L^p Hardy inequalities in the exterior of a ball

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October 22, 2025

Abstract

For $p>1$, $n\geq 2$ and $1<s<n$ we consider the weighted Hardy inequality

$$\int_{\bar{B}_1^c} \frac{|\nabla u(x)|^p}{(|x|-1)^{s-p}}\,dx \geq c(p,n,s)\int_{\bar{B}_1^c} \frac{|u(x)|^p}{(|x|-1)^{s}}\,dx, \qquad \forall u\in C_c^\infty(B_1^c),$$

in the complement of the unit ball. We first show that there is a critical value $n^*(p)$, that corresponds to having a bounded analytic to analytic connection of the ODE

$$\frac{dy}{dx}=\frac{n-1}{2p}\;\frac{(1+p(1-2x)y+(p-1)|y|^{\frac{p}{p-1}})}{x(1-x)}, \qquad 0<x<1.$$

This value is such that, for space dimensions $n\leq n^*(p)$ the best constant is equal to

$$c(p,n,s)=\min\left(\left(\frac{s-1}{p}\right)^p,\left(\frac{n-s}{p}\right)^p\right). \tag{0.1}$$

We next show that for $n>n^*(p)$ there exists a new parameter $1<s_1(p,n)<\frac{n+1}{2}$ which corresponds to having a bounded analytic to analytic connection of the ODE

$$\frac{dy}{dx}=\frac{-(n-1)xy+\frac{s-1}{p}\left[1+p\frac{n-s}{s-1}y+(p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0<x<1.$$

Whenever $s\in(1,s_1]\cup[n+1-s_1,n)$ the best constant is still given by (0.1), whereas for $s\in(s_1,n+1-s_1)$ the best constant takes on a smaller value. We also establish the following symmetry result

$$c(p,n,s)=c(p,n,n+1-s).$$

Our results extend for $p>1$, the ones obtained for $p=2$ in [8].

AMS Subject Classification: 35A23,34C37, 34B16, 35J92, 49R05, 35J20, 26D10.
Keywords: Hardy inequality, best constant, exterior of domain, distance function, p–Laplacian.

1 Introduction and main result

Let $p > 1$, $n \geq 2$ and Ω be a proper domain of $\mathbb{R}^n$. Under some rather weak regularity assumptions on the boundary $\partial\Omega$, there exists a positive constant C_Ω such that the following Hardy inequaliy holds true

$$\int_\Omega |\nabla u|^p dx \geq C_\Omega \int_\Omega \frac{|u|^p}{d^p} dx, \qquad \forall, u \in C_c^\infty(\Omega)$$

where $d(x) = \text{dist}(x, \partial\Omega)$, $x \in \Omega$, see e.g. [5, 6, 7, 17, 13, 2]. Under proper geometric assumptions on Ω one can identify the best constant C_Ω to be equal to $\left(\frac{p-1}{p}\right)^p$. For instance this is the case when Ω is convex, cf [14, 2] or even under the weaker assumption of weakly mean convexity, that is, domains that satisfy

$$-\Delta d(x) \geq 0, \qquad x \in \Omega,$$

in the sense of distributions, see [3, 2].

More general weighted Hardy inequalities are also considered

$$\int_\Omega \frac{|\nabla u|^p}{d^{s-p}} dx \geq C_{s,\Omega} \int_\Omega \frac{|u|^p}{d^s} dx, \qquad \forall u \in C_c^\infty(\Omega), \tag{1.1}$$

see e.g. [20, 16, 1, 2, 9]. Under convexity assumption on the domain Ω cf [1], or under weakly mean convexity assumption cf [18] the best constant $C_{s,\Omega}$ is shown to be

$$C_{s,\Omega} = \left(\frac{|s-1|}{p}\right)^p, \tag{1.2}$$

for $s > 1$.

When $s < 1$, in [2] section 3.4.3, was shown that

$$C_{s,\Omega} \geq \left(\frac{|s-1|}{p}\right)^p,$$

provided that Ω is such that Ω^c is a weakly mean convex domain. Under mild regularity assumptions on $\partial\Omega$ (in fact in a neighborhood of a point of the boundary is enough), one may test inequality (1.1) with a function that behaves like $d^{\frac{s-1}{p}+\varepsilon}(x)$ near the boundary of Ω and passing to the limit $\varepsilon \to 0^+$, one can easily conclude that $C_{s,\Omega} \leq \left(\frac{|s-1|}{p}\right)^p$ when $s \neq 1$, see [14, 3] for similar arguments. It then turns out that for the complement of a weakly mean convex domain there holds

$$C_{s,\Omega} = \left(\frac{|s-1|}{p}\right)^p, \qquad s < 1, \tag{1.3}$$

and therefore the constant of inequality (1.1), established in [2], is sharp.

In case $s > n$, it was shown that

$$C_{s,\Omega} \geq \left(\frac{s-n}{p}\right)^p,$$

without any geometrical assumption on Ω, besides $\Omega \neq \mathbb{R}^n$, cf [1, 9]. In general the lower bound $\left(\frac{s-n}{p}\right)^p$ is not sharp; see e.g. Theorem 1.5 below. If however we suppose that Ω^c is bounded, then testing with a function behaving near infinity like $d^{-\frac{s-n}{p}-\varepsilon}(x)$ and passing to the limit $\varepsilon \to 0^+$, one has that $C_{s,\Omega} \leq \left(\frac{s-n}{p}\right)^p$. In particular,

$$C_{s,\Omega} = \left(\frac{s-n}{p}\right)^p, \qquad\qquad s > n. \tag{1.4}$$

and the constant of inequality (1.1), established in [1] is sharp in case Ω^c is bounded.

In this work we focus in the case where $\Omega \subset \mathbb{R}^n$ is the complement of the unit ball, that is, $\Omega = \bar{B}_1^c$. This case was considered in [14, 4, 10]. By the proceeding comments when either $s < 1$ or else $s > n$ the best constant is given by (1.3) or (1.4) respectively. Therefore for $p > 1$ and $n \geq 2$, we concentrate in the range of the parameter $1 < s < n$ and we study the best constant $c(p, n, s)$ of the following Hardy inequality

$$\int_{\bar{B}_1^c} \frac{|\nabla u(x)|^p}{(|x|-1)^{s-p}}\, dx \geq c(p,n,s) \int_{\bar{B}_1^c} \frac{|u(x)|^p}{(|x|-1)^s}\, dx, \qquad \forall u \in C_c^\infty(B_1^c). \tag{1.5}$$

In the special case $p = 2$ the best constant was explicitly computed recently in [8]. The main result there, is the following

Theorem A *Let $n \geq 2$ and $s > 1$. The best constant of the Hardy inequality,*

$$\int_{\bar{B}_1^c} \frac{|\nabla u|^2}{(|x|-1)^{s-2}} dx \geq c(n,s) \int_{\bar{B}_1^c} \frac{u^2}{(|x|-1)^s} dx, \qquad \forall u \in C_c^\infty(\bar{B}_1^c),$$

(i) *in the case $n = 2, 3$, and $1 < s < n$, is given by*

$$c(n,s) = \begin{cases} \left(\frac{s-1}{2}\right)^2, & \text{if} \quad 1 < s \leq \frac{n+1}{2}, \\ \left(\frac{n-s}{2}\right)^2, & \text{if} \quad \frac{n+1}{2} < s < n. \end{cases},$$

and is not realized in the proper energy space,
(ii) *in the case $n > 3$ and $1 < s < n$, is given by*

$$c(n,s) = \begin{cases} \left(\frac{s-1}{2}\right)^2, & \text{if} \quad 1 < s \leq \frac{3n-5}{n-1}, \\ \frac{(n-2)(n-s-1)(s-2)}{(n-3)^2} & \text{if} \quad \frac{3n-5}{n-1} < s < \frac{n^2-3n+4}{n-1} \\ \left(\frac{n-s}{2}\right)^2, & \text{if} \quad \frac{n^2-3n+4}{n-1} \leq s < n. \end{cases}$$

Moreover, when

$$n > 3 \qquad \text{and} \qquad \frac{3n-5}{n-1} < s < \frac{n^2-3n+4}{n-1},$$

the best constant is realized by the function

$$u(x) = |x|^{2-n}(|x|-1)^{\frac{(n-2)(s-2)}{n-3}}, \qquad |x| > 1,$$

whereas in the other cases it is not realized in the proper energy space,
(iii) *in the case $s > n$, is given by*

$$c(n,s) = \left(\frac{n-s}{2}\right)^2,$$

and is not realized in the proper energy space.

By the proceeding discussion, testing inequality (1.5) either by a function behaving like $(|x|-1)^{\frac{s-1}{p}+\varepsilon}$ near the boundary of B_1^c or like $(|x|-1)^{-\frac{n-s}{p}-\varepsilon}$ near infinity, we conclude

$$c(p,n,s) \le \min\left(\left(\frac{s-1}{p}\right)^p, \left(\frac{n-s}{p}\right)^p\right).$$

As we shall see the best constant of (1.5) is related to the existence of positive radial smooth solutions of the Euler Lagrange equation in $(1,\infty)$.

$$\left(\frac{r^{n-1}|\phi'|^{p-2}\phi'}{(r-1)^{s-p}}\right)' + c(p,n,s)\ \frac{r^{n-1}|\phi|^{p-2}\phi}{(r-1)^s} = 0, \quad r>1. \tag{1.6}$$

When $1 < s \le \frac{n+1}{2}$ we define θ to be the unique solution of

$$c(p,n,s) = \theta^{p-1}(s-1-\theta(p-1)), \tag{1.7}$$

in the interval $\left[\frac{s-1}{p}, \frac{s-1}{p-1}\right)$. Changing variables by

$$y(x) = \frac{(r-1)^{p-1}}{\theta^{p-1}}\ \frac{|\phi'|^{p-2}\phi'}{|\phi|^{p-2}\phi}, \qquad x = \frac{1}{r}, \quad r>1, \tag{1.8}$$

y is defined for all $x \in (0,1)$ and satisfies the following singular ODE

$$\frac{dy}{dx} = \frac{-(n-1)xy + \left[s-1-\theta(p-1)+(n-s)y+\theta(p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0<x<1. \tag{1.9}$$

We initially study the case $s = \frac{n+1}{2}$, with $c(p,n,s) = \left(\frac{s-1}{p}\right)^p = \left(\frac{n-1}{2p}\right)^p$ in which case $\theta = \frac{n-1}{2p}$. The change of variables (1.8) leads to the study of solutions of the ODE

$$\frac{dy}{dx} = \frac{n-1}{2p}\ \frac{(1+p(1-2x)y+(p-1)|y|^{\frac{p}{p-1}})}{x(1-x)}, \qquad 0<x<1. \tag{1.10}$$

It turns out that for all values of the parameters $p>1$, $n\ge 2$, there are analytic at $(x,y)=(0,-1)$ solutions, that is, solutions $y(x)$ with $y(0)=-1$ and $y(x)$ is analytic in a neighborhood $(0,\varepsilon)$, for some $\varepsilon>0$. For fixed $p>1$ we are interested in the values of n (which we treat as a continuous parameter), for which the analytic solution exists for all $x\in(0,1)$ and connects to (1,1), that is $y(1)=1$. In a similar fashion, for all values of the parameters, one can produce solutions which are analytic at (1,1). As will shall see there is a *unique* value of the parameter n, that we denote by $n^*(p)$, with the property that the analytic at $(0,-1)$ solution is also analytic at (1,1), see Theorem 3.1. In particular, $n^*(p)$ is a continuous and strictly increasing function of p with $n^*(p)>2$ for $p>1$, cf Lemma 3.6. As shown in [8], $n^*(2)=3$, cf Theorem A. Our first result then reads

Theorem 1.1. *Let $p>1$, $2\le n\le n^*(p)$. Then*

$$c(p,n,s) = \begin{cases} \left(\frac{s-1}{p}\right)^p, & \text{if } 1<s\le\frac{n+1}{2}, \\ \left(\frac{n-s}{p}\right)^p, & \text{if } \frac{n+1}{2}<s<n. \end{cases}$$

We next consider the case where $p > 1$, $n > n^*(p)$ and $1 < s < \frac{n+1}{2}$ and we investigate when there holds $c(p,n,s) = \left(\frac{s-1}{p}\right)^p$, in which case $\theta = \frac{s-1}{p}$. The change of variables (1.8) leads to the following ODE

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{s-1}{p}\left[1 + p\frac{n-s}{s-1}y + (p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0 < x < 1.$$

For $x = 0$ the equation

$$H(y) := 1 + \frac{p(n-s)}{s-1}y + (p-1)|y|^{\frac{p}{p-1}} = 0,$$

has two negative roots satisfying

$$\rho_2 < -\left(\frac{n-s}{s-1}\right)^{p-1} < -1 < \rho_1 < 0.$$

There exists an analytic at $(0, \rho_2)$ solution. We are interested in these values of the parameter s for which the analytic at $(0, \rho_2)$ solution connects to $(1, 1)$. As a matter of fact there exists precisely one value of the parameter s, that we denote by $s_1(p, n)$, with $1 < s_1(p, n) < \frac{n+1}{2}$, for which the analytic at $(0, \rho_2)$ solution connects to $(1, 1)$ and at the same time it is analytic at $(1, 1)$ as well, cf Theorem 4.7 for the existence as well as for further properties of $s_1(p, n)$. It turns out that the best constant $c(p, n, s)$ retains the value $\left(\frac{s-1}{p}\right)^p$ as long as s is below $s_1(p, n)$ but it drops thereafter. More precisely, we have

Theorem 1.2. *Let $p > 1$, $n > n^*(p)$. Then*

$$\begin{aligned} c(p,n,s) &= \left(\frac{s-1}{p}\right)^p, \qquad 1 < s \le s_1(p,n) \\ c(p,n,s) &< \left(\frac{s-1}{p}\right)^p, \qquad s_1(p,n) < s \le \frac{n+1}{2}\ . \end{aligned}$$

By Theorem A, $s_1(2, n) = \frac{3n-5}{n-1}$. Concerning the case $\frac{n+1}{2} < s < n$ we have

Theorem 1.3. *(symmetry of the best constant) Let $p > 1$, $n \ge 2$ and $1 < s < n$. Then, the best constant satisfies*

$$c(p,n,s) = c(p,n,n+1-s).$$

In particular, if $p > 1$ and $n > n^*(p)$ then

$$\begin{aligned} c(p,n,s) &< \left(\frac{n-s}{p}\right)^p, \qquad \frac{n+1}{2} < s \le s_2(p,n) := n+1-s_1(p,n), \\ c(p,n,s) &= \left(\frac{n-s}{p}\right)^p, \qquad s_2(p,n) < s < n\ . \end{aligned}$$

By Theorem A, $s_2(2, n) = \frac{n^2-3n+4}{n-1}$. A depiction of the best constant $c(p, n, s)$ in case $n > n^*(p)$ is shown in fig. 1. See also fig. 3 in section 4.

Concerning the existence or nonexistence of minimizers for (1.5) we have

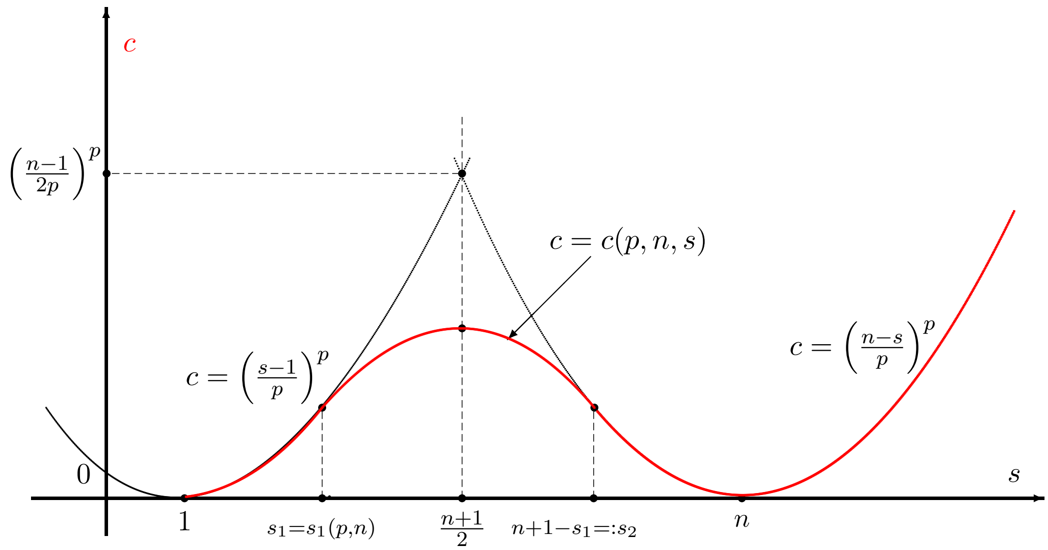

Figure 1: Case $p > 1$, $n > n^*(p)$, $s > 1$. A depiction of the best constant $c = c(p, n, s)$ (in red) for inequality (1.5).

