theta\, ds\right)^{\frac{1}{2}}\left(\int_0^1 \bar{g}'^2(s)s^\theta\, ds\right)^{\frac{1}{2}},$$

whence,

$$\int_0^1 |\bar{g}(s)-\bar{g}(1)|^2 s^\theta\, ds \leq \frac{4}{(\theta+1)^2}\int_0^1 \bar{g}'^2(s)s^\theta\, ds,$$

from which it follows that

$$\int_{\{|x'|<1,\ 0<s<1\}} |\bar{g}(s)-\bar{g}(1)|^2 s^\theta dx'ds \leq \frac{4}{(\theta+1)^2}\int_{\{|x'|<1,\ 0<s<1\}} (\partial_s g)^2 s^\theta dx'ds. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12) we obtain (2.9) with $R = 1$ and this completes the proof. □

We are now ready to study positive weak solutions of $\mathcal{L}u = 0$ in the proper energy space $H^1_{w_1,w_2}(C_R)$, for all $R \geq 1$. In particular, functions $u \in H^1_{w_1,w_2}(C_R)$ satisfy

$$\int_{C_R} (w_1\nabla_{x'}u\nabla_{x'}\varphi + w_1w_2\partial_\lambda u\partial_\lambda\varphi)\, dx'd\lambda = 0,$$

for any test function $\varphi \in H^1_{00,w_1,w_2}(C_R)$.

As usually, $u \in H^1_{w_1,w_2}(C_R)$ is a weak subsolution in C_R provided that

$$\int_{C_R} (w_1\nabla_{x'}u\nabla_{x'}\varphi + w_1w_2\partial_\lambda u\partial_\lambda\varphi)\, dx'd\lambda \leq 0,$$

for any $0 \leq \varphi \in H^1_{00,w_1,w_2}(C_R)$ and similarly for the weak supersolution, reversing the above inequality.

We note that by Lemma 2.1, under the assumption (H1) of the present section, we have

$$H^1_{00,w_1,w_2}(C_R) = H^1_{0,w_1,w_2}(C_R).$$

Theorem 1.3 (b) is a consequence of the following Harnack inequality, whose main ingredients are the local weighted Sobolev inequality, the local weighted Poincaré inequality and the density property stated above.

Theorem 2.4 *(Harnack inequality) Let $N \geq 2$ and suppose that w_1, w_2 satisfy (H1). Let $u \in H^1_{w_1,w_2}(C_R)$ be a nonnegative weak solution of $\mathcal{L}u = 0$ in C_R. Then for any $R \geq 1$ and any $0 < \alpha < 1$, there holds*

$$\sup_{C_{\alpha R}} u \leq C_H \inf_{C_{\alpha R}} u,$$

where $C_H = C_H(N, \alpha, \theta, c_1, c_2)$ is a positive constant.

The proof of this, uses several auxiliary results. We initially consider functions $u_\varepsilon = u + \varepsilon \geq \varepsilon > 0$, so that negative powers of u_ε are in the appropriate function spaces. In the final stage, when proving Theorem 2.4 we will send ε to zero. For simplicity we drop the subscript ε in the Lemmas that follow.

Lemma 2.5 *(Local boundedness) Let $N \geq 2$, suppose that w_1, w_2 satisfy (H1) and $u \in H^1_{w_1,w_2}(C_R)$ be a non negative subsolution of $\mathcal{L}u = 0$ in C_R. Then for any $R > 0$, $\alpha \in (0,1)$ and $p > 1$ there exists a positive constant $C = C(N, \theta, c_1, c_2)$ such that*

$$\sup_{C_{\alpha R}} u^p \leq C \left(\frac{(\frac{p}{p-1})^2 + 1}{(1-\alpha)^2 \alpha^{\frac{N-1}{\kappa}}} \right)^{\frac{\kappa}{\kappa-1}} \left(\frac{1}{V(C_R)} \int_{C_R} u^p w_1 dx' d\lambda \right), \tag{2.13}$$

where $\kappa = \frac{N+\theta}{N+\theta-2} > 1$.

Proof We consider a function $\xi(x', \lambda) = \xi(x') \geq 0$ such that $\xi \in C^\infty_0(B'_R)$, $B'_R = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$. Then, for $\beta > 0$, function $\varphi = \xi^2 u^\beta \in H^1_{0,w_1,w_2}(C_R)$ is an admissible test function. Using the computations

$$w_1 \nabla_{x'} u \nabla_{x'} (\xi^2 u^\beta) = \beta w_1 \xi^2 u^{\beta-1} \nabla_{x'} u \nabla_{x'} u + 2 \left(\sqrt{\epsilon} w_1^{\frac{1}{2}} \xi u^{\frac{\beta-1}{2}} \nabla_{x'} u \right) \left(\frac{1}{\sqrt{\epsilon}} w_1^{\frac{1}{2}} u^{\frac{\beta+1}{2}} \nabla_{x'} \xi \right),$$

$$w_1 w_2 \partial_\lambda u \partial_\lambda (\xi^2 u^\beta) = \beta w_1 w_2 \xi^2 u^{\beta-1} \partial_\lambda u \partial_\lambda u$$

the definition of weak solutions and Young inequality we arrive at

$$(\beta - \epsilon) \int_{C_R} \left(w_1 |\nabla_{x'} u|^2 + w_1 w_2 (\partial_\lambda u)^2 \right) u^{\beta-1} \xi^2 dx' d\lambda \leq \frac{1}{\epsilon} \int_{C_R} w_1 |\nabla_{x'} \xi|^2 u^{\beta+1} dx' d\lambda.$$

We choose $\epsilon = \beta/2$ and set $v := u^{\frac{\beta+1}{2}}$. The previous inequality yields

$$\int_{C_R} \left(|\nabla_{x'} (\xi v)|^2 + w_2 (\partial_\lambda (\xi v))^2 \right) w_1 dx' d\lambda \leq 2 \left[\left(\frac{\beta+1}{\beta} \right)^2 + 1 \right] \int_{C_R} |\nabla_{x'} \xi|^2 v^2 w_1 dx' d\lambda.$$

We next choose a function $\eta \in C^\infty(0,1)$ such that $\eta(y) = 1$ for $y \leq s$ and $\eta(y) = 0$ for $y \geq t$, with $0 < s < t < 1$. We take $\xi(x') = \eta\left(\frac{|x'|}{R}\right)$. Using the local weighted Sobolev inequality of Lemma 2.2, we get

$$\left(\int_{C_{sR}} u^{(\beta+1)\kappa} \ w_1 dx' d\lambda \right)^{\frac{1}{\kappa}} \leq C_S \frac{C_0 V^{\frac{1}{\kappa}-1}(C_{tR})}{(t-s)^2} \left[\left(\frac{\beta+1}{\beta} \right)^2 + 1 \right] \int_{C_{tR}} u^{\beta+1} \ w_1 dx' d\lambda$$

where C_0 is a universal constant and $\kappa = \frac{N+\theta}{N+\theta-2} > 1$ with $\theta \geq 1$ as defined in (H1). Making use of the *doubling property,* as well as of the fact that $V(C_{sR}) = c(sR)^{N-1}$, we deduce that

$$\left(\frac{1}{V(C_{sR})} \int_{C_{sR}} u^{(\beta+1)\kappa} \ w_1 dx' d\lambda \right)^{\frac{1}{\kappa}}$$
$$\leq \frac{C_0 C_S}{(t-s)^2 s^{\frac{N-1}{\kappa}}} \left[\left(\frac{\beta+1}{\beta} \right)^2 + 1 \right] \frac{1}{V(C_{tR})} \int_{C_{tR}} u^{\beta+1} \ w_1 dx' d\lambda.$$