Theorem 1.4. *Let $p > 1$.*

(i) If either $2 \le n \le n^*(p)$ and $1 < s < n$ or else $n > n^*(p)$ and $s \in (1, s_1] \cup [n+1-s_1, n)$, then the best constant $c(p, n, s)$ is not realized in the proper energy space .

(ii) If $n > n^*(p)$ and $s_1 < s < n + 1 - s_1$, then the best constant $c(p, n, s)$ is realized by a unique minimizer (modulo a multiplicative constant) which is radial.

We finally consider the case of the complement of an infinite circular cylinder. Let $n \ge 2$, $m \ge 1$ and

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m, \quad |x| > 1, \quad y \in \mathbb{R}^m\} = \bar{B}_1^c \times \mathbb{R}^m \ .$$

Theorem 1.5. *For $p > 1$, $n \ge 2$, $m \ge 1$ and $1 < s < n$, the following Hardy inequality holds true*

$$\int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)} u(x, y)|^p}{(|x| - 1)^{s-p}} \, dx \, dy \ge c(p, n, s) \int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|u|^p(x, y)}{(|x| - 1)^s} \, dx dy, \quad \forall u \in C_c^\infty(\bar{B}_1^c \times \mathbb{R}^m),$$

where the constant $c(p, n, s)$ is the one in (1.5) and it is sharp. This time however, the best constant is never realized in the proper energy space.

In our approach the change of variables (1.7), (1.8) plays a very important role. The choice of θ in (1.7) is suggested by the asymptotic behaviour of the solution of the Euler Lagrange equation (1.6) near the boundary, see Remark 2 after Theorem 5.2. The change of variables (1.8) normalizes the solution $\phi(r)$ and leads to a relatively simple ODE (1.9) that we analyse in detail. Through these change of variables two important parameters in the problem emerge, namely, $n^*(p)$ and $s_1(p, n)$. For these values of the parameters, the corresponding ODE (1.9) admits solution $y(x)$, $x \in (0, 1)$ which is analytic at both ends, $x = 0$ and $x = 1$, see Theorem 3.1 and section 3 for various properties of $n^*(p)$ and Theorem 4.7 and section 4 for various properties of $s_1(p, n)$.

2 Preliminaries

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $p > 1$, $s > 1$. Suppose that for some positive costant c we have a positive smooth function $\psi = \psi(x)$ that solves the Euler Lagrange*

$$\nabla \cdot \left(\frac{|\nabla \psi|^{p-2} \nabla \psi}{d^{s-p}} \right) + c \, \frac{\psi^{p-1}}{d^s} = 0, \qquad x \in \Omega. \tag{2.1}$$

Then, for the same value of the constant c we have the inequality

$$\int_\Omega \frac{|\nabla u(x)|^p}{d^{s-p}(x)} \, dx \geq c \int_\Omega \frac{|u(x)|^p}{d^s(x)} \, dx, \qquad \forall u \in C_c^\infty(\Omega). \tag{2.2}$$

Proof: Let any smooth vector field $\mathbf{F}$ in Ω and $u \in C_c^\infty(\Omega)$. In a similar way as in Theorem 4.1 of [3], starting from the quantity $\int_\Omega \nabla u \cdot \mathbf{F} \, dx$, an integration by parts and then Holder's inequality yield the following inequality

$$\int_\Omega \frac{|\nabla u(x)|^p}{d^{s-p}(x)} \, dx \geq \int_\Omega \left(\nabla \cdot \mathbf{F}(x) - (p-1) d^{\frac{s-p}{p-1}} |\mathbf{F}(x)|^{\frac{p}{p-1}} \right) |u(x)|^p \, dx.$$

We choose for $\mathbf{F}$,

$$\mathbf{F}(x) = - \frac{|\nabla \psi(x)|^{p-2} \nabla \psi(x)}{d^{s-p} \, \psi^{p-1}(x)}.$$

Using the fact that ψ satisfies (2.1), $\mathbf{F}$ is easily seen to satisfy

$$-\nabla \cdot \mathbf{F} + (p-1) d^{\frac{s-p}{p-1}} |\mathbf{F}|^{\frac{p}{p-1}} + \frac{c}{d^s} = 0,$$

from which the desired inequality follows.

□

Remark 1 In fact, it is well known that the converse of Proposition 2.1 holds true. That is, if inequality (2.2) holds true for some $c > 0$ then the corresponding Euler Lagrange equation (2.1) has a positive smooth solution in Ω.

Remark 2 In general there are several positive constants c for which (2.1) has a positive smooth solution. Our aim is to find the largest possible constant c for our problem at hand, where $\Omega = B_1^c$.

Remark 3 Since our domain is radial we study radial solutions of the Euler Lagrange. As we shall see in section 6, this is enough to get the best constant in the general case.

For $n \geq 2$, $p > 1$ and $1 < s < n$ it is easy to establish that for some positive constant c we have the inequality

$$\int_1^{+\infty} r^{n-1} \frac{|u'(r)|^p}{(r-1)^{s-p}} \, dr \geq c \int_1^{+\infty} r^{n-1} \frac{|u(r)|^p}{(r-1)^s} \, dr, \qquad \forall u \in C_c^1(1, +\infty).$$

We also set

$$c(p,n,s) = \inf_{u \in C_c^1(1,+\infty)} \frac{\int_1^{+\infty} r^{n-1} \frac{|u'(r)|^p}{(r-1)^{s-p}} \, dr}{\int_1^{+\infty} r^{n-1} \frac{|u(r)|^p}{(r-1)^s} \, dr}.$$

As a consequence, $\|u\| = \left(\int_1^{+\infty} r^{n-1}\frac{|u'(r)|^p}{(r-1)^{s-p}}\,dr\right)^{1/p}$ is a norm and we denote the completion of $C_c^1(1,+\infty)$ under this norm by $W_0^{1,p,s}(1,+\infty)$. In particular

$$c(p,n,s) = \inf_{u\in W_0^{1,p,s}(1,+\infty)} \frac{\int_1^{+\infty} r^{n-1}\frac{|u'(r)|^p}{(r-1)^{s-p}}\,dr}{\int_1^{+\infty} r^{n-1}\frac{|u(r)|^p}{(r-1)^{s}}\,dr}. \tag{2.3}$$

We next have

Theorem 2.2. *Let $n \geq 2$, $p > 1$ and $1 < s < n$. If*

$$c(p,n,s) < \min\left\{\left(\frac{s-1}{p}\right)^p, \left(\frac{n-s}{p}\right)^p\right\},$$

then, there exists a radial positive minimizer of (2.3) in $W_0^{1,p,s}(1,+\infty)$, which solves the corresponding Euler Lagrange.

The proof follows by quite similar arguments as in [4, 15, 19]. Alternatively, one can adapt the arguments in [12].

3 Existence and properties of $n^*(p)$

In this section we cosider the case $n \geq 2$, $p > 1$ and $s = \frac{n+1}{2}$. Although our interest is in the case where $n \geq 2$ is a natural number, in sections 3, 4, 5 and 6 we treat n as a continuous parameter in $[2,+\infty)$. We will study solutions of the ODE (1.10) which we recall:

$$\frac{dy}{dx} = \frac{n-1}{2p}\,\frac{(1+p(1-2x)y+(p-1)|y|^{\frac{p}{p-1}})}{x(1-x)}, \qquad 0<x<1, \quad y\in\mathbb{R}. \tag{3.1}$$

Elementary Calculus shows that

$$1+p(1-2x)y+(p-1)|y|^{\frac{p}{p-1}} \geq 1-|1-2x|^p > 0, \qquad x\in(0,1).$$

For $x=0$, the quantity

$$1+py+(p-1)|y|^{\frac{p}{p-1}} \geq 0,$$

has a double root at $y=-1$ and is strictly positive for $y\neq -1$.

Similarly, for $x=1$, the quantity

$$1-py+(p-1)|y|^{\frac{p}{p-1}} \geq 0,$$

has a double root at $y=1$ and is strictly positive for $y\neq 1$. In conclusion, the critical point of (3.1) are $(0,-1)$ and $(1,1)$. Moreover solutions of (3.1) are monotone increasing.

Our main interest is in the existence of bounded solutions of (3.1), that connect the points $(0,-1)$ and $(1,1)$ that is solutions $y(x)$ with the property $\lim_{x\to 0^+} y(x) = -1$ and $\lim_{x\to 1^-} y(x) = 1$. To this end we define

Definition: By AA connection we mean a solution $y(x)$ of (3.1) which is analytic in a neighborhood of $x=0$ with $y(0)=-1$ and at the same time is also analytic in a neighborhood of $x=1$ with $y(1)=1$; by AN connection we nean a solution which is analytic in a neighborhood of $x = 0$ with $y(0) = -1$ and is not analytic at $x = 1$, and similarly for NA.

For later use we denote by $F(p, n, x, y)$ the right hand side of (3.1), that is

$$F(p,n,x,y) = \frac{n-1}{2p} \, \frac{(1 + p(1-2x)y + (p-1)|y|^{\frac{p}{p-1}})}{x(1-x)}, \qquad 0 < x < 1,$$

We have the following monotonicity properties

$$\frac{\partial}{\partial p} F(p,n,x,y) = -\frac{n-1}{2p^2} \, \frac{(1 - |y|^{\frac{p}{p-1}} + |y|^{\frac{p}{p-1}} \ln |y|^{\frac{p}{p-1}})}{x(1-x)} < 0, \quad |y| \neq 1, \quad x \in (0,1), \tag{3.2}$$

as well as

$$\frac{\partial}{\partial n} F(p,n,x,y) = \frac{1 + p(1-2x)y + (p-1)|y|^{\frac{p}{p-1}}}{2px(1-x)} > 0, \quad |y| \neq 1, \quad x \in (0,1). \tag{3.3}$$

We define

$$\begin{aligned} \mathcal{N}(p) \;&=\; \{n \geq 2 : \text{ there exists an AN connection} \\ &\qquad \text{between } (0,-1) \text{ and } (1,1) \text{ of ODE (3.1)}\}. \end{aligned} \tag{3.4}$$

Our main result in this section is

Theorem 3.1. *Let $p > 1$. $\mathcal{N}(p)$ is a nonempty open bounded interval. We also define*

$$n^*(p) := \sup \mathcal{N}(p).$$

Then $2 < n^*(p) \notin \mathcal{N}(p)$ and for $n = n^*(p)$ the ODE (3.1) has a unique AA connection between $(0,-1)$ and $(1,1)$. Moreover, $n^*(p)$ is a continuous and strictly increasing function of p with the following upper bounds

(a) *if $1 < p \leq 2$ then $n^*(p) \leq 3$,*

(b) *if $p > 2$ then*

$$n^*(p) \leq 2p - 1 + 2\sqrt{(p-1)(p-2)}.$$

In addition we have (all connections refer to the ODE (3.1)).

(i) *for $2 \leq n < n^*$ there exist a AN connection y_1, between $(0,-1)$ and $(1,1)$ and a NA connection y_2 between the same points, such that $y_b(x) = -y_a(1-x)$,*

(ii) *for $n = n^*$ there is a unique solution y_0 connecting $(0,-1)$ to $(1,1)$ which in addition is an AA connection and satisfies $y_0(x) = -y_0(1-x)$.*

(iii) *for $n > n^*$ there are no connection between $(0,-1)$ and $(1,1)$. In particular the analytic solutions to either $(0,-1)$ or $(1,1)$ blow up at some finite x.*

See fig. 2. In the rest of the section we will present a series of Lemmas that will eventually lead to the proof of Theorem 3.1.

Lemma 3.2. *Let $n \geq 2$ and $p > 1$.*

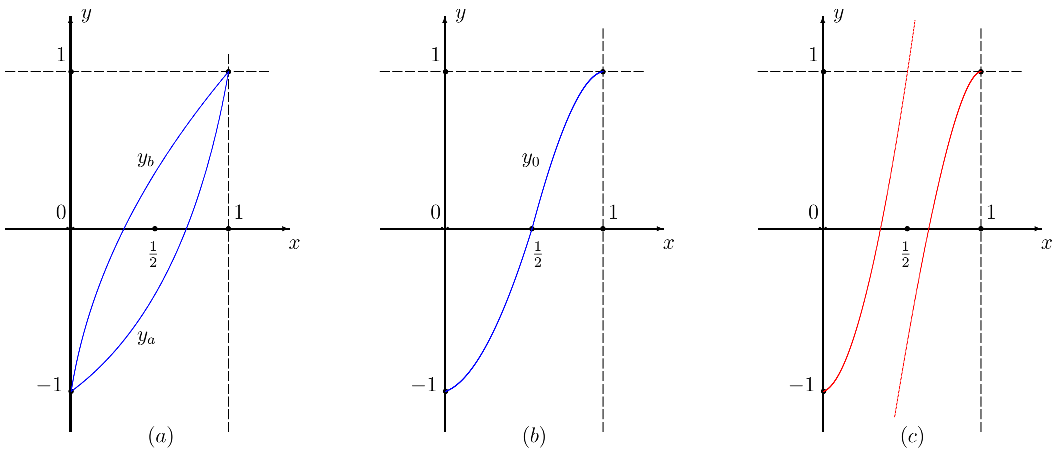

Figure 2: (a) Case $2 \le n < n^*(p)$: existence of both AN and NA connections, (b) case $n = n^*(p)$: existence of a unique connection, which is AA (c) case $n > n^*(p)$: nonexistence of connecting orbits. All connections refer to the ODE (3.1).

(a) There exists a unique analytic solution $y_a(x)$ of (3.1) near $(x, y) = (0, -1)$. Moreover for some $\varepsilon > 0$ and any $x \in [0, \varepsilon)$ there holds

$$y_a(x) = -1 + (n-1)x + \frac{n-1}{2}\left(\frac{(n-1)^2}{4(p-1)} - (n-2)\right)x^2 + O(x^3). \tag{3.5}$$

(b) If for some $\varepsilon \in (0, 1)$ there exists a solution $y(x)$ of (3.1) in $(0, \varepsilon)$ that in addition satisfies

$$y(x) \le y_a(x) \quad \textit{for} \quad x \in (0, \varepsilon) \quad \textit{and} \quad \lim_{x \to 0^+} y(x) = -1,$$

then necessarily $y(x) = y_a(x)$, $x \in (0, \varepsilon)$.

Similarly,

(c) There exists a unique analytic solution $y_b(x)$ of (3.1) near $(x, y) = (1, 1)$. Moreover for some $\varepsilon > 0$ and any $x \in [1 - \varepsilon, 1)$ there holds

$$y_b(x) = 1 + (n-1)(x-1) + \frac{n-1}{2}\left((n-2) - \frac{(n-1)^2}{4(p-1)}\right)(x-1)^2 + O((x-1)^3). \tag{3.6}$$

(d) If for some $\varepsilon \in (0, 1)$ there exists a solution $y(x)$ of (3.1) in $(1-\varepsilon, 1)$ that in addition satisfies

$$y(x) \ge y_b(x) \quad \textit{for} \quad x \in (1-\varepsilon, 1) \quad \textit{and} \quad \lim_{x \to 1^-} y(x) = 1,$$

then necessarily $y(x) = y_b(x)$, $x \in (1 - \varepsilon, 1)$.

(e) The two analytic solutions are connected via

$$y_b(x) = -y_a(1-x), \qquad x \in (1-\varepsilon, 1). \tag{3.7}$$

Moreover if y_a is an AN connection then y_b is NA connection, equality (3.7) holds for all $x \in (0, 1)$ and in addition

$$y_b(x) > y_a(x), \qquad x \in (0, 1),$$

implying in particular that the root x_* of y_a satisfies $x_* > \frac{1}{2}$.

Proof: (a) We write the ODE in the following way

$$xy'(x) = \frac{n-1}{2p}\,\frac{(1+p(1-2x)y+(p-1)|y|^{\frac{p}{p-1}})}{1-x} =: f(x,y), \qquad 0<x<1.$$

We next apply Proposition 1.1.1. p. 261 of [11], in a neighbourhood of the point $(x=0, y=-1)$. Since $f(0,-1)=0$ and $\frac{\partial f}{\partial y}(0,-1)=0$, which is not a positive integer, we have the existence of a unique analytic solution in a neighborhood of the point $(0,-1)$. The asymptotics follow easily.

(b) Suppose on the contrary there is another solution y satisfying $y_a(x) > y(x)$ in $(0,\varepsilon)$ which tends to -1 as $x \to 0^+$. We define $\phi(x) = y_a(x) - y(x) > 0$. Clearly $\lim_{x\to 0^+}\phi(x) = 0$ and using the convexity of $|t|^{\frac{p}{p-1}}$ we have

$$|y(x)|^{\frac{p}{p-1}} \geq |y_a(x)|^{\frac{p}{p-1}} + \frac{p}{p-1}|y_a(x)|^{\frac{p}{p-1}-2}y_a(x)(y(x)-y_a(x)), \quad x\in(0,\varepsilon).$$

This implies that for x close to zero, ϕ satisfies,

$$\phi'(x) \leq \frac{n-1}{2}\,\frac{1-2x-|y_a(x)|^{\frac{1}{p-1}}}{x(1-x)}\,\phi(x),$$

whence, after integration

$$\frac{\phi(\varepsilon)}{\phi(x)} \leq e^{\frac{n-1}{2}\int_x^{\varepsilon}\frac{1-2t-|y_a(t)|^{\frac{1}{p-1}}}{t(1-t)}\,dt}, \quad x\in(0,\varepsilon). \tag{3.8}$$

Using the asymptotics of y_a we have that

$$\lim_{t\to 0^+}\frac{1-2t-|y_a(t)|^{\frac{1}{p-1}}}{t(1-t)} = -2+\frac{n-1}{p-1},$$

implying that as $x\to 0^+$ the left hand side of (3.8) goes to infinity whereas the right hand side is finite, thus leading to a contradiction.

Parts (c) and (d) are quite similar and we the proof. The equality in part (e) follows from the uniqueness of the analytic solutions, whereas the inequality is a consequence of parts (b) and (d).

□

For completeness we also consider the limiting case $p=1$

Lemma 3.3. *Consider the ODE*

$$\frac{dy}{dx} = \frac{n-1}{2}\,\frac{(1+(1-2x)y)}{x(1-x)}, \qquad 0<x<1. \tag{3.9}$$

(i) *for $1<n<2$ there exist an AN monotonic connection y_1, between $(0,-1)$ and $(1,1)$ and a NA connection y_2 between the same points, such that $y_2(x) = -y_1(1-x)$,*

(ii) *For $n=2$, the function $y_0(x) = 2x-1$ is a monotonic AA connection between $(0,-1)$ and $(1,1)$.*

(iii) *for $n>2$ there are no monotonic connections between $(0,-1)$ and $(1,1)$.*

Proof: Part (ii) is an easy calculation.

For part (i) we note that y_0 is a supersolution of (3.9). In addition the analytic at $(0,-1)$ solution has asymptotics $y_a(x) = -1 + \frac{2(n-1)}{3-n} + O(x^2)$. Since $\frac{2(n-1)}{3-n} < 2$, y_a starts below y_0 and therefore stays below y_0 for $x \in (0,1)$, it increases monotonicaly to $(1,1)$ and it is not analytic to $(1,1)$. In fact the analytic at $(1,1)$ stays above y_0 and is a NA connection. The relation $y_2(x) = -y_1(1-x)$ is a consequence of the uniqueness of the analytic solutions.

For part (iii) suppose that for $n > 2$ there is a monotonic connection $y_1(x)$. Then $y_2(x) = -y_1(1-x)$ is another monotonic connection. They are either the same or else the one is above the other. If they are equal then $y_1(1/2) = y_2(1/2) = 0$. If, say, $y_2(x) < y_1(x)$ then their roots are symmetric with respect to $x = 1/2$. Hence the solution $y(x)$ of (3.9) with $y(1/2) = 0$ is such that $y_2(x) < y(x) < y_1(x)$ and therefore it is another monotonic connection. In all cases, there is a monotonic connection with $y(1/2) = 0$. By a simple comparison argument with $n = 2$ we see that

$$y(x) > 2x - 1, \qquad x \in (\frac{1}{2}, 1). \tag{3.10}$$

We next distinguish two cases, $n \geq 3$ and $3 > n > 2$.

Suppose first that $n \geq 3$. Differentiating (3.9) we obtain

$$y''(x) = \frac{n-1}{4(x(1-x))^2}[(n-3)(1-2x)(1+(1-2x)y(x)) - 4x(1-x)y(x)] < 0, \quad x \in (\frac{1}{2}, 1).$$

It follows that $y'(x)$ is strictly decreasing and therefore $\lim_{x\to 1^-} y'(x) = l \in [-\infty, \infty)$. If $l = -\infty$ we have an immediate contradiction, since we assumed y to be increasing. If $l > -\infty$, using L'Hopital rule in (3.9), we arrive at

$$l = \frac{n-1}{2}\,\frac{-2-l}{-1} \;\Leftrightarrow\; -(n-1) = \frac{n-3}{2} l,$$

which is impossible for $n = 3$ whereas when $n > 3$ it gives $l < 0$ and contradicts the fact that y increases.

Now $3 > n > 2$. Applying Proposition 1.1.1, p. 261 of [11], in a neighbourhood of the point $(x = 1, y = 1)$ since the ODE (3.9) can be written as

$$(x-1)y'(x) = -\frac{n-1}{2}\,\frac{(1+(1-2x)y)}{x} =: f(x,y)$$

with $f(1,1) = 0$ and

$$\frac{\partial f}{\partial y}(1,1) = \frac{n-1}{2} \notin \mathbb{N},$$

we conclude the existence of an analytic solution at $(1,1)$, that we denote by $y_a(x)$. We claim that the connection $y(x)$ is in fact $y_a(x)$. If not, then $\sigma(x) := y_a(x) - y(x)$ satisfies near $(1,1)$

$$\sigma'(x) = \frac{n-1}{2}\,\frac{(1-2x)\sigma(x)}{x(1-x)},$$

and therefore

$$\sigma = k(x(1-x))^{\frac{n-1}{2}}, \qquad k \in \mathbb{R}^*.$$

Suppose that $k > 0$. Then,

$$y_a'(x) - y'(x) = k\frac{n-1}{2}(1-2x)(x(1-x))^{\frac{n-3}{2}},$$

and it follows that $y'(x) \to +\infty$ as $x \to 1^-$, which contradicts (3.10). If $k < 0$ then $y'(x) \to -\infty$ as $x \to 1^-$ which is impossible since y is increasing. Hence in all cases $y(x) = y_a(x)$.

The slope of $y = y_a$ at $(1,1)$ is $\frac{2(n-1)}{3-n}$. On the other hand from (3.10) the slope of y_a should be smaller than 2, that is $\frac{2(n-1)}{3-n} \leq 2 \Leftrightarrow n \leq 2$ which is a contradiction.

□

We next have

Lemma 3.4. *Let $p > 1$. For $n = 2$ there exists AN connection of* (3.1) *between* $(0,-1)$ *and* $(1,1)$ *and in addition the set $\mathcal{N}(p)$ is an interval, open in its right end. Moreover for $1 < p < q$ we have the following monotonicity property $\mathcal{N}(p) \subset \mathcal{N}(q)$.*

Proof: We initially establish that for $n = 2$ and any $p > 1$, there exists an AN connection. Because of (3.2) the function $y(x) = 2x - 1$ is a supersolution of (3.1) for $n = 2$. By standard arguments, at $(0,-1)$ there exists an analytic solution with asymptotics $y_*(x) = -1 + x + O(x^2)$, therefore $y_*(x) < y(x)$ near $x = 0$. By comparison $y_*(x)$ stays below y for all $x \in (0,1)$. The function $y_*(x)$ is not analytic at $x = 1$, since if it were the analytic, then we would have $y_*(x) < 2x - 1$ which implies $y_*'(1) \geq 2$. But this is a contradiction, since the slope of the analytic solution at $(1,1)$ is one.

Let $n \in \mathcal{N} = \mathcal{N}(p)$ and y_1 the corresponding AN connection. If y_2 is the NA solution then by Lemma 3.2 we have that the root of y_1 is at a point $x_* > \frac{1}{2}$. In fact this characterizes the AN connections.

Next we show that the set $\mathcal{N}$ is open in the relative topology. Let $n \in \mathcal{N}$ and y_n the corresponding AN connection that becomes zero at $x_*(n) > \frac{1}{2}$. We use continuous dependence of the analytic solutions at $(0,-1)$ in the interval $(0, x_*(n))$ to get for n' close to n we have that $x_*(n') > \frac{1}{2}$ and therefore is $y_{n'}$ is not analytic at $(1,1)$.

We now establish that $\mathcal{N}$ is connected. To this end we use the monotonicity of F with respect to n, cf (3.3), which implies that if for some n we have an AN connection y_n then for $2 \leq \bar{n} < n$ y_n satisfies $y_n' > F(p, \bar{n}, x, y_n)$, $x \in (0,1)$. On the other hand the analytic at $(0,-1)$ solution satisfies $y_{\bar{n}}(x) = -1 + (\bar{n} - 1)x + O(x^2)$ and therefore $y_{\bar{n}}(x) < y_n(x)$ near $x = 0$. It follows by comparison that $y_{\bar{n}}(x) < y_n(x)$ for all $x \in (0,1)$, and therefore $y_{\bar{n}}(x)$ is an AN connection. Hence $\bar{n} \in \mathcal{N}$.

Finally we establish the monotonicity property. Let $1 < p < q$, $n \in \mathcal{N}(p)$ and y_p the AN connection. Then, as before, we obtain

$$y_p'(x) > F(q, n, x, y_p(x)), \qquad x \in (0,1).$$

Using the asymptotics of $y_p(x)$ for x close to 0^+ from Lemma 3.2, as well as the fact that y_p is a supersolution, we conclude that $y_q(x) < y_p(x)$ for all $x \in (0,1)$. Hence y_q is an AN connection and as a consequence $n \in \mathcal{N}(q)$.

□

Lemma 3.5. *For $p > 1$, the set $\mathcal{N}(p)$ is bounded above. In particular*

(i) *for $1 < p \leq 2$ an upper bound for $\mathcal{N}(p)$ is 3,*

(ii) *for $p > 2$, an upper bound for $\mathcal{N}(p)$ is $2p - 1 + 2\sqrt{(p-1)(p-2)}$.*

Proof: Part **(i)** follows from the fact that $\mathcal{N}(2) = [2,3)$ and the monotonicity property of $\mathcal{N}(p)$, cf Lemma 3.4.

(ii) We will show that for any

$$n > 2p - 1 + 2\sqrt{(p-1)(p-2)} \;\Leftrightarrow\; p - 1 < \frac{(n-1)^2}{4(n-2)}, \tag{3.11}$$

there is no connection between $(0, -1)$ and $(1, 1)$. A consequence of the first inequality is that $n > 3$.

We will first establish that there exists $k > 1$ and close to one as well as $\lambda < n - 1$ and close to $n - 1$ such that

$$\underline{y}(x) = \lambda(x - 1) + k, \qquad x \in (0, 1), \tag{3.12}$$

is a subsolution of (3.1). In fact we will choose λ to satisfy

$$\frac{4(n-2)(p-1)}{n-1} \;<\; \lambda \;<\; n-1, \tag{3.13}$$

$$\frac{2(n-1)}{p} \;<\; \lambda \;<\; n-1, \tag{3.14}$$

$$2 + \frac{4(p-2)(n-2)}{n-3} \;<\; \lambda \;<\; n-1, \tag{3.15}$$

$$\frac{8(n-1)^2(p-1)}{(n-3+2p)^2} \;<\; \lambda \;<\; n-1, \tag{3.16}$$

and k to satisfy

$$1 \;<\; k \;<\; \frac{1}{2}\left(\lambda - \sqrt{\lambda(\lambda - \frac{2(n-1)}{p})}\right), \tag{3.17}$$

$$1 \;<\; k \;<\; \frac{(n-3)\lambda}{2(n-3+2(p-2)(n-2))}, \tag{3.18}$$

$$1 \;<\; k \;\leq\; \frac{\lambda\left(n-3+2p-\sqrt{(n-3+2p)^2 - \frac{8(n-1)^2(p-1)}{\lambda}}\right)}{4(p-1)(n-1)}. \tag{3.19}$$

For later use we note that since $\lambda > 2$, if k satisfies (3.17), it easily follows that

$$k - \lambda < -1. \tag{3.20}$$

Similarly, if k satisfies (3.18) or (3.19), we also have (3.20). Hence under assumption (3.13)– (3.19), inequality (3.20) holds true.

Before proceeding to the proof we check that the stated inequalities are well defined. To this end we first establish that the range of λ is nonvoid. Indeed for (3.14) we note

$$\frac{4(n-2)(p-1)}{n-1} < n - 1 \;\Leftrightarrow\; p - 1 < \frac{(n-1)^2}{4(n-2)},$$

which is (3.11). The nonvoidness of (3.14) follows, since $p > 2$. For (3.15) we need

$$2 + \frac{4(p-2)(n-2)}{n-3} < n - 1 \;\Leftrightarrow\; p - 1 < \frac{(n-1)^2}{4(n-2)}.$$

For (3.16) we need,

$$\frac{8(n-1)^2(p-1)}{(n-3+2p)^2} < n-1 \;\Leftrightarrow\; (n-1-2(p-1))^2 > 0;$$

the inequality is strict because of (3.11).

We next check the nonvoidness of the k inequalities (3.17)– (3.19). The square roots are well defined by (3.14) and (3.16). For (3.14) we note that

$$1 < \frac{1}{2}\left(\lambda - \sqrt{\lambda(\lambda - \frac{2(n-1)}{p})}\right) \;\Leftrightarrow\; \left(\frac{n-1}{p} - 2\right)\lambda + 2 > 0.$$

This is immediate for $1 \leq p \leq \frac{n-1}{2}$, whereas in the case where $p > \frac{n-1}{2}$ the above inequality is equivalent to $\lambda < \frac{2p}{2p-(n-1)}$. It is enough to have $\frac{2p}{2p-(n-1)} > n-1$ which is equivalent to $p - 1 < \frac{(n-1)^2}{2(n-2)} - 1$. The last inequality follows from (3.13) since

$$\frac{(n-1)^2}{4(n-2)} < \frac{(n-1)^2}{2(n-2)} - 1.$$

The nonvoidness of (3.18) follows from (3.15). Concerning (3.19),

$$1 < \frac{\lambda\left(n-3+2p - \sqrt{(n-3+2p)^2 - \frac{8(n-1)^2(p-1)}{\lambda}}\right)}{4(p-1)(n-1)}$$

is equivalent to

$$\lambda\sqrt{(n-3+2p)^2 - \frac{8(n-1)^2(p-1)}{\lambda}} < \lambda(n-3+2p) - 4(p-1)(n-1).$$

The right hand side of this is positive because of (3.16). Squaring both sides, the last inequality turns out to be equivalent to $\lambda < n-1$.

We now proceed to the proof that $\underline{y}(x) = \lambda(x-1) + k$, $x \in (0,1)$ is a subsolution of (3.1). To this end we will show

$$\lambda x(1-x) \leq \frac{n-1}{2p}\,(1 + p(1-2x)\underline{y}(x) + (p-1)|\underline{y}(x)|^{\frac{p}{p-1}}), \qquad x \in (0,1).$$

Changing variables by $t = \lambda(x-1) + k$ this is equivalent to

$$\begin{aligned} A(t) \;\; &:= \;\; \frac{n-1}{2p} - k\left(1 - \frac{k}{\lambda}\right) + \frac{n-3}{2}\left(\frac{2k}{\lambda} - 1\right)t - \frac{n-2}{\lambda}t^2 \\ &\quad + \frac{n-1}{2}\frac{p-1}{p}|t|^{\frac{p}{p-1}} \geq 0, \qquad k - \lambda < t < k. \end{aligned} \tag{3.21}$$

By straighforward calculations we have

$$\begin{aligned} A'(t) \;&=\; \frac{n-3}{2}\left(\frac{2k}{\lambda} - 1\right) - \frac{2(n-2)}{\lambda}t + \frac{n-1}{2}|t|^{-\frac{(p-2)}{p-1}}t, \quad t \in (k-\lambda, k), \\ A''(t) \;&=\; -\frac{2(n-2)}{\lambda} + \frac{n-1}{2(p-1)}|t|^{-\frac{(p-2)}{p-1}}, \quad t \in (k-\lambda, k) \setminus \{0\}. \end{aligned}$$

Function A is convex in the intervals $(-t_0,0)$ and $(0,t_0)$ and therefore in $(-t_0,t_0)$, where

$$t_0=\left(\frac{\lambda(n-1)}{4(n-2)(p-1)}\right)^{\frac{p-1}{p-2}}.$$

We note that

$$t_0>1\ \Leftrightarrow\ \lambda>\frac{4(n-2)(p-1)}{n-1},$$

which is true by (3.13). We also have

$$\begin{aligned}A(0)&=\frac{n-1}{2p}-k+\frac{k^2}{\lambda},\\ A'(0)&=\frac{n-3}{2}\left(\frac{2k}{\lambda}-1\right).\end{aligned}$$

Because of (3.14) and (3.17) we have that $A(0)>0$ and $A'(0)<0$.