For $\alpha \in (0,1)$ and $\alpha \leq s < t \leq 1$ we have

$$\left(\frac{1}{V(C_{sR})} \int_{C_{sR}} u^{(\beta+1)\kappa} \ w_1 dx' d\lambda \right)^{\frac{1}{\kappa}}$$
$$\leq \frac{C_0 C_S}{(t-s)^2 \alpha^{\frac{N-1}{\kappa}}} \left[\left(\frac{\beta+1}{\beta} \right)^2 + 1 \right] \frac{1}{V(C_{tR})} \int_{C_{tR}} u^{\beta+1} \ w_1 dx' d\lambda.$$

for a universal constant C_0. Taking $s_{m+1} = \alpha + \frac{1-\alpha}{m+2}$ and $t_m = \alpha + \frac{1-\alpha}{m+1} = s_m, m = 0, 1, 2, \ldots$ the above inequality is written as

$$\left(\frac{1}{V(C_{s_{m+1}R})} \int_{C_{s_{m+1}R}} u^{(\beta+1)\kappa} \; w_1 dx' d\lambda \right)^{\frac{1}{\kappa}} \leq A_m \left(\frac{1}{V(C_{s_m R})} \int_{C_{s_m R}} u^{\beta+1} \; w_1 dx' d\lambda \right). \tag{2.14}$$

where

$$A_m := \frac{(m+1)^2(m+2)^2 C_0 C_S}{(1-\alpha)^2 \alpha^{\frac{N-1}{\kappa}}} \left[\left(\frac{\beta+1}{\beta} \right)^2 + 1 \right].$$

We define

$$I(m) := \left(\frac{1}{V(C_{s_m R})} \int_{C_{s_m R}} u^{\kappa^m(\beta+1)} \; w_1 dx' d\lambda \right)^{\frac{1}{\kappa^m}},$$

so that we rewrite (2.14) as

$$I(m+1) \leq A_m^{\frac{1}{\kappa^m}} I(m), \qquad m = 0, 1, 2, \ldots.$$

We next iterate the above inequality to obtain

$$I(\infty) \leq \left(\prod_{m=0}^{\infty} A_m^{\frac{1}{\kappa^m}} \right) I(0).$$

The infinite product is easily seen to be finite, therefore taking $p = \beta + 1 > 1$, we end up with

$$\sup_{C_{\alpha R}} u^p \leq C \left(\frac{(\frac{p}{p-1})^2 + 1}{(1-\alpha)^2 \alpha^{\frac{N-1}{\kappa}}} \right)^{\frac{\kappa}{\kappa-1}} \left(\frac{1}{V(C_R)} \int_{C_R} u^p w_1 dx' d\lambda \right).$$

□

We next have

Lemma 2.6 *Let $N \geq 2$ and suppose that w_1, w_2 satisfy (H1). Let $u \in H^1_{w_1,w_2}(C_R)$ be a non negative, supersolution in C_R. Then for any $R > 0$, any $0 < \alpha < 1$, there exists a positive constant $C = C(N, \theta, c_1, c_2)$ such that*

$$\sup_{C_{\alpha R}} u^p \leq C \left(\frac{(\frac{p}{p-1})^2 + 1}{(1-\alpha)^2 \alpha^{\frac{N-1}{\kappa}}} \right)^{\frac{\kappa}{\kappa-1}} \left(\frac{1}{V(C_R)} \int_{C_R} u^p w_1 dx' d\lambda \right), \tag{2.15}$$

for any $0 < p < 1$, and

$$\sup_{C_{\alpha R}} u^{-p} \leq C \left(\frac{(\frac{p}{p+1})^2 + 1}{(1-\alpha)^2 \alpha^{\frac{N-1}{\kappa}}} \right)^{\frac{\kappa}{\kappa-1}} \left(\frac{1}{V(C_R)} \int_{C_R} u^{-p} w_1 dx' d\lambda \right), \tag{2.16}$$

for any $p > 0$, where $\kappa = \frac{N+\theta}{N+\theta-2} > 1$.

Proof The proof is similar to the proof of the previous Lemma. We may suppose that u is bounded below by a positive constant. In particular $u^\beta \in H^1_{w_1,w_2}(C_R)$ for any $\beta < 0$. We again use $\varphi = \xi^2 u^\beta \in H^1_{0,w_1,w_2}(C_R)$ as a test function and we similarly arrive at the analogue of (2.14),

$$\left(\frac{1}{V(C_{s_{m+1}R})}\int_{C_{s_{m+1}R}} u^{(\beta+1)\kappa} \;\; w_1 dx' d\lambda\right)^{\frac{1}{\kappa}} \leq A_m \left(\frac{1}{V(C_{s_m R})}\int_{C_{s_m R}} u^{\beta+1} \;\; w_1 dx' d\lambda\right).$$

where

$$A_m := \frac{(m+2)^4 C_0 C_S}{(1-\alpha)^2 \alpha^{\frac{N-1}{\kappa}}}\left[\left(\frac{\beta+1}{\beta}\right)^2 + 1\right].$$

In case $\beta > -1$, we set $p = \beta + 1$ and we conclude the result as before, for the case $0 < p < 1$.

In case $\beta < -1$, then the iteration takes the form

$$\left(\frac{1}{V(C_{s_{m+1}R})}\int_{C_{s_{m+1}R}} u^{\kappa^{m+1}(\beta+1)} \;\; w_1 dx' d\lambda\right)^{\frac{1}{\kappa^{m+1}}}$$
$$\leq A_m^{\frac{1}{\kappa^m}} \left(\frac{1}{V(C_{s_m R})}\int_{C_{s_m R}} u^{\kappa^m(\beta+1)} \;\; w_1 dx' d\lambda\right)^{\frac{1}{\kappa^m}},$$

with $\beta + 1 = -p$ and again we conclude as before. □

The following Lemma is an adaptation of a similar result in Theorem 4.15 of [13].

Lemma 2.7 *Let $N \geq 2$ and suppose that w_1, w_2 satisfy (H1). Let $u \in H^1_{w_1,w_2}(C_R)$ be a non-negative supersolution in C_R. Then for any $\alpha \in (0,1)$ there exists $p_0 = p_0(\alpha, N, \theta, c_1, c_2) \in (0,1)$ such that*

$$\left(\int_{C_{\alpha R}} u^{p_0}\, w_1 dx' d\lambda\right)\left(\int_{C_{\alpha R}} u^{-p_0}\, w_1 dx' d\lambda\right) \leq 4V^2(C_{\alpha R}). \tag{2.17}$$

Proof We initially consider $u_\varepsilon = u + \varepsilon \geq \varepsilon > 0$. After proving the estimate, we send ε to zero. For simplicity in what follows we omit the subscript ε.

For some $p_0 = p_0(\alpha, N, \theta, c_1, c_2) > 0$ we will establish the following estimate

$$\int_{C_{\alpha R}} e^{p_0|v|} w_1 dx' d\lambda \leq 2\, V(C_{\alpha R}), \tag{2.18}$$

where,

$$v = \ln u - \ln L = \ln\frac{u}{L}, \quad \text{with} \quad \ln L := V^{-1}(C_{\alpha R})\int_{C_{\alpha R}} \ln u \;\; w_1 dx' d\lambda.$$

Once (2.18) is true for some p_0 we may assume that $p_0 \in (0,1)$. Since

$$e^{p_0|v|} = 1 + p_0|v| + \frac{(p_0|v|)^2}{2!} + \cdots + \frac{(p_0|v|)^n}{n!} + \cdots,$$

it is enough to establish

$$\int_{C_{\alpha R}} |v|^k w_1 dx' d\lambda \leq C_0^k e^k k!\, V(C_{\alpha R}), \tag{2.19}$$

where $C_0 = C_0(N, \theta, c_1, c_2, \alpha)$ is a positive constant. Then, (2.18) will follow by choosing $p_0 = (2C_0 e)^{-1}$, since

$$\int_{C_{\alpha R}} \frac{(p_0|v|)^k}{k!} w_1 dx' d\lambda \leq \frac{1}{2^k} V(C_{\alpha R}), \qquad k = 1, 2, \ldots.$$