We next study $A(t)$ in $k-\lambda\le t\le 0$. Since $A(0)>0$, $A'(0)<0$ and A is convex in $(-t_0,0)$ and concave in $(-\infty,-t_0)$ it follows that A is nonnegative provided that $A(k-\lambda)\ge 0$. But

$$A(k-\lambda)=\frac{2(n-1)}{p}\left(1+p(k-\lambda)+(p-1)|k-\lambda|^{\frac{p}{p-1}}\right)\ge 0.$$

Finally we study $A(t)$ in $0\le t\le k$. We first note that

$$A(k)=\frac{2(n-1)}{p}\left(1-pk+(p-1)|k|^{\frac{p}{p-1}}\right)\ge 0.$$

If A is decreasing in $[0,k]$ then clearly $A(t)\ge 0$ in $[0,k]$. If on the other hand function A has a minimum for some $\xi\in(0,k)$ then $A'(\xi)=0$. Straightforward calculations yield

$$A(\xi)=\frac{n-1}{2p}-k+\frac{k^2}{\lambda}+\frac{n-3}{2p}\left(\frac{2k}{\lambda}-1\right)\xi+\frac{(p-2)(n-2)}{p\lambda}\xi^2.$$

To proceed we define

$$B(t):=\frac{n-1}{2p}-k+\frac{k^2}{\lambda}+\frac{n-3}{2p}\left(\frac{2k}{\lambda}-1\right)t+\frac{(p-2)(n-2)}{p\lambda}t^2,\qquad t>0.$$

The minimum of B is taken at the point

$$t_1=\frac{(n-3)(\lambda-2k)}{4(n-2)(p-2)}.$$

As a consequence of (3.18) we have that $t_1\ge k$. Function B is convex and

$$\begin{aligned}B(0)&=A(0)>0,\\ B'(0)&=\frac{1}{p}A'(0)<0.\end{aligned}$$

It follows that $B(t)$ is decreasing in the interval $(0,k)$. Hence $A(\xi)=B(\xi)\ge B(k)$. We will show that $B(k)\ge 0$. We have

$$B(k)=\frac{n-1}{2p}-\left(1+\frac{n-3}{2p}\right)k+\frac{(p-1)(n-1)}{p\lambda}k^2.$$

The smallest root of this trinomial is in fact the right hand side of (3.19), and the positivity of $B(k)$ follows. Hence for $0 \leq t \leq k$ we have $A(t) \geq A(\xi) \geq 0$. This completes the proof of (3.21), and consequently of the fact that $\underline{y}$, as given by (3.12), is a subsolution of (3.1).

It remains to show that under condition (3.11) there is no connection between $(0, -1)$ and $(1, 1)$. Under condition (3.11) we have seen that there exist choices of k, λ satisfying (3.13) – (3.19), so that $\underline{y}$ as given by (3.12) is a subsolution of (3.1). Since $\underline{y}(0) = k - \lambda < -1$ any solution tending to $(0, -1)$, including the analytic at $(0, -1)$, stays above $\underline{y}$ and therefore blows up for some finite $x \in (0, 1)$. Similarly any solution tending to $(1, 1)$, including the analytic at $(1, 1)$, stays below $\underline{y}$ and blows up for some finite $x \in (0, 1)$. Therefore there is no connection between between $(0, -1)$ and $(1, 1)$.

□

We next have

Lemma 3.6. *For $p > 1$ and $n = n^*(p)$ the ODE (3.1) has an AA connection between $(0, -1)$ and $(1, 1)$. Moreover $n^*(p) > 2$, is a continuous and strictly increasing function of p.*

Proof: We first show that for $n = n^*(p)$ equation (3.1) has an AA connection. It follows from (3.3), that if n, $n' \in \mathcal{N}(p)$ with $n < n'$, the analytic at $(0, -1)$ solutions y_n and $y_{n'}$ respectively, satisfy

$$-1 < y_n(x) < y_n'(x) < 1, \qquad x \in (-1, 1).$$

Morever the analytic at $(0, -1)$ solution depends continuously on the parameters n and p in the compact intervals of $[-1, 1)$. As a consequence the analytic at $(0, -1)$, solution $y_{n^*(p)}$ is a connection between $(0, -1)$ and $(1, 1)$. Since $\mathcal{N}(p)$ is open by Lemma 3.4, $y_{n^*(p)}$ is also analytic at $(1, 1)$.

As a consequence of the monotonicity property of $\mathcal{N}(p)$ of Lemma 3.4, we have that if $1 < p < q$ then $n^*(p) \leq n^*(q)$. We will establish that we actually have strict inequality. Suppose on the contrary that $n^*(p) = n^*(q)$. We will study the analytic at $(0, -1)$ solution of (3.1) when the parameter is q, say, $y_q^a(x)$. Because of (3.2), $y_{n^*(p)}$ is a supersolution of the (3.1) with q in the place of p.

Using the he asymtotics of the analytic at $(0, -1)$ solution near $x = 0$, from Lemma 3.2, we get that

$$y_{n^*(p)}(x) > y_q^a(x), \tag{3.22}$$

initially near zero. Since $y_{n^*(p)}$ is a supersolution of (3.1) the above inequality holds for all $x \in (0, 1)$. We will show that y_q^a is not analytic at $(1, 1)$ and therefore $n^*(p) \in \mathcal{N}(q)$. Indeed, if it were analytic at $(1, 1)$, then the asymptotics of y_q^a near $x = 1$, implies that $y_q^a(x) > y_{n^*(p)}(x)$ near $x = 1$, which contradicts (3.22). This shows that $n^*(p) < n^*(q)$.

We finally establish the continuity of $n^*(p)$ with respect to p. Because of the monotonicity of $n^*(p)$, the following limits exist and we will show that

$$\lim_{r \to p^+} n^*(r) = n^*(p) = \lim_{r \to p^-} n^*(r).$$

Suppose that

$$\lim_{r \to p^+} n^*(r) = l > n^*(p).$$

Using the monotinicity (3.3), the analytic at $(0, -1)$ solution of (3.1) with $n = l$ has asymptotics $y_l^a(x) = -1 + (l - 1)x + O(x^2)$ and $y_l^a(x) > y_{n^*(p)}(x)$ as long as y_l^a exists.

We will show that y_l^a blows up at some $x \in (0,1)$. Indeed, if it connects to $(1,1)$ then it is not analytic at $(1,1)$, as one can see using the asymptotics of the analytic at $(1,1)$ and therefore $l \in \mathcal{N}(p) \Rightarrow l < n^*(p)$ which is a contradiction. Using the continuity of the analytic solutions with respect to r and n we conclude that for $|p-r|+|n-l|$ small, the analytic at $(0,-1)$ solution $y_{n,r}^a$ blows up as well. By the definition of the limit when r is close to p then $n^*(r)$ is close to l and this leads to a contradiction.

Suppose now that

$$n^*(p) > l = \lim_{r \to p^-} n^*(r).$$

Let y_p the AA connection of

$$y_p'(x) = F(p, n^*(p), x, y_p(x)),$$

and $y_{l,p}^a(x)$ the analytic at $(0,-1)$ solution of

$$y'(x) = F(p, l, x, y(x)).$$

Using the monotonicity property of (3.3) and the asymptotics of the analytic solution at $(0,-1)$ we conclude that $y_{l,p}^a(x) < y_p(x)$ in $(0,1)$ and therefore $y_{l,p}^a(x)$ is a connection between $(0,-1)$ and $(1,1)$. It cannot be AA connection since then $n^*(p) = l$ contrary to our assumption. Hence it is an AN connection. But then $y_{l,p}^a(x)$ becomes zero at $x_*(p) > \frac{1}{2}$. By continuous dependence of the analytic solution $y_{l,r}^a(x)$ at $(0,-1)$ of

$$y'(x) = F(r, l, x, y(x)),$$

in the interval $(0, x_*(n))$, we have that for $r < p$ and close to p, $y_{l,r}^a(x)$ will be close to $y_{l,p}^a(x)$ and therefore $x_*(r) > \frac{1}{2}$ and as a consequence $y_{l,r}^a(x)$ is an AN connection. This implies that $l < n^*(r)$ which is a contradiction.

$\square$

Proof of Theorem 3.1: The properties of the set $\mathcal{N}(p)$ as well as of $n^*(p)$ are established in Lemmas 3.4, 3.5 and 3.6. We next establish properties (i)–(iii).

The property of $n^*(p)$ of Part (ii) follows from Lemma 3.6 and the symmetry property of y_0 follows from Lemma 3.2. Part (i) follows from Lemma 3.4 in connection with Lemma 3.6. For part (iii) we note that if for some $n > n^*(p)$ we had a connection y then we would also have either an AN connection or else an AA connection. Indeed by Lemma 3.2 part(b) the analytic at $(0,-1)$ solution y_1 stays below y and is a connection. If it were an AN connection then we contradict the properties of $\mathcal{N}(p)$. If on the other hand it were an AA connection, and if we denote by y_0 the AA connection corresponding to $n^*(p)$ then since we have

$$y_1'(x) = F(p, n, x, y_1(x)) > F(p, n^*(p), x, y_1(x)), \qquad x \in (0,1),$$

in connection with the asymptotics of analytic solutions at $(0,-1)$ imply that $y_0(x) < y_1(x)$, $x \in (0,1)$. However, this inequality violates the asymptotics of analytic solutions at $(1,1)$.

$\square$

4 Existence and properties of $s_1(p, n)$

In this section we consider the case $n \geq 2$, $p > 1$ and $1 < s < \frac{n+1}{2}$. We will study the existence of bounded connecting orbits for the ODE (1.9) with the choice $\theta = (s-1)/p$, that is:

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{s-1}{p}\left[1 + p\frac{n-s}{s-1}y + (p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0 < x < 1. \tag{4.1}$$

For $x = 0$ the equation

$$H(y) := 1 + \frac{p(n-s)}{s-1}y + (p-1)|y|^{\frac{p}{p-1}} = 0,$$

has two negative roots satisfying

$$\rho_2 < -\left(\frac{n-s}{s-1}\right)^{p-1} < -1 < \rho_1 < 0.$$

We easily establish the following monotonicity properties

$$\frac{\partial \rho_2(p)}{\partial p} < 0, \qquad \frac{\partial \rho_2(s)}{\partial s} > 0, \qquad \frac{\partial \rho_2(n)}{\partial n} < 0. \tag{4.2}$$

Similarly, for $x = 1$, the quantity

$$1 - py + (p-1)|y|^{\frac{p}{p-1}} \geq 0,$$

has a double root at $y = 1$ and is strictly positive for $y \neq 1$.

There are three critical points of the ODE, namely $(0, \rho_1)$, $(0, \rho_2)$, $(1, 1)$, that will be important to our analysis. There are other critical points, that is, points at which the numerator of the right hand side of (4.1) is zero,

$$-(n-1)xy + \frac{s-1}{p}H(y) = 0, \qquad 0 < x < 1.$$

Clearly, they lie on the curve

$$x = P(y) := \frac{s-1}{p(n-1)} \frac{H(y)}{y}, \qquad 0 < x < 1, \tag{4.3}$$

which equivalently can be written as

$$x = P(y), \qquad \rho_2 < y < \rho_1.$$

If there is a pair (x_0, y_0) with $x_0 \in (0, 1)$ and $y_0 \in (\rho_2, \rho_1)$ such that

$$-(n-1)x_0y_0 + \frac{s-1}{p}H(y_0) = 0,$$

then the solution of the ODE with $y(x_0) = y_0$ is such that for all $x \in (0, x_0)$ there holds

$$-(n-1)xy + \frac{s-1}{p}H(y) < 0,$$

and therefore y is decreasing in $(0, x_0)$ with $\lim_{x\to 0^+} y(x) = \rho_1$. On the other hand for $x \in (x_0, 1)$, $y(x)$ is increasing for as long as it exists.

We are primarily interested in the existence of bounded solutions of (4.1), that connect the points $(0, \rho_2)$ and $(1, 1)$. To this end, similarly to the previous section, we define

Definition: (i) y is a connection between $(0, \rho_2)$ and $(1, 1)$ when $y(x)$ solves (4.1) in $(0, 1)$ with the property $\lim_{x\to 0^+} y(x) = \rho_2$ and $\lim_{x\to 1^-} y(x) = 1$.

(ii) y is an AA connection between $(0, \rho_2)$ and $(1, 1)$ when it is a connection between $(0, \rho_2)$ and $(1, 1)$ and in addition is analytic near $x = 0$ and near $x = 1$.

(iii) y is an AN connection between $(0, \rho_2)$ and $(1, 1)$ when it is a connection between $(0, \rho_2)$ and $(1, 1)$ and in addition is analytic near $x = 0$ but it is not analytic at $x = 1$.

(iv) We similarly define NA connections as well as connections between $(0, \rho_1)$ and $(1, 1)$.

If we denote by $F(p, n, s, x, y)$ the right hand side of (4.1) we have the following monotonicity properties

$$\frac{\partial}{\partial p} F(p, n, s, x, y) = -\frac{s-1}{p^2} \frac{(1 - |y|^{\frac{p}{p-1}} + |y|^{\frac{p}{p-1}} \ln |y|^{\frac{p}{p-1}})}{x(1-x)} < 0, \quad |y| \neq 1, \quad x \in (0, 1), \tag{4.4}$$

as well as

$$\frac{\partial}{\partial s} F(p, n, s, x, y) = \frac{1 - py + (p-1)|y|^{\frac{p}{p-1}}}{px(1-x)} > 0, \quad y \neq 1, \quad x \in (0, 1). \tag{4.5}$$

Lemma 4.1. *Let $n \geq 2$ and $p > 1$.*
(a) There exists a unique analytic solution $y_a(x)$ of (4.1) near $(x, y) = (0, \rho_2)$. Moreover for some $\varepsilon > 0$ and any $x \in [0, \varepsilon)$ there holds

$$y_a(x) = \rho_2 + \alpha x + \beta x^2 + O(x^3), \tag{4.6}$$

with

$$\begin{aligned} \alpha &= \frac{(n-1)\rho_2}{n - s - 1 - (s-1)|\rho_2|^{\frac{1}{p-1}}} = -\frac{(n-1)\rho_2}{1 - \frac{s-1}{p} H'(\rho_2)}, \\ \beta &= \frac{(n-2)\alpha - \frac{s-1}{2(p-1)}|\rho_2|^{\frac{p}{p-1}-2}\alpha^2}{n - s - 2 - (s-1)|\rho_2|^{\frac{1}{p-1}}}. \end{aligned}$$

(b) If for some $\varepsilon \in (0, 1)$ there exists a solution $y(x)$ of (4.1) in $(0, \varepsilon)$ that in addition satisfies

$$\lim_{x\to 0^+} y(x) = \rho_2,$$

then necessarily $y(x) = y_a(x)$, $x \in (0, \varepsilon)$.
(c) There exists a unique analytic solution $y_b(x)$ of (4.1) near $(x, y) = (1, 1)$. Moreover for some $\varepsilon > 0$ and any $x \in [1 - \varepsilon, 1)$ there holds

$$y_b(x) = 1-(n-1)(1-x)+\frac{n-1}{2}\left[(n-2) - \frac{(s-1)(n-1)}{2(p-1)}\right](1-x)^2+O((x-1)^3). \tag{4.7}$$

(d) If for some $\varepsilon \in (0,1)$ there exists a solution $y(x)$ of (4.1) in $(1-\varepsilon, 1)$ that in addition satisfies

$$y(x) \geq y_b(x) \quad for \quad x \in (1-\varepsilon, 1) \quad and \quad lim_{x \to 1^-} y(x) = 1,$$

then necessarily $y(x) = y_b(x)$, $x \in (1-\varepsilon, 1)$.

Proof: (a) We write the ODE in the following way

$$xy'(x) = \frac{-(n-1)xy + \frac{s-1}{p} H(y)}{1-x} =: f(x,y), \qquad 0 < x < 1.$$

We next apply Proposition 1.1.1. p. 261 of [11], in a neighbourhood of the point $(x = 0, y = \rho_2)$. Since $f(0, \rho_2) = 0$ and $\frac{\partial f}{\partial y}(0, \rho_2) = (s-1)\left(\frac{n-1}{s-1} - |\rho_2|^{\frac{1}{p-1}}\right) < 0$ we have the existence of a unique analytic solution in a neighborhood of the point $(0, \rho_2)$. The asymptotics follow easily.