To prove (2.19) we take as a test function $\varphi = \xi^2 u^{-1}$ in the definition of u as a supersolution, with $\xi(x') = \eta\left(\frac{|x'|}{R}\right)$, $\eta \in C^\infty(0,1)$, $\eta(y) = 1$ for $y \leq \alpha$, $0 < \alpha < 1$ and $\eta(y) = 0$ for $y \geq 1$. We get

$$\begin{aligned}
&\int_{C_{\alpha R}} \left(|\nabla_{x'} v|^2 + w_2(\partial_\lambda v)^2\right) w_1 dx' d\lambda \leq \int_{C_R} \left(|\nabla_{x'} v|^2 + w_2(\partial_\lambda v)^2\right) \xi^2 w_1 dx' d\lambda \\
&\leq 4 \int_{C_R} |\nabla_{x'} \xi|^2 w_1 dx' d\lambda. \qquad (2.20)
\end{aligned}$$

We next use the *local weighted Poincaré inequality* of Lemma 2.3, to get

$$\int_{C_{\alpha R}} v^2 w_1 dx' d\lambda \leq 4C_P(\alpha R)^2 \int_{C_R} |\nabla_{x'} \xi|^2 w_1 dx' d\lambda \leq \frac{C}{\alpha^{N-3}(1-\alpha)^2} V(C_{\alpha R}),$$

and by Holder inequality

$$\int_{C_{\alpha R}} |v| w_1 dx' d\lambda \leq V^{\frac{1}{2}}(C_{\alpha R}) \left(\int_{C_{\alpha R}} v^2 w_1 dx' d\lambda\right)^{\frac{1}{2}} \leq \frac{C}{\alpha^{\frac{N-3}{2}}(1-\alpha)} V(C_{\alpha R}). \quad (2.21)$$

These prove (2.19) for $k = 1, 2$.

It remains to prove it for any integer $k \geq 3$. We now use as a test function in the definition of u as a weak supersolution, the function $\varphi = \xi^2 u^{-1} |v_m|^{2\beta}$, for $v_m = -m$ if $v \leq -m$, $v_m = v$ if $|v| < m$, and $v_m = m$ if $v \geq m$. We have $\varphi \in H^1_{0,w_1,w_2}(C_R) \cap L^\infty(C_R)$ and

$$\begin{aligned}
&\int_{C_R} \xi^2 |v_m|^{2\beta} (|\nabla_{x'} v|^2 + w_2(\partial_\lambda v)^2) w_1 dx' d\lambda \\
&\leq (2\beta) \int_{C_R} (\nabla_{x'} v \nabla_{x'} |v_m| + w_2 \partial_\lambda v \partial_\lambda |v_m|) |v_m|^{2\beta-1} \xi^2 w_1 dx' d\lambda \\
&+2 \int_{C_R} \xi \nabla_{x'} v \nabla_{x'} \xi |v_m|^{2\beta} w_1 dx' d\lambda.
\end{aligned}$$

Notice that $\nabla_{x'} v \nabla_{x'} |v_m| = \nabla_{x'} v_m \nabla_{x'} |v_m| \leq |\nabla_{x'} v_m|^2$ a.e. in C_R, and similarly for the partial derivative with respect to λ. Young's inequality implies

$$(2\beta)|v_m|^{2\beta-1} \leq \frac{2\beta-1}{2\beta} |v_m|^{2\beta} + \frac{1}{2\beta}(2\beta)^{2\beta} = \left(1 - \frac{1}{2\beta}\right) |v_m|^{2\beta} + (2\beta)^{2\beta-1}.$$

Hence we obtain

$$\begin{aligned}
&\int_{C_R} \xi^2 |v_m|^{2\beta} (|\nabla_{x'} v|^2 + w_2(\partial_\lambda v)^2) w_1 dx' d\lambda \\
&\leq \left(1 - \frac{1}{2\beta}\right) \int_{C_R} (|\nabla_{x'} v_m|^2 + w_2(\partial_\lambda v_m)^2) |v_m|^{2\beta} \xi^2 w_1 dx' d\lambda \\
&+(2\beta)^{2\beta-1} \int_{C_R} (|\nabla_{x'} v_m|^2 + w_2(\partial_\lambda v_m)^2) \xi^2 w_1 dx' d\lambda
\end{aligned}$$

$$+2\int_{C_R}\xi\nabla_{x'}v\nabla_{x'}\xi|v_m|^{2\beta}w_1dx'd\lambda.$$

Thus, since $\nabla_{x'}v=\nabla_{x'}v_m$ for $|v|<m$ and $\nabla_{x'}v_m=0$ for $|v|>m$, and similarly for the partial derivative with respect to λ, we deduce

$$\begin{aligned}&\int_{C_R}\xi^2|v_m|^{2\beta}(|\nabla_{x'}v|^2+w_2(\partial_\lambda v)^2)w_1dx'd\lambda\\&\le(2\beta)^{2\beta}\int_{C_R}(|\nabla_{x'}v_m|^2+w_2(\partial_\lambda v_m)^2)\xi^2w_1dx'd\lambda\\&+4\beta\int_{C_R}\xi\nabla_{x'}v\nabla_{x'}\xi|v_m|^{2\beta}w_1dx'd\lambda.\end{aligned}$$

Therefore by Cauchy inequality on the second term of the right hand side, we obtain that

$$\begin{aligned}&\int_{C_R}\xi^2|v_m|^{2\beta}(|\nabla_{x'}v|^2+w_2(\partial_\lambda v)^2)w_1dx'd\lambda\\&\le(2\beta)^{2\beta}\int_{C_R}(|\nabla_{x'}v_m|^2+w_2(\partial_\lambda v_m)^2)\xi^2w_1dx'd\lambda\\&+\beta^2\int_{C_R}|\nabla_{x'}\xi|^2|v_m|^{2\beta}w_1dx'd\lambda.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_{C_R}\xi^2|v_m|^{2\beta}(|\nabla_{x'}v_m|^2+w_2(\partial_\lambda v_m)^2)w_1dx'd\lambda\\&\le(2\beta)^{2\beta}\int_{C_R}(|\nabla_{x'}v_m|^2+w_2(\partial_\lambda v_m)^2)\xi^2w_1dx'd\lambda\\&+\beta^2\int_{C_R}|\nabla_{x'}\xi|^2|v_m|^{2\beta}w_1dx'd\lambda.\end{aligned}$$

In the following we write $v=v_m$ and then let $m\to+\infty$. By (2.20) we deduce that

$$\begin{aligned}&\int_{C_R}\xi^2|v|^{2\beta}(|\nabla_{x'}v|^2+w_2(\partial_\lambda v)^2)w_1dx'd\lambda\\&\le4(2\beta)^{2\beta}\int_{C_R}|\nabla_{x'}\xi|^2w_1dx'd\lambda+\beta^2\int_{C_R}|\nabla_{x'}\xi|^2|v|^{2\beta}w_1dx'd\lambda.\end{aligned}\tag{2.22}$$

We next use Young's inequality

$$\begin{aligned}|\nabla_{x'}(\xi|v|^\beta)|^2&\le2|\nabla_{x'}\xi|^2|v|^{2\beta}+2\beta^2\xi^2|v|^{2\beta-2}|\nabla_{x'}v|^2\\&\le2|\nabla_{x'}\xi|^2|v|^{2\beta}+2\xi^2|\nabla_{x'}v|^2\left(\frac{\beta-1}{\beta}|v|^{2\beta}+\frac{1}{\beta}\beta^{2\beta}\right),\end{aligned}$$

and

$$\begin{aligned}(\partial_\lambda(\xi|v|^\beta))^2&=\beta^2\xi^2|v|^{2\beta-2}(\partial_\lambda v)^2\\&\le\xi^2(\partial_\lambda v)^2\left(\frac{\beta-1}{\beta}|v|^{2\beta}+\frac{1}{\beta}\beta^{2\beta}\right).\end{aligned}$$

From (2.22) and (2.20),

$$\int_{C_R}(|\nabla_{x'}(\xi|v|^\beta)|^2+w_2(\partial_\lambda\xi|v|^\beta)^2)w_1dx'd\lambda$$

$$
\begin{aligned}
&\leq C\left\{(2\beta)^{2\beta}\int_{C_R}|\nabla_{x'}\xi|^2 w_1 dx'd\lambda\right.\\
&\left.+\beta^2\int_{C_R}|\nabla_{x'}\xi|^2|v|^{2\beta}w_1 dx'd\lambda\right\}.
\end{aligned}
$$