(b) Suppose on the contrary there are two such solutions $y_1(x) > y_2(x)$ in $(0, \varepsilon)$. We define $\phi(x) = y_1(x) - y_2(x)$. Clearly $\lim_{x \to 0^+} \phi(x) = 0$ and is easily seen that ϕ satisfies the following ODE for some ξ such that $y_1(x) > \xi(x) > y_2(x)$,

$$\phi'(x) = \frac{-(n-1)x\phi(x) + \frac{s-1}{p} H'(\xi(x))\phi(x)}{x(1-x)}, \qquad 0 < x < \varepsilon.$$

From this we easily derive

$$\frac{\phi(x)}{(1-x)^{n-1}} = \frac{\phi(\varepsilon)}{(1-\varepsilon)^{n-1}} e^{\frac{s-1}{p} \int_\varepsilon^x \frac{H'(\xi(t))}{t(1-t)} dt}, \qquad 0 < x < \varepsilon.$$

Taking the limit $x \to 0^+$ we arrive at a contradiction: the left hand side tends to zero and the right hand side tends to infinity since $H'(\xi(t)) \to H'(\rho_2) < 0$.

(c) It is similar to (a). This time $\frac{\partial f}{\partial y}(1,1) = 0$.

(d) Suppose on the contrary there is another solution y satisfying $y(x) > y_b(x)$ in $(1-\varepsilon, 1)$, which tends to 1 as $x \to 1^-$. We define $\phi(x) := y(x) - y_b(x) > 0$. Clearly $\lim_{x \to 1^-} \phi(x) = 0$ and using the convexity of $|t|^{\frac{p}{p-1}}$ we have

$$|y(x)|^{\frac{p}{p-1}} \geq |y_b(x)|^{\frac{p}{p-1}} + \frac{p}{p-1} |y_b(x)|^{\frac{p}{p-1}-2} y_b(x)(y(x) - y_b(x)), \quad x \in (1-\varepsilon, 1).$$

Hence, for x close to one, ϕ satisfies,

$$\phi'(x) \geq \frac{-(n-1)x + (n-s) + (s-1)|y_b(x)|^{\frac{1}{p-1}}}{x(1-x)} \phi(x),$$

whence

$$\int_{1-\varepsilon}^x \frac{\phi'(t)}{\phi(t)} dt \geq \int_{1-\varepsilon}^x \frac{-(n-1)t + (n-s) + (s-1)|y_b(t)|^{\frac{1}{p-1}}}{t(1-t)} dt.$$

As $x \to 1^-$ the left hand side tends to $-\infty$ whereas the right hand side is finite since, using the asymptotics of y_b,

$$\lim_{t \to 1^-} \frac{-(n-1)t + (n-s) + (s-1)|y_b(t)|^{\frac{1}{p-1}}}{t(1-t)} = \frac{(n-1)(p-s)}{p-1},$$

leading to a contradiction.

□

We next prove some auxilliary Lemmas.

Lemma 4.2. *Let $n > 2$, $p > 1$. If*

$$0 < s - 1 < \frac{p(n-1)(n-2)}{1 + p(n-2) + (p-1)(n-2)^{\frac{p}{p-1}}},$$

then

$$\rho_2 < -(n-2).$$

Proof: It is a consequence of the fact that under our assumption on s, one easily computes that $H(-(n-2)) < 0$.

□

Lemma 4.3. *Let $n > 2$, $n \neq 3$ and $p > 2$. Then,*

$$2\frac{n-2}{n-1} < \frac{p(n-1)(n-2)}{1 + p(n-2) + (p-1)(n-2)^{\frac{p}{p-1}}},$$

whereas for $n = 3$ we have equality.

Proof: Setting $t = \frac{p}{p-1}$, the above inequality is equivalent to

$$Q(t) := t(n-2)^2 - 2(n-2)^t + 2 - t > 0, \qquad 1 < t < 2.$$

Function Q is strictly concave for $n \neq 3$, with $Q(1) > 0$ and $Q(2) = 0$, whence the result.

□

Lemma 4.4. *Let $n > 2$, $p > 1$. In addition, we assume that*

(i) if $p \geq 2$ then $0 < s - 1 < 2\frac{n-2}{n-1}$,

(ii) if $1 < p < 2$ then $0 < s - 1 < \min\left(\frac{2(p-1)(n-2)}{n-1}, \frac{p(n-1)(n-2)}{1+p(n-2)+(p-1)(n-2)^{\frac{p}{p-1}}}\right)$.

Then, $\bar{y}(x) = 1 - (n-1)(1-x)$ is a supersolution to the ODE (4.1).

Proof: We need to prove that

$$\frac{d\bar{y}}{dx} \geq \frac{-(n-1)x\bar{y} + \frac{s-1}{p}H(\bar{y})}{x(1-x)}, \quad 0 < x < 1.$$

After straightforward calculations this is equivalent to

$$Q(p,y) := \frac{n-2}{n-1} - \frac{s-1}{p} + \left(-\frac{2(n-2)}{n-1} + s - 1\right)y + \frac{n-2}{n-1}y^2 - \frac{(s-1)(p-1)}{p}|y|^{\frac{p}{p-1}} \geq 0,$$

for $-(n-2) \leq y \leq 1$.

We next consider the cases

(a) $p = 2$. We compute

$$Q(2,y) = \left(\frac{n-2}{n-1} - \frac{s-1}{2}\right)(y-1)^2 \geq 0.$$

(b) $p > 2$. In this case

$$\frac{\partial Q}{\partial p} = \frac{s-1}{p^2}\left[1 - |y|^{\frac{p}{p-1}} + |y|^{\frac{p}{p-1}} \ln |y|^{\frac{p}{p-1}}\right] \geq 0,$$

since $g(t) := 1 - t + t \ln t > 0$ for $t > 0$, $t \neq 1$ and $g(t) = 0$ for $t = 1$. As a consequence

$$Q(y,p) \geq Q(y,2) \geq 0.$$

(c) $1 < p < 2$. We now have

$$\begin{aligned}
\frac{\partial Q}{\partial y} &= -2\frac{n-2}{n-1} + s - 1 + 2\frac{n-2}{n-1}y - (s-1)|y|^{\frac{2-p}{p-1}}y, \\
\frac{\partial^2 Q}{\partial y^2} &= 2\frac{n-2}{n-1} - \frac{s-1}{p-1}|y|^{\frac{2-p}{p-1}}.
\end{aligned}$$

By our assumption

$$s - 1 < \frac{2(p-1)(n-2)}{n-1}.$$

As a consequence, Q is a convex function of y for $y \in [-1,1]$. Now, since $Q(p,1) = \frac{\partial Q}{\partial y}(p,1) = 0$ it follows that

$$\frac{\partial Q}{\partial y}(p,y) < 0, \qquad Q(p,y) > 0, \quad y \in [-1,1).$$

We next consider the case $n > 3$ and study Q in the interval $[-(n-2), -1)$. The positivity of Q is equivalent to

$$R(y) := \frac{(1-y)^2}{1 - py + (p-1)|y|^{\frac{p}{p-1}}} > \frac{(s-1)(n-1)}{p(n-2)}, \qquad -(n-2) \leq y < -1.$$

The derivative of R is

$$R'(y) = \frac{(1-y)\left[-py(|y|^{\frac{2-p}{p-1}} - 1) + (2-p)(|y|^{\frac{p}{p-1}} - 1)\right]}{(1 - py + (p-1)|y|^{\frac{p}{p-1}})^2} > 0, \quad \text{for} \quad y < -1.$$

Hence

$$R(y) > R(-(n-2)) = \frac{(n-1)^2}{1 + p(n-2) + (p-1)(n-2)^{\frac{p}{p-1}}} > \frac{(s-1)(n-1)}{p(n-2)},$$

by our assumptions, and therefore

$$Q(p,y) > 0, \qquad -(n-2) \leq y < -1.$$

The proof of the Lemma in now complete.

□

Lemma 4.5. *Let $n > 2$, $p > 1$. In addition, we assume that*

(i) if $p \geq 2$ then $0 < s - 1 < 2\frac{n-2}{n-1}$,

(ii) if $1 < p < 2$ then $0 < s - 1 < \min\left(\frac{2(p-1)(n-2)}{n-1}, \frac{p(n-1)(n-2)}{1+p(n-2)+(p-1)(n-2)^{\frac{p}{p-1}}}\right)$.

Then, there exists an AN connection between $(0, \rho_2)$ and $(1,1)$.

Proof: By Lemmas 4.2 and 4.3, under our hypotheses there holds

$$\rho_2 < -(n-2),$$

for any $p > 1$. By Lemma 4.4, function $\bar{y}(x) = 1 - (n-1)(1-x)$ is a supersolution to the ODE (4.1). By Lemma 4.1 there exists an analytic solution at $(0, \rho_2)$, $y_a(x)$ that in addition satisfies $y_a(x) < \bar{y}(x)$ near $x = 0$. By comparison $y_a(x) < \bar{y}(x)$ for all $x \in (0,1)$ and therefore $\lim_{x\to 1_-} y_a(x) = 1$. We claim that y_a is an AN connection, that is, it is not analytic at $(1,1)$. Indeed, using the asymptotics of the analytic at $(1,1)$ solution y_b, see (4.7) we have that for x close to one,

$$y_a(x) < \bar{y}(x) = 1 - (n-1)(1-x) < y_b(x);$$

for the last inequality we also used the fact that

$$s - 1 < \frac{2(p-1)(n-2)}{n-1}.$$

Consequently y_a is not analytic at $(1,1)$.

□

We define

$$\mathcal{S}(p,n) = \Big\{ s : 1 < s < \frac{n+1}{2} \text{ and in addition there exists an AN connection between } (0,\rho_2) \text{ and } (1,1) \text{ of the ODE (4.1).} \Big\} \tag{4.8}$$

Lemma 4.6. *Let $n > 2$ and $p > 1$. The set $\mathcal{S}(p,n)$ is a nonempty interval, open in its right end.*

Proof: The set $\mathcal{S}(p,n)$ is nonempty by Lemma 4.5.

To prove the openess we argue as follows: If for some s we have an AN connection $y_a(x)$ between $(0,\rho_2)$ and $(1,1)$ then for the same s there exists an NA connection $y_b(x)$ between $(0,\rho_1)$ and $(1,1)$. Let τ_a be the unique point for which $y_a(\tau_a) = 0$ and similarly for τ_b. Then necessarily $0 < \tau_b < \tau_a < 1$. Using the continuous dependence of y_a in the interval $(0,\tau_a)$, it follows that a small variation in s, will result in a small variation in the root τ_a. By a similar argument for $y_b(x)$, a small variation in s, will result in an analytic at $(1,1)$ solution, with root near τ_b. We conclude that for small variations of s the two solutions are distinct. Hence, the solution close to the AN connection y_b, remains AN and the set $\mathcal{S}(p,n)$ is open.

Let $s_1 \in \mathcal{S}(p,n)$ with y_{1a} the corresponding AN solution, analytic at $(0,\rho_2)$ and $1 < s_2 < s_1$ with y_{2a} the analytic at $(0,\rho_2)$ solution for $s = s_2$. Using the monotonicities of the F cf (4.5), as well as of ρ_2 cf (4.2), we conclude by comparison that $y_{1a}(x) > y_{2a}(x)$. If we denote by y_{1b}, y_{2b} the corresponding analytic at $(1,1)$ solutions, then for x close to one, using the asymptotics of the y_{ib}, $i = 1,2$, we have

$$y_{2b}(x) \geq y_{1b}(x) > y_{1a}(x) > y_{2a}(x).$$

As a consequence, y_{2a} is not analytic at $(1,1)$ and therefore $s_2 \in \mathcal{S}(p,n)$. Hence $\mathcal{S}(p,n)$ is an interval.

□

Theorem 4.7. *Let $n > 2$, $p > 1$ and $1 < s < \frac{n+1}{2}$. $\mathcal{S}(p,n)$ is a nonempty open bounded interval. We also define*

$$s_1(p,n) := \sup \mathcal{S}(p,n).$$

Then,

(a) *in the case $2 \le n \le n^*(p)$, there holds $s_1(p,n) = \frac{n+1}{2}$; moreover,*

(i) *if $2 \le n < n^*(p)$ and $1 < s \le \frac{n+1}{2}$, then there exists an AN connection.*

(ii) *if $n = n^*(p)$ and $1 < s < \frac{n^*(p)+1}{2}$, then there exists an AN connection.*

(iii) *if $n = n^*(p)$ and $s = \frac{n^*(p)+1}{2}$, then there exists an AA connection.*

(b) *in the case $n > n^*(p)$, there holds $1 < s_1(p,n) < \frac{n+1}{2}$; moreover,*

(i) *if $1 < s < s_1(p,n)$, then there exists an AN connection.*

(ii) *if $s = s_1(p,n)$, then there exists an AA connection.*

(iii) *if $s_1(p,n) < s \le \frac{n+1}{2}$, then there do not exist connections.*

All the above connections are between $(0,\rho_2)$ and $(1,1)$ and refer to the ODE (4.1).

(c) *Whenever there exists such an AN connection, then, for the same value of the parameters (p,n,s), there exists a connection between $(0,\rho_1)$ and $(1,1)$. analytic at $(1,1)$.*

See fig. 3. For the case $n > n^*$ see also fig. 4.

Proof: Part (a)(iii) is contained in Theorem 3.1. We prove (a)(ii). We denote by y_0 the AA connection coresponding to the case (a)(iii) and for $1 < s < \frac{n^*(p)+1}{2}$ we denote by y_a the analytic at $(0,\rho_2)$ solution. From the monotonicity of ρ_2 we have $y_a(0) < y_0(0)$ and by continuity $y_a(x) < y_0(x)$ near $x = 0$. Since $\frac{\partial F}{\partial s} \ge 0$, cf (4.5), we conclude that $y_0(x)$ is a supersolution of the ODE (4.1) for $s < \frac{n^*(p)+1}{2}$, and by comparison it follows that $y_a(x) < y_0(x)$ for all $x \in (0,1)$. Hence y_a connects to $(1,1)$. We will show that it is not analytic at $(1,1)$. If y_a was analytic then, using the asymptotics of the analytic solutions at $(1,1)$, we would obtain $y_a(x) > y_0(x)$ near $(1,1)$ which is a contradiction. Therefore y_a is an AN connection. We next prove (a)(i). We will use part (i) of Theorem 3.1. In particular, choosing $s_0 = \frac{n+1}{2}$ and $2 < n < n^*(p)$ there exists an AN connection y_0. Since $\frac{\partial F}{\partial s} \ge 0$, cf (4.5), y_0 is a supersolution of the ODE (4.1) for $1 < s < s_0$. By a similar argument as before the analytic at $(0,\rho_2)$ solution y_a satisfies $y_a(x) \le y_0(x)$, $x \in (0,1)$. The asymptotics of the analytic solutions at $(1,1)$ ansure that y_a is not analytic at $(1,1)$ and therefore it is an AN connection.

For part (b), suppose on the contrary, that $s_1 = \frac{n+1}{2}$. For $s = \frac{n+1}{2}$ and $n > n^*$ it follows from part (iii) of Theorem 3.1 that the analytic at $(0,\rho_2) = (0,-1)$ will blow up. Using the continuous dependence on s of the analytic solution in $[0,1)$ we conclude that for values of s smaller but close to $\frac{n+1}{2}$ the analytic solution will also blow up. Consequently, $s_1 < \frac{n+1}{2}$.

Part (b)(i) is clear from the definition of $\mathcal{S}(p,n)$. We prove part (b)(iii). Suppose there exists a connection for some $s_0 > s_1$ between $(0,\rho_2)$ and $(1,1)$. By Lemma 4.6 it is not an AN connection, therefore it is AA connection, say y_{s_0}. Arguing as in the proof of (a)(i) we establish that for any $s_1 < s < s_0$ the analytic at $(0,\rho_2)$ solution say y_s satisfies $y_s(x) < y_{s_0}(x)$ for $x \in (0,1)$ and connects to $(1,1)$. Now $y_s(x)$ can not be an AA connection, since the asymptotics near $(1,1)$ would lead to $y_s(x) > y_{s_0}(x)$ for

x near 1, which is a contradiction. This complete the proof of part (b)(iii). We next prove part (b)(ii). We first prove that for $s = s_1$ there is a connection. If there was no connection, then by a similar argument as in the proof of (b), we would have no connection for s smaller but close to s_1, which is a contradiction. This connection is necessarily AA by the openess of $\mathcal{S}(p, n)$.

For part (c), let y_a be the AN connection and y_b be the analytic at $(1, 1)$ solution of (4.1). By Lemma 4.1(d), $y_b(x) > y_a(x)$ for x close to one, and by comparison $y_b(x) > y_a(x)$ for all $x \in (0, 1)$. It follows that $y_b(x)$ connects to either $(0, \rho_2)$ or $(0, \rho_1)$. By Lemma 4.1(b) it cannot connect to $(0, \rho_2)$, since we would have $y_b(x) = y_a(x)$ and y_a would have been an AA connection. Hence y_b connects $(0, \rho_1)$ and $(1, 1)$.

□

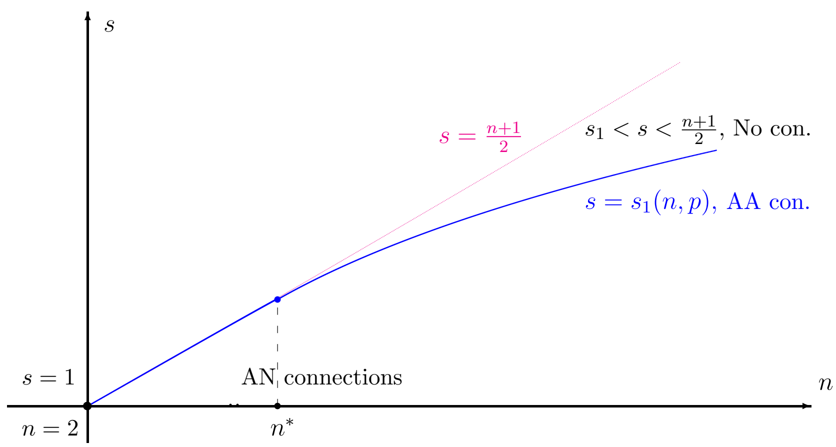

Figure 3: Case $1 < s < \frac{n+1}{2}$. Regions of existence and nonexistence of connections for Theorem 4.7. There are AN connections below the blue line, a unique AA connection on the blue line and no connections above the blue line. As stated in Corollary 4.12, the best constant $c(p, n, s)$ for the Hardy inequality (1.5) is equal to $\left(\frac{s-1}{p}\right)^p$ below or on the bleu line and strictly less than this value above the bleu line.

Lemma 4.8. *Let $p > 1$ and $n > n^*(p)$. For $(\bar{p}, \bar{n})$ close to (p, n) there holds*

$$\bar{n} > n^*(\bar{p}).$$

As a consequence $s_1(\bar{p}, \bar{n})$ is well defined.

Proof: By continuous dpendence of $n^*(p)$ with respect to p we have that given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\bar{p} - p| < \delta \qquad \Rightarrow \qquad |n^*(\bar{p}) - n^*(p)| < \varepsilon.$$

For $0 < \varepsilon < \frac{n - n^*(p)}{2}$ and $0 < \delta \leq \varepsilon$ we have for $|\bar{p} - p| + |\bar{n} - n| < \delta$,

$$\bar{n} > n - \delta > n^*(p) + \varepsilon > n^*(\bar{p}).$$

□

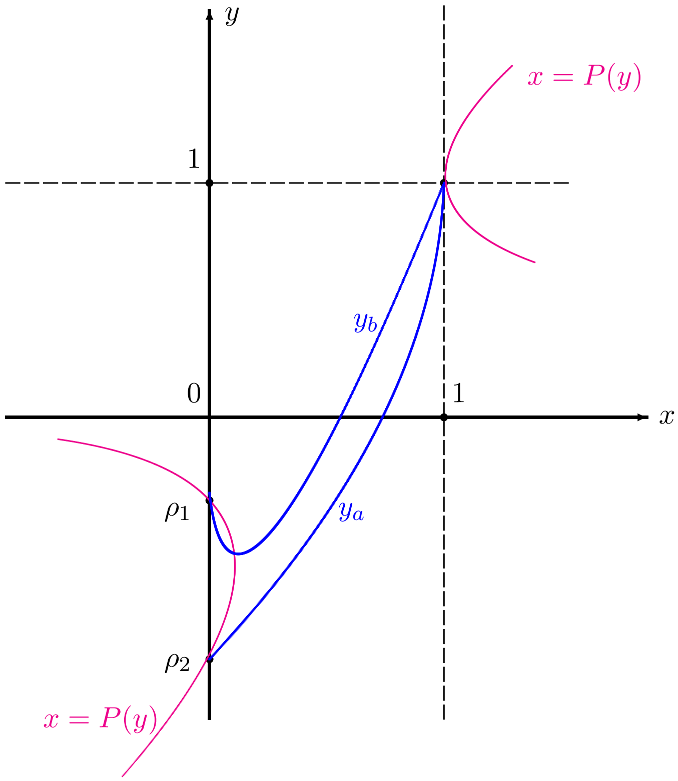

Figure 4: Let $n > n^*$. For $1 < s < s_1$, there exists an AN connection y_a and a connection between $(0, \rho_1)$ and $(1, 1)$, denoted by y_b, analytic at $(1, 1)$. For $s = s_1$ the two coincide to an AA connection between $(0, \rho_2)$ and $(1, 1)$. For $s > s_1$ there is no connection between either $(0, \rho_1)$ or $(0, \rho_2)$ and $(1, 1)$. All connections refer to the ODE (4.1).

Lemma 4.9. *Let $p > 1$ and $n > n^*(p)$. For $(\bar{p}, \bar{n})$ close to (p, n) $s_1(\bar{p}, \bar{n})$ is well defined by the previous Lemma. In addition we have*

$$\lim_{(\bar{p},\bar{n})\to(p,n)} s_1(\bar{p}, \bar{n}) = s_1(p, n).$$

Proof: Suppose on the contrary that there exists a $\varepsilon_0 > 0$ and a sequence $((\bar{p}_k, \bar{n}_k)$ such that

$$|\bar{p}_k - p| + |\bar{n}_k - n| < \frac{1}{k} \quad \text{and} \quad |s_1(\bar{p}_k, \bar{n}_k) - s_1(p, n)| > \varepsilon_0, \quad k = 1, 2, \ldots$$

Then for suitable subsequences we will have that at least one of the following inequalities holds

$$s_1(\bar{p}_k, \bar{n}_k) > s_1(p, n) + \varepsilon_0 =: s_0, \tag{4.9}$$

or

$$s_1(\bar{p}_k, \bar{n}_k) < s_1(p, n) - \varepsilon_0 =: \bar{s}_0. \tag{4.10}$$

In both cases we will reach a contradiction.

Suppose first that (4.9) holds. Since $s_0 > s_1(p, n)$ the analytic at $(0, \rho_2)$ solution of (4.1) for $s = s_0$ say, $y_{a,p,n}(x)$ blows up. Therefore for some point $x_0 \in (0, 1)$ we have that $y_{a,p,n}(x_0) = 2$. Using the continuous dependence of the analytic at $(0, \rho_2)$ solution on the parameters (p, n), given $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that

$$|\bar{p} - p| + |\bar{n} - n| < \delta_1 \quad \Rightarrow \quad |y_{a,\bar{p},\bar{n}}(x) - y_{a,p,n}(x)| < \varepsilon, \quad x \in [0, x_0],$$

which in particular implies that $y_{a,\bar{p},\bar{n}}(x)$, for ε small enough, blows up as well and therefore $s_1(\bar{p}, \bar{n}) < s_0$, contradicting (4.9).

Suppose now that (4.10) holds. Since $\bar{s}_0 < s_1(p, n)$ then the analytic at $(0, \rho_2)$ solution of (4.1) for $s = \bar{s}_0$ say, $y_{a,p,n}(x)$ is an AN connection. If $y_{b,p,n}(x)$ is the analytic at $(1, 1)$ solution for $s = \bar{s}_0$ we have that $y_{b,p,n}(x) > y_{a,p,n}(x)$, $x \in (0, 1)$. By continuous dependence of of the analytic solution $y_{a,p,n}(x)$ for $x \in [0, \frac{1}{2}]$, given $xe > 0$ there exists a $\delta_2 > 0$ such that

$$|\bar{p} - p| + |\bar{n} - n| < \delta_2 \quad \Rightarrow \quad |y_{a,\bar{p},\bar{n}}(x) - y_{a,p,n}(x)| < \varepsilon, \quad x \in [0, 1/2].$$

By a similar argument for $y_{b,p,n}(x)$, there exists a $\delta_3 > 0$ such that

$$|\bar{p} - p| + |\bar{n} - n| < \delta_3 \quad \Rightarrow \quad |y_{b,\bar{p},\bar{n}}(x) - y_{b,p,n}(x)| < \varepsilon, \quad x \in [1/2, 1].$$

Hence, for ε small enough there exists δ small such that

$$|\bar{p} - p| + |\bar{n} - n| < \delta \quad \Rightarrow \quad y_{b,\bar{p},\bar{n}}(1/2) > y_{a,\bar{p},\bar{n}}(1/2).$$

As a consequence $y_{b,\bar{p},\bar{n}}(x) > y_{a,\bar{p},\bar{n}}(x)$, for $x \in (1/2, 1)$ and necessarilly $y_{a,\bar{p},\bar{n}}$ is an AN connection. It follows that $\bar{s}_0 < s_1(\bar{p}, \bar{n})$, contradicting (4.10).

□

Lemma 4.10. *Let $p > 1$ and $n > n^*(p)$. Then, $s_1(p, n)$ is a strictly increasing function of p.*

Proof: Throughout the proof, n is fixed. Let $p_2 > p_1 > 1$ and y_0 be the AA connection corresponding to $s_1(p_1, n)$. Using the monotonicity of F cf (4.4) it follows that y_0 is a supersolution of the equation $y'(x) = F(p_1, s_1(p_1, n), n, y(x))$, therefore since $\rho_2(p_2) < \rho_2(p_1)$, see (4.2), the analytic at $(0, \rho_2)$ solution of $y_a'(x) = F(p_2, s_1(p_1, n), n, y_a(x))$, satisfies $y_a(x) < y_0(x)$ for all $x \in [0, 1)$. Using the asymptotics of analytic solutions at $(1, 1)$ it follows that y_a is not an AA connection, hence it is an AN connection. Consequently $s_1(p_2, n) > s_1(p_1, n)$.

□

We next consider the relation between the existence of positive smooth solutions of the Euler Lagrange equation

$$\left(\frac{r^{n-1}|\phi'|^{p-2}\phi'}{(r-1)^{s-p}} \right)' + \left(\frac{s-1}{p} \right)^p \frac{r^{n-1}|\phi|^{p-2}\phi}{(r-1)^s} = 0, \quad r > 1, \tag{4.11}$$

and the existence of connections in the ODE (4.1). In particular we have

Theorem 4.11. *Let $p > 1$, $n \geq 2$ and $1 < s < \frac{n+1}{2}$.*
(i) Suppose that equation (4.11) has a positive smooth solution in $(1, +\infty)$. Then, equation (4.1) has a solution $y_b(x)$, $x \in (0, 1)$ which is analytic at (1,1) and connects to either $(0, \rho_1)$ or $(0, \rho_2)$.

(ii) Suppose that equation (4.1) *has a solution* $y_b(x)$*,* $x \in (0,1)$*, which is analytic at (1,1) and connects to either* $(0,\rho_1)$ *or* $(0,\rho_2)$*. Then, function*

$$\phi(r) = \exp\left[\frac{s-1}{p}\int_2^r \frac{|y_b(1/t)|^{\frac{2-p}{p-1}} y_b(1/t)}{t-1}\,dt\right], \qquad r > 1, \tag{4.12}$$

is a smooth positive solution of the Euler Lagrange equation (4.11), which in addition satisfies for a positive constant k_1*,*

$$\lim_{r\to 1^+} \frac{\phi(r)}{(r-1)^{\frac{s-1}{p}}} = k_1. \tag{4.13}$$

Proof: (i) Let $\phi(r)$ be a positive smooth solutions of equation (4.11). Then, function

$$y(x) = \frac{(r-1)^{p-1}}{\left(\frac{s-1}{p}\right)^{p-1}} \frac{|\phi'|^{p-2}\phi'}{|\phi|^{p-2}\phi}, \qquad x = \frac{1}{r}, \quad r > 1,$$

is defined for all $x \in (0,1)$ and solves (4.1). Consider now the analytic at $(1,1)$ solution y_b of (4.1). We will show that it exists for all $x \in (0,1)$ and consequently $\lim_{x\to 0^+} y_b(x)$ is equal either to ρ_1 or to ρ_2. Suppose on the contrary that y_b blows up at some point $x_0 \in (0,1)$. Then, the analytic at $(0,\rho_2)$ solution y_a of (4.1) will by increasing and it will blow up at a point $x_1 \in (0,1)$ since it cannot tend to 1 by Lemma 4.1(d). As a consequence, any solution of the ODE blows up either forward or backward or in both directions and therefore no solution exists for all $x \in (0,1)$ contradicting the existence of y. Hence y_b is defined and it is bounded for all $x \in (0,1)$.

There are two possibilities: Suppose first that y_b is increasing for all $x \in (0,1)$, which in particular means that it does not meet the curve $x = P(y)$; in this case it tends either to ρ_1 or to ρ_2. In the second case, y_b meets the curve $x = P(y)$ at some point (x_0, y_0) with $x_0 \in (0,1)$, it changes monotonicity and it tends to ρ_1. In either case y_b tends either to ρ_1 or to ρ_2 as $x \to 0^+$.

(ii) The fact that ϕ solves (4.11) is a straightforward calculation. The existence of the limit (4.13), follows by using the asymptotics of y_b at (1,1), cf Lemma 4.1(c).

□

The following is a direct consequence of Theorems 4.7 and 4.11 in connection with Proposition 2.1.

Corollary 4.12. *(i) Let* $p > 1$ *and suppose that either* $2 \le n \le n^*(p)$ *and* $1 < s \le \frac{n+1}{2}$ *or else* $n > n^*(p)$ *and* $1 < s \le s_1(p,n)$*. Then equation* (4.11) *has a positive smooth solution in* $(1,+\infty)$ *with*

$$\lim_{r\to 1^+} \frac{\phi(r)}{(r-1)^{\frac{s-1}{p}}} = k_1, \tag{4.14}$$

for a positive constant k_1*. The best constant of the corresponding Hardy inequality is given by*

$$c(p,n,s) = \left(\frac{s-1}{p}\right)^p.$$

(ii) Let $p > 1$*,* $n > n^*(p)$ *and* $s_1(p,n) < s \le \frac{n+1}{2}$*. Then, equation* (4.11) *has a positive smooth solution* ϕ *in some finite interval* $(1,R)$ *satisfying the limit* (4.14)*, with* $\phi(R) = 0$*. The best constant of the corresponding Hardy inequality satisfies*

$$c(p,n,s) < \left(\frac{s-1}{p}\right)^p.$$

See fig. 3.

5 Existence of minimizers

In this section we consider the case where $p>1$, $n>n^*(p)$ and $s_1(p,n)<s\le \frac{n+1}{2}$. For these range of the parameters, by Corollary 4.12(ii), the best constant satisfies

$$c(p,n,s)<\left(\frac{s-1}{p}\right)^p\le\left(\frac{n-s}{p}\right)^p, \tag{5.1}$$

in which case, by Theorem 2.2, $c(p,n,s)$ is achieved by a positive function $\phi(r)$, $r>1$, in the proper energy space $W_0^{1,p,s}(1,+\infty)$. In particular ϕ satisfies the Euler Lagrange

$$\left(\frac{r^{n-1}|\phi'|^{p-2}\phi'}{(r-1)^{s-p}}\right)'+c(p,n,s)\ \frac{r^{n-1}|\phi|^{p-2}\phi}{(r-1)^s}=0,\quad r>1. \tag{5.2}$$

We define θ to be the unique solution of

$$c(p,n,s)=\theta^{p-1}(s-1-\theta(p-1)), \tag{5.3}$$

in the interval $\left(\frac{s-1}{p},\frac{s-1}{p-1}\right)$. Changing variables by

$$y_e(x)=\frac{(r-1)^{p-1}}{\theta^{p-1}}\ \frac{|\phi'|^{p-2}\phi'}{|\phi|^{p-2}\phi},\qquad x=\frac{1}{r},\quad r>1, \tag{5.4}$$

y_e is defined for all $x\in(0,1)$ and satisfies

$$\frac{dy}{dx}=\frac{-(n-1)xy+\left[s-1-\theta(p-1)+(n-s)y+\theta(p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)},\qquad 0<x<1. \tag{5.5}$$

We will study solutions of (5.5) that exist for all $x\in(0,1)$.