Applying the *local weighted Sobolev inequality* of Lemma 2.2, with $\kappa=\frac{N+\theta}{N+\theta-2}$ we deduce that

$$
\begin{aligned}
&\left(\int_{C_R}(\xi|v|^{\beta})^{2\kappa}w_1 dx'd\lambda\right)^{\frac{1}{\kappa}}\leq C_S V(C_R)^{\frac{1}{\kappa}-1}R^2 C\left\{(2\beta)^{2\beta}\int_{C_R}|\nabla_{x'}\xi|^2 w_1 dx'd\lambda\right.\\
&\left.+\beta^2\int_{C_R}|\nabla_{x'}\xi|^2|v|^{2\beta}w_1 dx'd\lambda\right\}.
\end{aligned}
$$

Choose the cut off function $\xi=1$ on B_{r_iR}, $\xi=0$ on $B_R\setminus B_{r_{i-1}R}$ with $r_i=\tau+\frac{1}{2^i}(\tau'-\tau)$, $0<\tau<\tau'<1$, $\beta_i=\kappa^{i-1}$ we deduce that

$$
\begin{aligned}
&\left(V^{-1}(C_{r_iR})\int_{C_{r_iR}}|v|^{2\kappa^i}w_1 dx'd\lambda\right)^{\frac{1}{\kappa}}\\
&\quad\leq\frac{C2^{2(i-1)}}{(\tau'-\tau)^2}\left\{(2\kappa^{i-1})^{2\kappa^{i-1}}+\kappa^{2i}V^{-1}(C_{r_{i-1}R})\int_{C_{r_{i-1}R}}|v|^{2\kappa^{i-1}}w_1 dx'd\lambda\right\}.
\end{aligned}
$$

Naming I_i to the power $2\kappa^{i-1}$ the left hand side, we have for $i=2,3,4,\cdots$ that

$$
I_i=C^{\frac{i}{2\kappa^{i-1}}}\left\{2\kappa^{i-1}+I_{i-1}\right\}
$$

with $C=C(N,\theta,c_1,c_2,\tau,\tau')>0$. Iterating the above inequality and observing that $\sum_{i=0}^{\infty}\frac{i}{\kappa^i}<\infty$, we obtain

$$
I_i\leq C\sum_{j=1}^{i}\kappa^{j-1}+CI_0\leq C\kappa^i+CI_0.
$$

Now for $\beta\geq 2$ there exists a i such that $2\kappa^{i-1}\leq\beta<2\kappa^i$. Hence

$$
\left(V^{-1}(C_{\tau R})\int_{C_{\tau R}}|v|^{\beta}w_1 dx'd\lambda\right)^{\frac{1}{\beta}}\leq CI_i\leq C\kappa^i+CI_0\leq C\beta+CI_0\leq C\beta,
$$

due to (2.21). This is the desired estimate since

$$
\int_{C_{\tau R}}|v|^{\beta}w_1 dx'd\lambda\leq V(C_{\tau R})C^{\beta}\beta^{\beta}\leq V(C_{\tau R})C^{\beta}e^{\beta}\beta!
$$

due to the Sterling formula for integer β. This proves (2.18) with $\alpha=\tau$.

To complete the proof we apply (2.18) to obtain

$$
\int_{C_{\alpha R}}e^{p_0v}w_1 dx'd\lambda\leq 2\,V(C_{\alpha R}),\qquad\int_{C_{\alpha R}}e^{-p_0v}w_1 dx'd\lambda\leq 2\,V(C_{\alpha R}).
$$

Multiplying the two and recalling the definition of v the result follows. □

We next have

Proof of Theorem 2.4 Let $p_0 \in (0, 1)$ as given by Lemma 2.7. From Lemma 2.6 and for some constant $C_1 = C_1(\alpha, N, \theta, c_1, c_2)$,

$$\sup_{C_{\alpha^2 R}} u^{-1} = \frac{1}{\inf_{C_{\alpha^2 R}} u} \leq C_1 \left(\frac{1}{V(C_{\alpha R})} \int_{C_{\alpha R}} u^{-p_0}\, w_1 dx' d\lambda \right)^{\frac{1}{p_0}},$$

hence,

$$C_1^{-1} \left(\frac{1}{V(C_{\alpha R})} \int_{C_{\alpha R}} u^{-p_0}\, w_1 dx' d\lambda \right)^{-\frac{1}{p_0}} \leq \inf_{C_{\alpha^2 R}} u,$$

Similarly we have

$$\sup_{C_{\alpha^2 R}} u \leq C_2 \left(\frac{1}{V(C_{\alpha R})} \int_{C_{\alpha R}} u^{p_0}\, w_1 dx' d\lambda \right)^{\frac{1}{p_0}}.$$

Using Lemma 2.7 we conclude that

$$\sup_{C_{\alpha^2 R}} u \leq C_H \inf_{C_{\alpha^2 R}} u.$$

with $C_H = C_1 C_2 4^{\frac{1}{p_0}}$ independent of R. □

We finally have

Proof of Theorem 1.3 (b): Take instead of u the function $v := u - \inf_S u$ which is still a nonnegative solution of $\mathcal{L}v = 0$ in S, having now $\inf_S v = 0$. From the definition of infimum, for any $\epsilon > 0$, there exists $x_\epsilon = (x'_\varepsilon, \lambda) \in S$ such that $v(x_\epsilon) < \epsilon$. As a consequence of the Harnack inequality, Theorem 2.4, for any $R \geq |x'_\epsilon|/\alpha$, we have $\sup_{C_{\alpha R}} v \leq C_H \inf_{C_{\alpha R}} v < C_H \epsilon$; thus $v(x) \leq C_H \epsilon$ in all $C_{\alpha R}$ for any R big enough, whence $0 \leq v(x) \leq C_H \epsilon$ for all $x \in S$. The claim follows from the arbitrariness of ϵ. □

3 Critical and supercritical cases: Proof of Theorem 1.4

To obtain the result we make use of the oscillation decrease method, cf Sect. 4.3 of [13], as adapted in Theorem 1.4 of [19] to the anisotropic setting.

For $R \geq 2$ we define

$$L_R := \{(x', \lambda) \in S : |x'| < R,\ \varphi(\lambda) < R^m\}$$

where

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t) dt. \tag{3.1}$$

Let us define the notion of weak solutions of $\mathcal{L}u = 0$ that we consider throught this section. The norm of the energy space is given by

$$||u||^2_{H^1_{w_1,w_2}(L_R)} := \int_{L_R} \left(u^2 + |\nabla_{x'} u|^2 + w_2 (\partial_\lambda u)^2\right) w_1 dx' d\lambda.$$

We define the space $\mathring{H}^1{}_{w_1,w_2}(L_R)$ is the completion of $C_0^\infty(L_R)$ under the above norm. Then, u is a weak solution if $u \in H^1_{w_1,w_2}(L_R)$ and for any $\psi \in \mathring{H}^1{}_{w_1,w_2}(L_R)$ there holds

$$\int_{L_R} (w_1 \nabla_{x'} u \nabla_{x'} \psi + w_1 w_2 \partial_\lambda u \partial_\lambda \psi)\, dx' d\lambda = 0,$$

As usually, $u \in H^1_{w_1,w_2}(L_R)$ is a weak subsolution in L_R provided that

$$\int_{L_R} (w_1 \nabla_{x'} u \nabla_{x'} \psi + w_1 w_2 \partial_\lambda u \partial_\lambda \psi)\, dx' d\lambda \leq 0,$$

for any $0 \leq \psi \in \mathring{H}^1{}_{w_1,w_2}(L_R)$ and similarly for the weak supersolution.

It is easily seen that we have the following *volume doubling property*, that is, there exists a positive constant C_D (independent of R) such that

$$V(L_{2R}) \leq C_D V(L_R)$$

for every $R \geq 2$, where $V(D) := \int_D w_1 dx' d\lambda$. We also note that

$$\int_{L_R} \left(|\nabla_{x'} \varphi|^2 + w_2 (\partial_\lambda \varphi)^2\right) w_1 dx' d\lambda = \omega_{N-1} R^{N-1} (R^m - 1). \tag{3.2}$$

Our first result reads

Lemma 3.1 (local weighted Sobolev) *Suppose w_1, w_2 satisfy (H2) for some $\theta > 0$ and $m > 2$. Then for $q = \frac{2(N+\theta)}{N-2+\theta}$ there exists a positive constant C_S such that for any $R \geq 2$ and for all $f \in C_0^\infty(L_R)$ there holds*

$$\left(\int_{L_R} |f|^q w_1 dx' d\lambda\right)^{\frac{2}{q}} \leq C_S R^2 (V(L_R))^{\frac{2}{q}-1} \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2\right) w_1 dx' d\lambda. \tag{3.3}$$

Proof The proof is similar to the proof of Lemma 2.2, we therefore sketch it. By scaling in the x'–variables

$$x' = Ry', \qquad g(y', \lambda) = f(y'R, \lambda) \in C_0^\infty(|y'| < 1, \varphi < R^m),$$

estimate (3.3) takes the following equivalent form

$$\left(\int_{\{|y'|<1,\ \varphi<R^m\}} |g|^q w_1 dy' d\lambda\right)^{\frac{2}{q}} \leq C_S \int_{\{|y'|<1,\ \varphi<R^m\}} \left(|\nabla_{y'} g|^2 + R^2 w_2 (\partial_\lambda g)^2\right) w_1 dy' d\lambda;$$

Since $\varphi(\lambda) < R^m$ is equivalent to $\varphi^{\frac{2}{m}}(\lambda) < R^2$, it is enough to establish

$$\left(\int_{\{|y'|<1,\ \varphi<R^m\}} |g|^q w_1 dy' d\lambda\right)^{\frac{2}{q}} \leq C_S \int_{\{|y'|<1,\ \varphi<R^m\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2\right) w_1 dy' d\lambda;$$

In fact we will prove a stronger inequality, namely for $g \in C_0^\infty(|y'| < 1,\ |\lambda| < 1)$.

$$\left(\int_{\{|y'|<1,\ |\lambda|<1\}} |g|^q w_1 dy' d\lambda\right)^{\frac{2}{q}} \leq C_S \int_{\{|y'|<1,\ |\lambda|<1\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2\right) w_1 dy' d\lambda.$$

It is enough to prove the result in the upper half cylinder, that is for $g \in C_0^\infty(|y'| < 1,\ |\lambda| < 1)$,

$$\left(\int_{\{|y'|<1,\ 0<\lambda<1\}} |g|^q w_1 dy' d\lambda\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ 0<\lambda<1\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2 (\partial_\lambda g)^2\right) w_1 dy' d\lambda. \tag{3.4}$$

To do this we change variables by

$$\begin{aligned} s = s(\lambda) &= \left(\int_\lambda^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t) dt\right) \left(\int_0^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t) dt\right)^{-1}, \\ h(y', s) &= g(y', \lambda), \quad \lambda \in (0, 1). \end{aligned} \tag{3.5}$$

Taking into account (H2), inequality (3.4) takes the form

$$\left(\int_{\{|y'|<1,\ 0<s<1\}} |h|^q s^\theta dy' ds\right)^{\frac{2}{q}} \le C_S \int_{\{|y'|<1,\ 0<s<1\}} \left(|\nabla_{y'} h|^2 + (\partial_s h)^2\right) s^\theta dy' ds,$$

where $h \in C_0^\infty(|y'| < 1)$, with $h(y', 0) = 0$. This is true by Proposition 2.1 of [9]. □

Next, after recalling that

$$\bar{f} := \frac{1}{V(L_R)} \int_{L_R} f(x', \lambda)\, w_1 dx' d\lambda$$

we have

Lemma 3.2 (local weighted Poincaré) *Suppose w_1, w_2 satisfy (H2) for some $\theta > 0$ and $m > 2$. Then, there exists a positive constant C_P such that, for every $R \ge 2$ and every $f \in C^1(\overline{L_R})$ satisfying $f = 0$ on $\{|x'| \le R,\ \varphi = R^m\}$, there holds*

$$\int_{L_R} |f - \bar{f}|^2 w_1 dx' d\lambda \le C_P R^2 \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2\right) w_1 dx' d\lambda. \tag{3.6}$$

Moreover, if in addition f is such that

$$V\left(\{(x', \lambda) \in L_R : f(x', \lambda) = 0\}\right) \ge \frac{1}{2} V(L_R),$$

then, we also have

$$\int_{L_R} f^2(x', \lambda) w_1 dx' d\lambda \le C_P R^2 \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2 (\partial_\lambda f)^2\right) w_1 dx' d\lambda. \tag{3.7}$$

Proof Assuming that (3.6) has been established, we first show that it implies (3.7). To this end we show that if f satisfies $V(\{f = 0\} \cap L_R) \ge \frac{1}{2} V(L_R)$, we then have

$$\int_{L_R} f^2 w_1 dx' d\lambda \le 2 \int_{L_R} |f - \bar{f}|^2 w_1 dx' d\lambda. \tag{3.8}$$

Indeed (3.8) follows easily from the following computation:

$$\begin{aligned} \int_{L_R} |f - \bar{f}|^2 w_1 dx' d\lambda &= \int_{L_R} f^2 w_1 dx' d\lambda - \frac{(\int_{L_R} f w_1 dx' d\lambda)^2}{V(L_R)} \\ &= \int_{L_R} f^2 w_1 dx' d\lambda - \frac{(\int_{\{f \ne 0\} \cap L_R} f w_1 dx' d\lambda)^2}{V(L_R)} \end{aligned}$$

$$\geq \int_{L_R} f^2 w_1 dx' d\lambda - \frac{\left(\int_{\{f\neq 0\}\cap L_R} f^2 w_1 dx' d\lambda\right) V(\{f\neq 0\}\cap L_R)}{V(L_R)}$$
$$= \int_{L_R} f^2 w_1 dx' d\lambda - \frac{\left(\int_{L_R} f^2 w_1 dx' d\lambda\right) V(\{f\neq 0\}\cap L_R)}{V(L_R)}$$
$$\geq \frac{1}{2}\int_{L_R} f^2 w_1 dx' d\lambda,$$

and (3.8) follows. In the sequel we will give the proof of (3.6).

Since

$$\int_{L_R} |f-\bar f|^2 w_1 dy' d\lambda = \min_{\xi\in\mathbb{R}} \int_{L_R} |f-\xi|^2 w_1 dy' d\lambda,$$

it is enough to prove that for every $f\in C^1(\overline{L_R})$ satisfying $f=0$ on $\{|x'|\leq R,\ \varphi=R^m\}$, the following inequality holds

$$\int_{L_R} |f-\xi|^2 w_1 dx' d\lambda \leq C_P R^2 \int_{L_R} \left(|\nabla_{x'} f|^2 + w_2(\partial_\lambda f)^2\right) w_1 dx' d\lambda, \tag{3.9}$$

for a particular choice of the constant ξ that we will specify later.

Once again we rescale by

$$x' = Ry', \qquad g(y',\lambda) = f(y'R,\lambda),$$

and (3.9) takes the following equivalent form

$$\int_{\{|y'|<1,\ \varphi<R^m\}} |g-\xi|^2 w_1 dy' d\lambda \leq C_P \int_{\{|y'|<1,\ \varphi<R^m\}} \left(|\nabla_{y'} g|^2 + R^2 w_2(\partial_\lambda g)^2\right) w_1 dy' d\lambda$$

for any $g\in C^1(\overline{\{|y'|<1,\ \varphi<R^m\}})$ such that $g=0$ on $\{|y'|\leq 1,\ \varphi=R^m\}$. This inequality will follow after establishing

$$\int_{\{|y'|<1,|\lambda|<1\}} |g-\xi|^2 w_1 dy' d\lambda \leq C_P \int_{\{|y'|<1,\ |\lambda|<1\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2(\partial_\lambda g)^2\right) w_1 dy' d\lambda \tag{3.10}$$

for any $g\in C^1(\overline{\{|y'|<1,\ |\lambda|<1\}})$.