For $x=0$ the equation

$$s-1-\theta(p-1)+(n-s)y+\theta(p-1)|y|^{\frac{p}{p-1}}=0,$$

has two negative roots satisfying

$$\rho_2<-\left(\frac{n-s}{\theta p}\right)^{p-1}<\rho_1<0. \tag{5.6}$$

Similarly, for $x=1$, the quantity

$$s-1-\theta(p-1)-(s-1)y+\theta(p-1)|y|^{\frac{p}{p-1}}=0, \tag{5.7}$$

has two roots, namely, τ and 1, satisfying

$$0<\tau<\left(\frac{s-1}{\theta p}\right)^{p-1}<1. \tag{5.8}$$

Now there are four critical points of the ODE, namely $(0,\rho_1)$, $(0,\rho_2)$, $(1,\tau)$, $(1,1)$ that will be important to our analysis. There are other critical points, that is, points at which the numerator of the right hand side of (5.5) is zero. They lie on a two–branch curve that we can be described as

$$x = P(y) := \frac{n-s}{s-1} + \frac{s-1-\theta(p-1)}{n-1}\frac{1}{y} + \frac{\theta(p-1)}{n-1}\frac{|y|^{\frac{p}{p-1}}}{y}, \qquad y \neq 0,$$

For the branch with $y < 0$, the maximum value of $P(y)$ is taken at the value

$$y_1 = -\left(\frac{s-1-\theta(p-1)}{\theta}\right)^{\frac{p-1}{p}},$$

and it is equal to

$$P(y_1) = \frac{n-s}{n-1} - \frac{p}{n-1}\left(\theta^{p-1}(s-1-\theta(p-1))\right)^{\frac{1}{p}} = \frac{n-s}{n-1} - \frac{p}{n-1}c^{\frac{1}{p}}(p,n,s).$$

It is worth noticing that, by means of (5.1),

$$0 < x_1 := P(y_1) < 1.$$

For future reference we also note that

$$P'(\rho_2) = -\frac{p(s-1-\theta(p-1)) + (n-s)\rho_2}{(p-1)(n-1)\rho_2^2}. \tag{5.9}$$

Similarly, for the branch $y > 0$, the minimum value of $P(y)$ is taken at the value

$$y_2 = \left(\frac{s-1-\theta(p-1)}{\theta}\right)^{\frac{p-1}{p}},$$

and it is equal to

$$P(y_2) = \frac{n-s}{n-1} + \frac{p}{n-1}\left(\theta^{p-1}(s-1-\theta(p-1))\right)^{\frac{1}{p}} = \frac{n-s}{n-1} + \frac{p}{n-1}c^{\frac{1}{p}}(p,n,s).$$

It is easy to see that

$$0 < x_1 = P(y_1) < x_2 = P(y_2) < 1.$$

We also note that

$$P'(1) = \frac{\theta p - (s-1)}{n-1}. \tag{5.10}$$

If there is a pair (x_0, y_0) such that $0 < x_0 = P(y_0)$ with $\rho_2 < y_0 < \rho_1$, then the solution $y(x)$ of the ODE (5.5) with $y(x_0) = y_0$ is such that for all $x \in (0, x_0)$ there holds $0 < x < P(y(x))$ and therefore y is decreasing for $x \in (0, x_0)$ and $y_0 < y(x) < \rho_1$. It is easy to show that $\lim_{x\to 0^+} y(x) = \rho_1$.

Similarly, if there is a pair (x_0, y_0) such that $x_2 \leq x_0 = P(y_0) < 1$ with $\tau < y_0 < 1$, then the solution $y(x)$ of the ODE (5.5) with $y(x_0) = y_0$ is such that for all $x \in (x_0, 1)$ there holds $x > P(y(x))$ and therefore y is decreasing for $x \in (x_0, 1)$ and $\tau < y(x) \leq y_0$. Necessarily $\lim_{x\to 1^-} y(x) = \tau$.

Finally we note that if for a pair (x_0, y_0) with $x_0 \in (0,1)$ and such that

$$-(n-1)x_0y_0 + s - 1 - \theta(p-1) + (n-s)y_0 + \theta(p-1)|y_0|^{\frac{p}{p-1}} > 0,$$

then, the solution $y(x)$ of the ODE with $y(x_0) = y_0$ is increasing until it crosses the curve $x = P(y)$. See fig 5.

Similarly to Lemma 4.1 we have

Lemma 5.1. *Let $p > 1$, $n > n^*(p)$ and $s_1(p,n) < s < \frac{n+1}{2}$.*
(a) There exists a unique analytic solution $y_a(x)$ of (5.5) near $(x,y) = (0,\rho_2)$. Moreover for some $\varepsilon > 0$ and any $x \in [0,\varepsilon)$ there holds

$$y_a(x) = \rho_2 + \frac{(n-1)\rho_2}{n-s-1-\theta p|\rho_2|^{\frac{1}{p-1}}}\, x + O(x^2). \tag{5.11}$$

(b) If for some $\varepsilon \in (0,1)$ there exists a solution $y(x)$ of (5.5) in $(0,\varepsilon)$ that in addition satisfies

$$\lim_{x\to 0^+} y(x) = \rho_2,$$

then necessarily $y(x) = y_a(x)$, $x \in (0,\varepsilon)$.
(c) There exists a unique analytic solution $y_b(x)$ of (5.5) near $(x,y) = (1,1)$. Moreover for some $\varepsilon > 0$ and any $x \in [1-\varepsilon, 1)$ there holds

$$y_b(x) = 1 - \frac{n-1}{1+\theta p-(s-1)}(1-x) + O((x-1)^2). \tag{5.12}$$

(d) If for some $\varepsilon \in (0,1)$ there exists a solution $y(x)$ of (5.5) in $(1-\varepsilon,1)$ that in addition satisfies

$$\lim_{x\to 1^-} y(x) = 1,$$

then necessarily $y(x) = y_b(x)$, $x \in (1-\varepsilon, 1)$.
(e) If for some solution y there exists a point $x_1 \in (0,1)$ such that $y(x_1) > 1$ then for some $x_2 \in (x_1, 1)$ there holds $\lim_{x\to x_2^-} y(x) = +\infty$. Similarly if for some $x_3 \in (0,1)$ such that $y(x_3) < \rho_2$ then for some $x_4 \in (0,x_3)$ we have $\lim_{x\to x_4^+} y(x) = -\infty$.

The proof is quite similar to the proof of Lemma 4.1. There is a difference in part (d) due to the fact that the root 1 of equation (5.7) is simple with $\theta p - (s-1) > 0$.

□

Theorem 5.2. *Let $p > 1$, $n > n^*(p)$ and $s_1(p,n) < s < \frac{n+1}{2}$. If ϕ is the energetic positive solution of (5.2), then y_e defined by (5.4) is the unique AA connection between $(0,\rho_2)$ and $(1,1)$ of the ODE (5.5). Conversely, if y_e is an AA connection of the ODE (5.5) between $(0,\rho_2)$ and $(1,1)$, then any positive function ϕ defined via the change of variables (5.4) is an energetic positive solution of (5.2).*

Proof: step 1: Suppose the ODE (5.5) has a solution $y(x)$ existing for all $x \in (0,1)$. Then, (i) the analytic at (1,1) solution y_b connects to either $(0,\rho_1)$ or else to $(0,\rho_2)$. (ii) Similarly, the analytic at (1,1) solution y_a satisfies $y_a(x) \le y_b(x)$ for $x \in (0,1)$ and connects to either (1,1) or else to $(1,\tau)$.

We prove (i), case (ii) being quite similar. We consider the analytic at (1,1) solution y_b. Since the slope of $y = P^{-1}(x)$ (upper branch) at $x = 1$ is above the slope of y_b,

$$\frac{n-1}{\theta p-(s-1)} > \frac{n-1}{1+\theta p-(s-1)},$$

we conclude that y_b is increasing and stays above the curve $x = P(y)$ near $(1,1)$.

We claim that $y_b(x) \ge \rho_2$ for all $x \in (0,1)$. Indeed, if this is not the case then there exists an $x_3 \in (0,1)$ such that $y_b(x_3) < \rho_2$ and by Lemma 5.1 solution y_b blows up at some $x_4 \in (0,x_3)$. On the other hand the analytic at $(0,\rho_2)$ solution y_a cannot

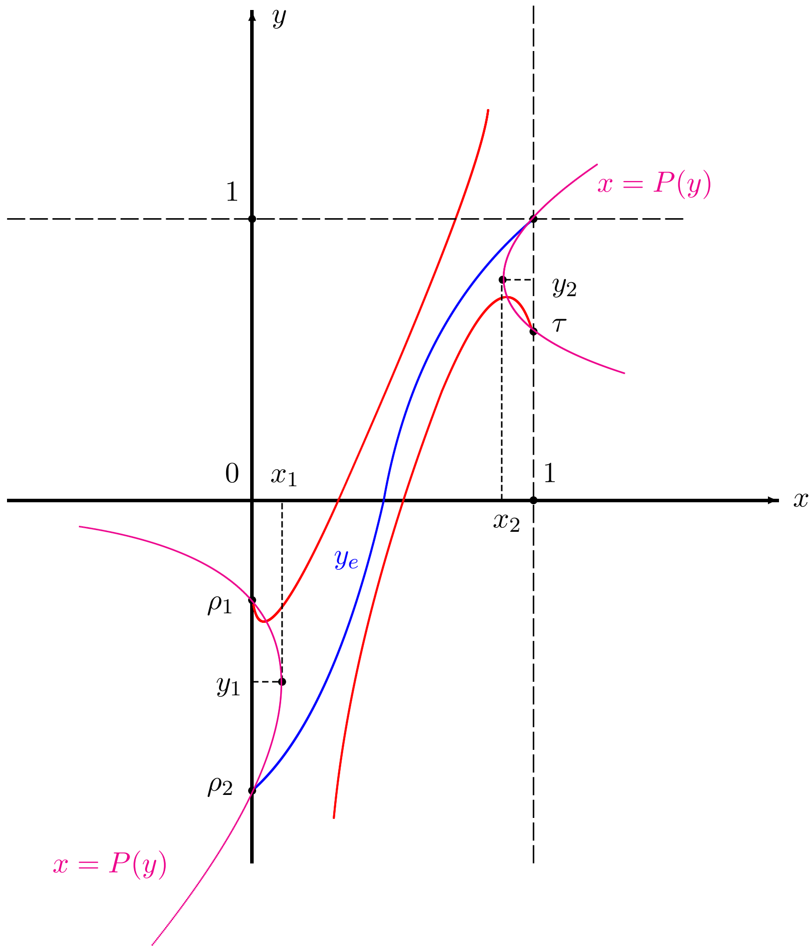

Figure 5: Case $n > n^*(p)$, $s_1(p,n) < s \leq \frac{n+1}{2}$. There is a unique solution y_e of the ODE (5.5) connecting $(0, \rho_2)$ and $(1, 1)$ (blue trajectory) and it is analytic at both ends. All solutions passing through either $(0, \rho_1)$ or $(1, \tau)$ blow up (red trajectories)

cross y_b and cannot tend to 1 by Lemma 5.1(d), so it has to exceed one and eventually blow up at a finite point in $(0, 1)$. As a consequence, any solution of the ODE blows up either forward or backward or in both directions and therefore no solution exists for all $x \in (0, 1)$ contradicting the existence of y. Hence $y_b(x) \geq \rho_2$.

There are two cases: Suppose first that y_b is increasing for all $x \in (0, 1)$, which in particular means that it does not meet the curve $x = P(y)$ in $(0, x_1]$. In such a case it tends either to ρ_1 or to ρ_2. In the second case, y_b meets the curve $x = P(y)$ at some point (x_0, y_0) with $x_0 \in (0, x_1]$, it changes monotonicity and it tends to ρ_1.

step 2: If y is a solution of (5.5) that is defined for all $x \in (0, 1)$ then

$$y_a(x) \leq y(x) \leq y_b(x), \qquad x \in (0, 1).$$

Suppose on the contrary that for some $x_0 \in (0, 1)$ we have that $y(x_0) > y_b(x_0)$. Then $y(x)$ is increasing, it cannot tend to 1 as $x \to 1^-$ and therefore it will exceed one and eventually will blow up before $x = 1$ contradicting the existence of $y(x)$ in (0,1). By a similar argument we cannot have $y(x_0) < y_a(x_0)$.

step 3: If y is a solution of (5.5) for which $\lim_{x\to 0^+} y(x) = \rho_1$, then any positive function $\psi(r)$ defined via the transformation

$$y(x) = \frac{(r-1)^{p-1}}{\theta^{p-1}} \frac{|\psi'(r)|^{p-2}\psi'(r)}{|\psi(r)|^{p-2}\psi(r)}, \qquad x = \frac{1}{r}, \quad r \text{ near } \infty,$$

is not energetic near ∞, that is

$$\int^{\infty} \frac{r^{n-1}|\psi(r)|^p\, dr}{(r-1)^s} = +\infty.$$

Suppose that $\lim_{x\to 0^+} y(x) = \rho_1$. Then, for ε small and r large enough

$$\frac{(r-1)^{p-1}}{\theta^{p-1}} \frac{|\psi'(r)|^{p-2}\psi'(r)}{|\psi(r)|^{p-2}\psi(r)} \geq \rho_1 - \varepsilon.$$

Solving this we conclude that for some positive constant $c > 0$,

$$\psi(r) \geq (r-1)^{-\theta|\rho_1-\varepsilon|^{\frac{1}{p-1}}},$$

from which the statement follows taking into account that $-\left(\frac{n-s}{\theta p}\right)^{p-1} < \rho_1$ which is true by (5.6).

step 4: Similarly to step 3, if y is a solution of (5.5) for which $\lim_{x\to 1^-} y(x) = \tau$, then any positive function $\psi(r)$ defined as above for r close to 1, is not energetic at one, that is

$$\int^{1^+} \frac{r^{n-1}|\psi(r)|^p\, dr}{(r-1)^s} = +\infty.$$

Starting with $\lim_{x\to 1^-} y(x) = \tau$, this time we end up with

$$\psi(r) \geq (r-1)^{-\theta|\tau+\varepsilon|^{\frac{1}{p-1}}},$$

and the statement follows taking into account that $\tau < \left(\frac{s-1}{\theta p}\right)^{p-1}$ which is true by (5.8).

step 5: Completion of the proof. We first show that if ϕ is a positive energetic solution, y_e is an AA connection between $(0,\rho_2)$ to $(1,1)$. Indeed, since ϕ is energetic at infinity, $\lim_{x\to 0^+} y_e(x) = \rho_2$ and therefore $y_e(x) = y_a(x)$. Since ϕ is energetic at $r = 1$ then $\lim_{x\to 1^-} y_e(x) = 1$ and therefore $y_e(x) = y_a(x) = y_b(x)$.

Now suppose we have an AA connection that is, $y_a(x) = y_b(x)$. Then an argument quite similar to the ones in steps 3 and 4 and the fact that

$$\rho_2 < -\left(\frac{n-s}{\theta p}\right)^{p-1},$$

shows that ϕ is energetic at infinity. Quite similarly, because of

$$\left(\frac{s-1}{\theta p}\right)^{p-1} < 1,$$

ϕ is energetic at $r = 1^+$.

□

Remark 1. Contrary to the above Theorem, in the case where $n \geq n^*(p)$, $c(p,n,s) = \left(\frac{s-1}{p}\right)^p$, which corresponds to $\theta = \frac{s-1}{p}$ and $s = s_1(p,n)$, the AA connections of (5.5), see Theorem 4.7 part a(iii) and b(ii), correspond through the change of variables (5.4) to *non energetic* positive solutions ϕ of (5.2).

Remark 2. Using the asymptoics of y_e one can establish the existense of positive constants k_1, k_∞ such that, any positive solution ϕ of the Euler Lagrange equation (5.2) satisfies

$$\lim_{r \to 1^+} \frac{\phi(r)}{(r-1)^\theta} = k_1, \qquad \lim_{r \to \infty} \phi(r)\,(r-1)^{\theta|\rho_2|^{\frac{1}{p-1}}} = k_\infty.$$

6 Symmetries

Throughout this section we consider the case $n \geq 2$, $p > 1$ and $\frac{n+1}{2} < s < n$. Our aim is to establish symmetry properties of the best constant $c(p,n,s)$ as well as of solutions of (1.7).

We first investigate the case where the best constant $c(p,n,s) = \left(\frac{n-s}{p}\right)^p < \left(\frac{s-1}{p}\right)^p$. In this case the Euler Lagrange equation

$$\left(\frac{r^{n-1}|\phi'|^{p-2}\phi'}{(r-1)^{s-p}}\right)' + \left(\frac{n-s}{p}\right)^p \frac{r^{n-1}|\phi|^{p-2}\phi}{(r-1)^s} = 0, \quad r > 1, \tag{6.1}$$

has a positive smooth solution in $(1,\infty)$.