To prove (3.10), once again we work in the upper half cylinder and choose

$$\xi = \frac{\int_{|y'|<1} g(y',0)dy'}{\omega_{N-1}}.$$

We will show that for every $g\in C^1(\overline{\{|y'|<1, 0<\lambda<1\}})$ there holds

$$\int_{\{|y'|<1,\ 0<\lambda<1\}} |g-\xi|^2 w_1 dy' d\lambda$$
$$\leq C_P \int_{\{|y'|<1,\ 0<\lambda<1\}} \left(|\nabla_{y'} g|^2 + \varphi^{\frac{2}{m}} w_2(\partial_\lambda g)^2\right) w_1 dy' d\lambda, \tag{3.11}$$

Using (H2) and making the change of variables (3.5) we are lead to prove that for $\xi = \frac{\int_{|y'|<1} h(y',1)dy'}{\omega_{N-1}}$ and $\theta>0$ there holds,

$$\int_{\{|y'|<1,\ 0<s<1\}} |h-\xi|^2 s^\theta dy' ds \leq C_P \int_{\{|y'|<1,\ 0<s<1\}} \left(|\nabla_{y'} h|^2 + (\partial_s h)^2\right) s^\theta dy' ds,$$

taking into account that $h(y', 0) = 0$. This inequality is quite similar to (2.9).

We next have

Theorem 3.3 *(Density Theorem) Suppose w_1, w_2 satisfy (H2) for some $\theta > 0$ and $m > 2$. Let u be a weak subsolution in L_R with*

$$0 \le u \le 2,$$

and $u \in H^1_{w_1,w_2}(L_R)$ for every $R \ge 2$. Suppose in addition that

$$V(\{u \ge 1\} \cap L_R) \ge \frac{1}{2} V(L_R).$$

Then, there exists $\alpha \in (0, 1)$ and $R_0 \ge 2$ such that for any $R \ge R_0$, there holds

$$\inf_{L_{\alpha R}} u \ge \delta.$$

Constants α, R_0, δ depend only on N, θ, m, c_1, c_2.

Proof For $\varepsilon > 0$ we define $u_\varepsilon = u + \varepsilon \ge \varepsilon > 0$. We will work with u_ε in place of u so that all subsequent quantities exist. For convenience we drop the subscript ε.

For $\alpha \in (0, 1)$ we define

$$\psi(x) = \begin{cases} u(x) + \left(\frac{\alpha}{2}\right)^m, & A_R, \\ u(x) + \frac{\varphi(\lambda)}{R^m}, & L_R - A_R \end{cases} \tag{3.12}$$

where,

$$A_R = \left\{ |x'| < R,\ \varphi(\lambda) < \left(\frac{\alpha R}{2}\right)^m \right\}.$$

For $\beta \ge 0$, we also define

$$\phi := (-\ln \psi - \beta)_+.$$

Function ϕ is also a subsolution, since in the region $\{\phi > 0\}$ we have $-\mathcal{L}\phi \le 0$. We also notice that when $\varphi(\lambda) = R^m$ then $\psi = u+1$ and therefore $\phi = 0$ there. We next test $-\mathcal{L}\phi \le 0$ with the function $\phi\xi^2$ where $\xi = \xi(x')$ is a standard cut off function in $B'_R = \{|x'| < R\}$. We integrate by parts and eventually arrive at

$$\int_{L_R} (w_1 |\nabla_{x'}(\phi\xi)|^2 + w_1 w_2 \partial_\lambda (\phi\xi)^2) dx' d\lambda \le C \int_{L_R} |\nabla_{x'}\xi|^2 \phi^2 w_1 dx' d\lambda.$$

Making use of the *local weighted Sobolev inequality* in Lemma 3.1 we have

$$\left(\int_{L_R} |\phi\xi|^q w_1 dx' d\lambda \right)^{\frac{2}{q}} \le C R^2 V^{\frac{2}{q}-1}(L_R) \int_{L_R} |\nabla_{x'}\xi|^2 \phi^2 w_1 dx' d\lambda$$

for $q = \frac{2(N+\theta)}{N+\theta-2}$. Holder inequality implies

$$\int_{L_R} (\phi\xi)^2 w_1 dx' d\lambda \le C R^2 V^{\frac{2}{q}-1}(L_R) \left(\int_{L_R} |\nabla_{x'}\xi|^2 \phi^2 w_1 dx' d\lambda \right) V^{1-\frac{2}{q}}(\operatorname{supp}(\phi\xi) \cap L_R). \tag{3.13}$$

We will use (3.13) for various choices of functions. More precisely, for $\eta_k \in C^1(0,1)$ with $\eta_k(y) = 1$ if $y \le \frac{1}{2} + \frac{1}{2^{k+2}}$ and $\eta(y) = 0$ if $y \ge \frac{1}{2} + \frac{1}{2^{k+1}}$, we set $\xi_k(x') = \eta_k\left(\frac{|x'|}{R}\right)$ and

$$\beta_k := E(1 - 2^{-k}), \quad \phi_k = (-\ln\psi - \beta_k)_+, \qquad \text{for } k = 0, 1, \dots,$$

for some positive constant E to be made more precise later. We also set

$$b_k := \frac{1}{V(L_R)} \int_{L_R} (\phi_k \xi_k)^2 w_1 dx' d\lambda,$$

Using (3.13) as well as the facts that

ϕ_k is decreasing in k,

$1 \le \xi_{k-1}$ in $\operatorname{supp}(\xi_k)$,

$1 \le \frac{2^k}{E}\phi_{k-1}$ in $\operatorname{supp}(\xi_k\phi_k)$,

we obtain that, for $\sigma = 1 - \frac{2}{q} = \frac{2}{N+\theta} \in (0,1)$, we have,

$$b_k \le \frac{C}{E^{2\sigma}}(2^{2k} b_{k-1})^{1+\sigma}, \qquad k = 1, 2, \dots.$$

Choosing $E = D b_0^{\frac{1}{2}}$ for appropriately large constant D, as in [19], we inductively prove that

$$b_k \le \frac{b_0}{2^{\frac{2k(1+\sigma)}{\sigma}}}, \qquad k = 1, 2, \dots.$$

It follows that $\lim_{k\to+\infty} b_k = 0$ and since $\lim_{k\to+\infty}\beta_k = E$, we conclude that $\sup_{L_{\frac{R}{2}}}(-\ln\psi - E) \le 0$. Recalling that $\phi_0 = (-\ln\psi)_+$ and the choice of the constant E, we have

$$\sup_{L_{\frac{R}{2}}} \phi_0 \le C V^{-\frac{1}{2}}(L_R) \left(\int_{L_R} \phi_0^2 w_1 dx' d\lambda \right)^{\frac{1}{2}}.$$

Next we will use the *local weighted Poincaré inequality* (3.7). Note that $\{\phi_0 = 0\} = \{-\ln\psi \le 0\} = \{\psi \ge 1\}$ contains the set $\{u \ge 1\}$. The weighted volume of the latter set satisfies the assumption of Lemma 3.2, we therefore arrive at

$$\sup_{L_{\frac{R}{2}}} \phi_0 \le C R V^{-\frac{1}{2}}(L_R) \left(\int_{L_R} (w_1|\nabla_{x'}\phi_0|^2 + w_1 w_2 (\partial_\lambda \phi_0)^2) dx' d\lambda \right)^{\frac{1}{2}}.$$

Using the specific form of ψ cf (3.12) and splitting suitably the integral cf (3.12) we estimate

$$\begin{aligned} &\int_{L_R} (w_1|\nabla_{x'}\phi_0|^2 + w_1 w_2(\partial_\lambda\phi_0)^2) dx' d\lambda \\ &\quad \le \frac{C}{R^{2m}} \int_{L_R} (w_1|\nabla_{x'}\varphi|^2 + w_1 w_2 (\partial_\lambda \varphi)^2) dx' d\lambda + \frac{C}{R^2} V(L_R), \end{aligned}$$

Using also the estimate (3.2),

$$\int_{L_R} (w_1|\nabla_{x'}\varphi|^2 + w_1 w_2(\partial_\lambda\varphi)^2) dx' d\lambda \le C R^{N-1+m},$$

we eventually conclude

$$\sup_{L_{\frac{R}{2}}} \phi_0 \le CRV^{-\frac{1}{2}}(L_R)R^{\frac{N-1-m}{2}} + C = C(R^{1-\frac{m}{2}} + 1).$$

Since $m > 2$ the first term in the right hand side tends to zero as R tends to $+\infty$. Therefore, there exists $R_0 > 0$ such that $\sup_{L_{\frac{R}{2}}} \phi_0 \le C$, for any $R \ge R_0$.

Whence in a smaller set $\{|x'| < \frac{R}{2}, \varphi(\lambda) < \left(\frac{\alpha}{2}R\right)^m\}$, we deduce that $e^{-\phi_0} \le \psi \le u + \left(\frac{\alpha}{2}\right)^m$. As a consequence for α small enough, there exists $\delta > 0$ such that $\inf_{L_{\frac{\alpha R}{2}}} u \ge \delta$, for any $R \ge R_0$ and this completes the proof. □

As usual we define

$$\mathrm{osc}_{L_R} u := \sup_{L_R} u - \inf_{L_R} u = M_R - m_R.$$

Lemma 3.4 *(Oscillation decrease) Suppose w_1, w_2 satisfy (H2) for some $\theta > 0$ and $m > 2$. Let u be a weak solution in L_R with*

$$0 \le u \le 2,$$

and $u \in H^1_{w_1,w_2}(L_R)$ for every $R \ge 2$. Then, there exists $\alpha \in (0, 1)$ and $R_0 \ge 2$ such that

$$osc_{L_{\alpha R}} u \le \left(1 - \frac{\delta}{2}\right) osc_{L_R} u \tag{3.14}$$

for some $\delta > 0$ and all $R \ge R_0$. The constants α, R_0, δ depend only on the constants of the problem, that is on N, θ, m, c_1, c_2.

In fact, the α and δ appearing in the Lemma are the same as the ones given by Theorem 3.3.

Proof In L_R we define the following nonnegative solutions

$$u_1 = \frac{u - m_R}{M_R - m_R} \text{ and } u_2 = \frac{M_R - u}{M_R - m_R}.$$

Note that $u_1 + u_2 = 1$ and $0 \le 2u_i \le 2$, $i = 1, 2$. Therefore either $V(\{2u_1 \ge 1\} \cap L_R) \ge \frac{1}{2}V(L_R)$ or else $V(\{2u_2 \ge 1\} \cap L_R) \ge \frac{1}{2}V(L_R)$. Suppose the first one is true. We then apply Theorem 3.3 to $2u_1$ to deduce that there exists $\alpha \in (0, 1)$ and $R_0 \ge 2$ such that for any $R \ge R_0$, there holds

$$inf_{L_{\alpha R}} 2u_1 \ge \delta \;\; \Leftrightarrow \;\; inf_{L_{\alpha R}} u_1 \ge \frac{\delta}{2}.$$

Therefore

$$\frac{m_{aR} - m_R}{M_R - m_R} \ge \frac{\delta}{2} \quad \Rightarrow \quad \mathrm{osc}_{L_{\alpha R}} u = M_{\alpha R} - m_{\alpha R} \le \left(1 - \frac{\delta}{2}\right) \mathrm{osc}_{L_R} u.$$

We argue similarly in the case $V(\{2u_2 \ge 1\} \cap L_R) \ge \frac{1}{2}V(L_R)$. □

We are now ready to give the proof of Theorem 1.4

Proof of Theorem 1.4 Let u be a bounded solution. We may assume that it is nonnegative and we consider the function $v = \frac{2u}{\|u\|_{L^\infty(S)}}$ so that $0 \le v \le 2$.

For R large enough we apply iteratively the Lemma 3.4 to obtain

$$\mathrm{osc}_{L_R} v \le \left(1 - \frac{\delta}{2}\right)^k \mathrm{osc}_{L_{\frac{R}{\alpha^k}}} v \le 2\left(1 - \frac{\delta}{2}\right)^k, \quad k = 1, 2, \dots.$$

Therefore

$$\mathrm{osc}_{L_R} u \le \left(1 - \frac{\delta}{2}\right)^k ||u||_{L^\infty(S)}, \quad k = 1, 2, \dots.$$

Passing to the limit $k \to \infty$ we conclude that u is constant in L_R for any $R \ge R_0$, hence in S.

4 The distance function weight and final remarks

In this section we first make specific choices of the weights w_1, w_2 and give the proof of Theorems 1.1 and 1.2. We next present some extensions of our results.

We make the following choices

$$w_1(\lambda) = (1 - |\lambda|)^\alpha \quad \text{and} \quad w_2(\lambda) = (1 - |\lambda|)^\nu.$$

Proof of Theorem 1.1 It is a consequence of Theorem 1.3. For part (a) we note that $\alpha > -1$ is equivalent to $w_1 \in L^1(0,1)$ and $\nu < 1 - \alpha$ is equivalent to $(w_1 w_2)^{-1} \in L^1(0,1)$. Since $\alpha > -1$ and $\nu < 1 - \alpha$ it follows that $\nu < 2$, which is equivalent to $w_2^{-\frac{1}{2}} \in L^1(0,1)$. Similarly, for part (b) when $1 - \alpha \le \nu < 2$ then $(w_1 w_2)^{-1} \notin L^1(0,1)$, and (H1) is satisfied by choosing $\theta = \frac{\alpha + \frac{\nu}{2}}{1 - \frac{\nu}{2}} \ge 1$. □

We next have

Proof of Theorem 1.2 It is a consequence of Theorem 1.4. As we have seen, $\alpha > -1$ is equivalent to $w_1 \in L^1(0,1)$. We next note that $(w_2)^{-\frac{1}{2}} \notin L^1(0,1)$ corresponds to $\nu \ge 2$. Then,

$$\varphi(\lambda) = 1 + \int_0^{|\lambda|} (w_1 w_2)^{-1}(t)dt = 1 + \int_0^{|\lambda|} (1 - |t|)^{-\alpha-\nu} dt \sim (1 - |\lambda|)^{-(\alpha+\nu-1)},$$

for $|\lambda| \sim 1$. When $\frac{\alpha+\nu-1}{m} - \frac{\nu}{2} + 1 > 0 \Leftrightarrow m(\nu - 2) < 2(\alpha + \nu - 1)$, then

$$\varphi^{-\frac{1}{m}}(\lambda) w_2^{-\frac{1}{2}}(\lambda) \sim (1 - |\lambda|)^{\frac{\alpha+\nu-1}{m} - \frac{\nu}{2}} \in L^1(0,1).$$

Moreover in this case

$$\int_{|\lambda|}^1 \varphi^{-\frac{1}{m}}(t) w_2^{-\frac{1}{2}}(t) dt \sim (1 - |\lambda|)^{\frac{\alpha+\nu-1}{m} - \frac{\nu}{2} + 1}.$$

It follows that (H2) holds if we choose θ such that

$$\theta\left(\frac{\alpha + \nu - 1}{m} - \frac{\nu}{2} + 1\right) = \alpha + \frac{\nu}{2} - \frac{\alpha + \nu - 1}{m}. \tag{4.1}$$

If m is such that

$$\frac{\alpha+\nu-1}{2} > \frac{\alpha+\nu-1}{m} > \frac{\nu}{2} - 1,$$

then, since $\alpha > -1$ and $\nu \geq 2$ we have that $\frac{\alpha+\nu-1}{\alpha+\frac{\nu}{2}} < 2$ and it follows from (4.1) the positivity of θ. Hence, all hypothesis of Theorem 1.4 are satisfied and the result follows. □

We next give examples of weights w_1, w_2 that satisfy the hypotheses of Theorem 1.3(b) and are not covered by Theorem 1.1(b). To this end we consider weights which when $\frac{1}{2} \leq |\lambda| < 1$, they behave like

$$w_1(\lambda) \approx (1-|\lambda|)^{\alpha}(\ln(1-|\lambda|)^{-1})^{\beta},$$
$$w_2(\lambda) \approx (1-|\lambda|)^{\nu}(\ln(1-|\lambda|)^{-1})^{\mu},$$

whereas for $|\lambda| \leq \frac{1}{2}$ both weights are positive, bounded and stay away from zero. Straightforward elementary calculations show that

Lemma 4.1 *All assumptions of Theorem 1.3(b) are satisfied provided that either $\alpha = -1$, $\beta < -1$, $\nu = 2$, $2 < \mu \leq 1 - \beta$ and choosing $\theta = \frac{2\beta+\mu}{2-\mu} \geq 1$, or, $\alpha > -1$, $\nu = 1 - \alpha$, $\beta = -\mu$, $\mu \in \mathbb{R}$, and choosing $\theta = 1$, or, $\alpha > -1$, $2 > \nu > 1 - \alpha$, $\beta = -\frac{\mu(1+\alpha)}{2-\nu}$, $\mu \in \mathbb{R}$, and choosing $\theta = \frac{2\alpha+\nu}{2-\nu} \geq 1$.*

Remark (i) Our Liouville type results in the supercritical case, that is $\alpha > -1$ and $\nu > 2$ of Theorem 1.2, can be transformed to Liouville type results for the isotropic equation of the form

$$div((1+|s|)^{\tau}\nabla v(x',s)) = 0, \quad \text{in } \mathbb{R}^N \quad \text{for} \quad \tau = \frac{2\alpha+\nu}{2-\nu}; \tag{4.2}$$

here τ can be any number in the interval $(-\infty, -1)$. This can be done via the change of variables

$$s = \int_0^{\lambda} (1-|t|)^{-\frac{\nu}{2}}\, dt, \qquad v(x',s) = u(x,\lambda).$$

It follows that every bounded weak solution of (4.2) is constant.

(ii) In the critical case, that is $\nu = 2$, using the same change of variables, one can obtain Liouville type results for the isotropic equation of the form

$$div(e^{\tau|s|}\nabla v(x',s)) = 0 \text{ in } \mathbb{R}^N, \qquad \tau = -(1+\alpha),$$

here τ can be any number in the interval $(-\infty, 0)$.

We note that the above results do not follow by the ones by Bella and Schäffner [2] but they do follow from Theorem 2.1 in [M1].

(iii) Similarly, in the subcritical case, that is $1-\alpha \leq \nu < 2$, one obtains Liouville type results as in Theorem 1.1(b), for the equation

$$div((1-|s|)^{\tau}\nabla v(x',s)) = 0, \quad \text{in } \mathbb{R}^{N-1} \times (-1,1), \qquad \tau = \frac{2\alpha+\nu}{2-\nu};$$

here τ can be any number in the interval $[1, +\infty)$.

In an other direction, we note that the same results with Theorems 1.1 and 1.2 hold for more general operators that can be thought of as perturbations of the operators we considered so far. More precisely, let

$$\mathcal{L}'_{\alpha,\nu}\, u := div(B_{\alpha,\nu}(x',\lambda)\nabla u)$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{N-1} \frac{\partial}{\partial x_i}\left(A_{i,j}(1-|\lambda|)^{\alpha}\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial \lambda}\left(A_{N,N}(1-|\lambda|)^{\alpha+\nu}\frac{\partial u}{\partial \lambda}\right)\\
&\quad + \sum_{j=1}^{N-1} \frac{\partial}{\partial \lambda}\left(A_{N,j}(1-|\lambda|)^{\alpha+\frac{\nu}{2}}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{N-1} \frac{\partial}{\partial x_i}\left(A_{i,N}(1-|\lambda|)^{\alpha+\frac{\nu}{2}}\frac{\partial u}{\partial \lambda}\right),
\end{aligned}
\tag{4.3}
$$

in

$$S = \mathbb{R}^{N-1} \times (-1,1) \qquad N \geq 2,$$

where the $N \times N$ matrix $A = (A_{i,j})$ has bounded and measurable entries $A_{i,j} = A_{i,j}(x', \lambda)$ for $(x', \lambda) \in S$, and it is symmetric and uniformly elliptic, that is, for some constants $0 < c_0 \leq C_0$ the following inequalities hold true for any $\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $(x', \lambda) \in S$:

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{N} A_{i,j}(x',\lambda)\xi_i\xi_j \leq C_0|\xi|^2.$$

Equivalently, we have

$$
\begin{aligned}
&c_0(1-|\lambda|)^{\alpha}(|\xi'|^2 + (1-|\lambda|)^{\nu}|\xi_N|^2)\\
&\leq \sum_{i,j=1}^{N} (B_{\alpha,\nu})_{i,j}(x',\lambda)\xi_i\xi_j \leq C_0(1-|\lambda|)^{\alpha}(|\xi'|^2 + (1-|\lambda|)^{\nu}|\xi_N|^2).
\end{aligned}
$$

Clearly, the model operator $\mathcal{L}_{\alpha,\nu}$, defined in (1.2), follows from $\mathcal{L}'_{\alpha,\nu}$ in the special case where $A_{i,j} = \delta_{i,j}$, that is, when A is the identity matrix. By quite similar arguments one can prove

Theorem 4.2 *Let* $\alpha > -1$.
(a) *If in addition* $1-\alpha \leq \nu < 2$, *then any nonnegative weak solution of* $\mathcal{L}'_{\alpha,\nu}u = 0$ *in* S *is constant.*

(b) *If in addition* $\nu \geq 2$ *and*

$$
\begin{aligned}
&A_{i,N} \textit{ do not depend on } x_i, \quad i = 1, 2, \ldots, N\text{-}1,\\
&A_{N,N} \textit{does not depend on } \lambda,
\end{aligned}
$$

every bounded weak solutions of $\mathcal{L}'_{\alpha,\nu}u = 0$ *in* S *is constant.*

We can also consider the more general operator

$$
\begin{aligned}
\mathcal{L}'\, u &:= div(B(x',\lambda)\nabla u)\\
&= \sum_{i,j=1}^{N-1} \frac{\partial}{\partial x_i}\left(A_{i,j}w_1\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial \lambda}\left(A_{N,N}w_1 w_2\frac{\partial u}{\partial \lambda}\right)\\
&\quad + \sum_{j=1}^{N-1} \frac{\partial}{\partial \lambda}\left(A_{N,j}w_1\sqrt{w_2}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{N-1} \frac{\partial}{\partial x_i}\left(A_{i,N}w_1\sqrt{w_2}\frac{\partial u}{\partial \lambda}\right),
\end{aligned}
\tag{4.4}
$$

in

$$S = \mathbb{R}^{N-1} \times (-1,1) \quad N \geq 2$$

where the $N \times N$ matrix $A = (A_{i,j})$ has bounded and measurable entries $A_{i,j} = A_{i,j}(x', \lambda)$ for $(x', \lambda) \in S$, and it is symmetric and uniformly elliptic, that is, for some constants $0 < c_0 \le C_0$ the following inequalities hold true for any $\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and for any $(x', \lambda) \in S$:

$$c_0|\xi|^2 \le \sum_{i,j=1}^{N} A_{i,j}(x', \lambda)\xi_i\xi_j \le C_0|\xi|^2.$$

Equivalently

$$c_0 w_1(|\xi'|^2 + w_2|\xi_N|^2) \le \sum_{i,j=1}^{N} B_{i,j}(x', \lambda)\xi_i\xi_j \le C_0 w_1(|\xi'|^2 + w_2|\xi_N|^2).$$

We recall that $w_i = w_i(|\lambda|)$, $i = 1, 2$. The model operator $\mathcal{L}$ in (1.1) follows from $\mathcal{L}'$ in the special case where A is the $N \times N$ identity matrix. By quite similar arguments we have

Theorem 4.3 **(a)** *If w_1, w_2 satisfy (H1) for some $\theta \ge 1$, then any nonnegative weak solution of $\mathcal{L}'u = 0$ in S, is constant.*
(b) *If w_1, w_2 satisfy (H2) for some $\theta > 0$ and $m > 2$ and in addition,*

$$A_{i,N} \text{ do not depend on } x_i, \quad i = 1, 2, \dots, N\text{-}1,$$
$$A_{N,N} \text{ does not depend on } \lambda,$$

then any bounded weak solution of $\mathcal{L}'u = 0$ in S is constant.

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