Similarly to section 5, by making the change of variables

$$y(x) = \frac{(r-1)^{p-1}}{\left(\frac{n-s}{p}\right)^{p-1}} \frac{|\phi'|^{p-2}\phi'}{|\phi|^{p-2}\phi}, \qquad x = \frac{1}{r}, \quad r > 1, \tag{6.2}$$

function y satisfies

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{n-s}{p}\left[1 + py + (p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0 < x < 1. \tag{6.3}$$

As usual we are interested in the points at which

$$-(n-1)xy + \frac{n-s}{p}\left[1 + py + (p-1)|y|^{\frac{p}{p-1}}\right] = 0.$$

For $x = 0$ the equation has -1 as a double root, whereas for $x = 1$ the equation becomes

$$1 - \frac{(s-1)p}{n-s}y + (p-1)|y|^{\frac{p}{p-1}} = 0,$$

and it has two positive roots satisfying

$$0 < \tau_1 < 1 < \left(\frac{s-1}{n-s}\right)^{p-1} < \tau_2.$$

The critical points of interest now are $(0,-1)$, $(1,\tau_1)$ and $(1,\tau_2)$. The other critical points lie on the curve

$$x = P(y) = \frac{n-s}{p(n-1)} \frac{1+py+(p-1)|y|^{\frac{p}{p-1}}}{y}.$$

As in section 5 one can establish the existence of a local analytic at $(0,-1)$ solution $y_a(x)$ as well as the existence of a local analytic at $(1,\tau_1)$ solution $y_b(x)$. We next state the analogue of Theorem 4.11.

Theorem 6.1. *Let $p>1$, $n\geq 2$ and $\frac{n+1}{2}<s<n$.*
(i) Suppose that equation (6.1) has a positive smooth solution in $(1,+\infty)$. Then, equation (6.3) has a solution $y_a(x)$, $x\in(0,1)$ which is analytic at (0,-1) and connects to either $(1,\tau_1)$ or $(1,\tau_2)$.
(ii) Suppose that equation (6.3) has a solution $y_a(x)$, $x\in(0,1)$, which is analytic at (0,-1) and connects to either $(1,\tau_1)$ or $(1,\tau_2)$. Then, function

$$\phi(r) = \exp\left[\frac{n-s}{p}\int_2^r \frac{|y_a(1/t)|^{\frac{2-p}{p-1}} y_a(1/t)}{t-1}\,dt\right], \qquad r>1, \tag{6.4}$$

is a smooth positive solution of the Euler Lagrange equation (6.1), which in addition satisfies for a positive constant k_∞,

$$\lim_{r\to\infty} \phi(r)\,(r-1)^{\frac{n-s}{p}} = k_\infty. \tag{6.5}$$

As a matter of fact there is a strong connection between solutions of the ODE (6.3) when $\frac{n+1}{2}<s<n$ and solutions of (4.1) when $1<s<\frac{n+1}{2}$.

For $s\in\left(\frac{n+1}{2},n\right)$ we define

$$\bar{s} = n+1-s \in \left(1, \frac{n+1}{2}\right).$$

We next have

Lemma 6.2. *Let y be a solution of (6.3) with $\frac{n+1}{2}<s<n$. Then*

$$\bar{y}(x) = -y(1-x),$$

is a solution of (4.1) where the value of the parameter s in (4.1) is $\bar{s}$. Conversely, if $\bar{y}(x)$ solves (4.1) with $\bar{s}\in\left(1,\frac{n+1}{2}\right)$ then $y(x)=-\bar{y}(1-x)$ solves (6.3) with $s=n+1-\bar{s}\in\left(\frac{n+1}{2},n\right)$.

In particular, if y is a solution of (6.3) which is analytic at $(0,-1)$ and connects to $(1,\tau_1)$ then $\bar{y}$ is a solution of (4.1) which is analytic at $(1,1)$ and connects to $(0,\rho_1)$.

Similarly, if y is an AA connection of the ODE (6.3) between (0,-1) and $(1,\tau_2)$ then $\bar{y}$ is an AA connection of the ODE (4.1) between $(0,\rho_2)$ and $(1,1)$.

Proof: It is straightforward calculation.

$$\begin{aligned}
\bar{y}'(x) &= y'(1-x) \\
&= \frac{-(n-1)(1-x)y(1-x) + \frac{n-s}{p}\left[1 + py(1-x) + (p-1)|y(1-x)|^{\frac{p}{p-1}}\right]}{x(1-x)} \\
&= \frac{-(n-1)x\bar{y}(x) + \frac{n-s}{p}\left[1 + p\,\frac{s-1}{n-s}\,\bar{y}(x) + (p-1)|\bar{y}(x)|^{\frac{p}{p-1}}\right]}{x(1-x)} \\
&= \frac{-(n-1)x\bar{y}(x) + \frac{\bar{s}-1}{p}\left[1 + p\,\frac{n-\bar{s}}{\bar{s}-1}\,\bar{y}(x) + (p-1)|\bar{y}(x)|^{\frac{p}{p-1}}\right]}{x(1-x)},
\end{aligned}$$

here we also used the fact that $\bar{s} = n + 1 - s$. To proof of the converse is quite similar.

The rest of the statements is a direct consequence of this duality.

□

We define

$$s_2(p,n) := n + 1 - s_1(p,n)$$

The following is a direct consequence of Theorem 6.1 and Lemma 6.2 in connection with Proposition 2.1.

Corollary 6.3. *(i) Let $p > 1$ and suppose that either $2 \le n \le n^*(p)$ and $\frac{n+1}{2} < s < n$ or else $n > n^*(p)$ and $s_2(p,n) \le s < n$. Then equation (6.1) has a positive smooth solution in $(1,+\infty)$ with*

$$\lim_{r\to\infty} \phi(r)\,(r-1)^{\frac{n-s}{p}} = k_\infty, \tag{6.6}$$

for a positive constant k_∞. The best constant of the corresponding Hardy inequality is given by

$$c(p,n,s) = \left(\frac{n-s}{p}\right)^p .$$

(ii) Let $p > 1$, $n > n^*(p)$ and $\frac{n+1}{2} < s < s_2(p,n)$. Then, equation (6.1) has a positive smooth solution ϕ in some finite interval $(1,R)$ with $\phi(R) = 0$. In particular there is no positive smooth solution in the whole interval $(1,+\infty)$. The best constant of the corresponding Hardy inequality satisfies

$$c(p,n,s) < \left(\frac{n-s}{p}\right)^p .$$

We next consider the case $\frac{n+1}{2} < s < s_2(p,n)$ where we have existence of energetic solutions ϕ of the corresponding Euler Lagrange,

$$\left(\frac{r^{n-1}|\phi'|^{p-2}\phi'}{(r-1)^{s-p}}\right)' + c(p,n,s)\,\frac{r^{n-1}|\phi|^{p-2}\phi}{(r-1)^s} = 0, \quad r > 1. \tag{6.7}$$

This time we define $\theta \in \left(\frac{n-s}{p}, \frac{n-s}{p-1}\right)$ to be the unique solution of

$$c(p,n,s) = \theta^{p-1}(n - s - \theta(p-1)),$$

and y_e by

$$y(x) = \frac{(r-1)^{p-1}}{\theta^{p-1}}\,\frac{|\phi'|^{p-2}\phi'}{|\phi|^{p-2}\phi}, \qquad x = \frac{1}{r}, \quad r > 1.$$

Then y_e is a solution of

$$\frac{dy}{dx} = \frac{-(n-1)xy + \left[n - s - \theta(p-1) + (n-s)y + \theta(p-1)|y|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0 < x < 1. \tag{6.8}$$

defined for all $x \in (0,1)$.

We next recall that $\bar{s} = n + 1 - s \in \left(s_1(p,n), \frac{n+1}{2}\right)$ and we note that

$$c(p,n,\bar{s}) = \theta^{p-1}(\bar{s} - 1 - \theta(p-1)) \ .$$

Let $\bar{y}(x)$ be the analytic at (1,1) solution that connects to $(0, \rho_1)$ and satisfies

$$\frac{d\bar{y}}{dx} = \frac{-(n-1)x\bar{y} + \left[\bar{s} - 1 - \theta(p-1) + (n-\bar{s})\bar{y} + \theta(p-1)|\bar{y}|^{\frac{p}{p-1}}\right]}{x(1-x)}, \qquad 0 < x < 1.$$

We then have

Lemma 6.4. *Let $p > 1$, $n > n^*(p)$ and $\frac{n+1}{2} < s < s_2(p,n)$. Then*

$$y(x) = -\bar{y}(1-x), \qquad x \in (0,1),$$

is analytic at $(0,-1)$, connects to $(1,\tau_1)$ and satisfies (6.8). The converse statement is also true.

Proof: A straightforward calculation yields, for $t = 1 - x$

$$\begin{aligned}
y'(x) &= \bar{y}'(t) \\
&= \frac{-(n-1)t\,\bar{y}(t) + \left[\bar{s} - 1 - \theta(p-1) + (n-\bar{s})\bar{y}(t) + \theta(p-1)|\bar{y}(t)|^{\frac{p}{p-1}}\right]}{t(1-t)} \\
&= \frac{-(n-1)x\,y(x) + \left[\bar{s} - 1 - \theta(p-1) + (\bar{s}-1)y(x) + \theta(p-1)|y(x)|^{\frac{p}{p-1}}\right]}{x(1-x)} \\
&= \frac{-(n-1)x\,y(x) + \left[n - s - \theta(p-1) + (n-s)y(x) + \theta(p-1)|y(x)|^{\frac{p}{p-1}}\right]}{x(1-x)}.
\end{aligned}$$

Hence y satisfies (6.8) and it is in fact analytic at $(0,-1)$ and connects to $(1,\tau_1)$.

□

7 Proofs of the main Theorems

In this section we will provide the proofs of our main Theorems, stated in the Introduction.

Proof of Theorem 1.1: It follows by combining Theorems 4.7 and 1.3.

Proof of Theorem 1.2: It is a direct consequence of Corollary 4.12.

Proof of Theorem 1.3: It follows by combining Lemmas 6.2 and 6.4.

Before the proof of Theorems 1.4 and 1.5 we present two auxiliary Lemmas.

Lemma 7.1. *Let $p > 1$ and $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^n$, $n \geq 1$. Then*

$$|\mathbf{a}|^p - p\,\mathbf{a}\cdot\mathbf{b}|\mathbf{b}|^{p-2} + (p-1)|\mathbf{b}|^p \geq 0.$$

Equality holds if and only if $\mathbf{a} = \mathbf{b}$.

Proof: The proof is a consequence of the fact that the function

$$F(\mathbf{a}) := |\mathbf{a}|^p - p\,\mathbf{a}\cdot\mathbf{b}|\mathbf{b}|^{p-2} + (p-1)|\mathbf{b}|^p,$$

is strictly convex and attains its minimum value, which is equal to zero, at the point $\mathbf{a} = \mathbf{b}$.

□

Lemma 7.2. *Let $a, b \geq 0$. Then there exists a constant $c = c(p) > 0$ such that*
(i) if $1 < p \leq 2$ then

$$(a^2 + b^2)^{\frac{p}{2}} \leq a^p + c\ b^p.$$

(ii) if $p > 2$ then

$$(a^2 + b^2)^{\frac{p}{2}} \leq a^p + c\ (a^{p-2}b^2 + b^p).$$

Proof: For $1 < p \leq 2$ it is a consequence of

$$(t+1)^{\frac{p}{2}} \leq t^{\frac{p}{2}} + c, \qquad t \geq 0,$$

whereas for $p > 2$ this is a consequence of the inequality

$$(t+1)^{\frac{p}{2}} \leq t^{\frac{p}{2}} + c(t^{\frac{p}{2}-1} + 1), \qquad t \geq 0.$$

In both cases we set $t = a^2/b^2$.

□

We next have

Proof of Theorem 1.4: Part (i). In this case either $c(p,n,s) = \left(\frac{s-1}{p}\right)^p$ by Theorem 4.7 and Corollary 4.12, in which case there exists a radial positive solution of the Euler Lagrange behaving like $\phi(r) \sim (r-1)^{\frac{s-1}{p}}$ near $r = 1$ cf (4.14) or else $c(p,n,s) = \left(\frac{n-s}{p}\right)^p$, $\frac{n+1}{2} < s < n$, by Theorem 6.1, in which case here exists a radial positive solution of the Euler Lagrange behaving like $\phi(r) \sim (r-1)^{-\frac{n-s}{p}}$ at infinity, cf (6.5). In both cases ϕ is not in the proper energy space, as can easily be checked using the asymptotics near one in the first case and near infinity in the second case. In both cases we will establish that there exist no other energetic minimizer.

We may assume that $c(p,n,s) = \left(\frac{s-1}{p}\right)^p$, the other case being quite similar. Then ϕ satisfies

$$\left(\frac{r^{n-1}|\phi'|^{p-2}\phi'}{(r-1)^{s-p}}\right)' + \left(\frac{s-1}{p}\right)^p \frac{r^{n-1}|\phi|^{p-2}\phi}{(r-1)^s} = 0, \quad r > 1. \tag{7.9}$$

Suppose on the contrary, that there exists an energetic positive smooth function f, not necessarily radial, that satisfies (4.11)

$$\int_{\bar{B}_1^c} \frac{|\nabla f|^p}{(|x|-1)^{s-p}}\,dx = \left(\frac{s-1}{p}\right)^p \int_{\bar{B}_1^c} \frac{|f|^p}{(|x|-1)^s}\,dx. \tag{7.10}$$

In addition it solves the Euler Lagrange equation (2.1), that is

$$\nabla\cdot\left(\frac{|\nabla f|^{p-2}\nabla f}{(|x|-1)^{s-p}}\right)+\left(\frac{s-1}{p}\right)^p\frac{|f|^{p-2}f}{(|x|-1)^s}=0,\qquad x\in\bar{B}_1^c.$$

We multiply (7.9) by f^p/ϕ^{p-1} and integrate over $B_R\setminus B_\rho$, $1<\rho<R$, to get

$$\int_{B_R\setminus B_\rho}\frac{f^p}{\phi^{p-1}}\nabla\left(\frac{|\nabla\phi|^{p-2}\nabla\phi}{(|x|-1)^{s-p}}\right)dx+\left(\frac{s-1}{p}\right)^p\int_{B_R\setminus B_\rho}\frac{f^p}{(|x|-1)^s}dx=0.$$

After an integration by parts we arrive at

$$\begin{aligned}&-p\int_{B_R\setminus B_\rho}\frac{f^{p-1}\nabla f}{\phi^{p-1}}\cdot\frac{\nabla\phi|\nabla\phi|^{p-2}}{(|x|-1)^{s-p}}dx\\&+(p-1)\int_{B_R\setminus B_\rho}\frac{f^p}{\phi^p}\frac{|\nabla\phi|^p}{(|x|-1)^{s-p}}dx+\left(\frac{s-1}{p}\right)^p\int_{B_R\setminus B_\rho}\frac{f^p}{(|x|-1)^s}dx\\&+\frac{(R-1)^{p-1}|\phi'(R)|^{p-2}\phi'(R)}{\phi^{p-1}(R)}\;(R-1)\int_{\partial B_R}\frac{|f|^p}{(|x|-1)^s}dS_x\\&-\frac{(\rho-1)^{p-1}|\phi'(\rho)|^{p-2}\phi'(\rho)}{\phi^{p-1}(\rho)}\;(\rho-1)\int_{\partial B_\rho}\frac{|f|^p}{(|x|-1)^s}dS_x=0\end{aligned}\tag{7.11}$$

We will use the fact that the quantity

$$\frac{(r-1)^{p-1}|\phi'(r)|^{p-2}\phi'(r)}{\phi^{p-1}(r)},$$

is uniformly bounded for $r\in(1,+\infty)$ by Theorem 4.11. On the other hand since

$$\int_{B_1^c}\frac{f^p}{(|x|-1)^s}dx<\infty,$$

there exists subsequences $R_j\to+\infty$, $\rho_j\to1^+$ such that

$$\lim(R_j-1)\int_{\partial B_{R_j}}\frac{|f|^p}{(|x|-1)^s}dS_x=0,\qquad\lim(\rho_j-1)\int_{\partial B_{\rho_j}}\frac{|f|^p}{(|x|-1)^s}dS_x=0.$$

Because of these, the last two integrals of (7.11) tend to zero along the subsequences R_j and ρ_j. The remaining three integrals are finite. Indeed, the third integral is clearly finite in B_1^c, the second integral is dominated by the third one, and for the first one we note that

$$\begin{aligned}\left|\int_{B_R\setminus B_\rho}\frac{f^{p-1}\nabla f}{\phi^{p-1}}\cdot\frac{\nabla\phi|\nabla\phi|^{p-2}}{(|x|-1)^{s-p}}dx\right|&\le\int_{B_R\setminus B_\rho}\frac{f^{p-1}|\nabla f|}{\phi^{p-1}}\frac{|\nabla\phi|^{p-1}}{(|x|-1)^{s-p}}dx\\=\int_{B_R\setminus B_\rho}\frac{|\nabla f|}{(|x|-1)^{\frac{s-p}{p}}}\frac{f^{p-1}}{(|x|-1)^{\frac{s(p-1)}{p}}}\frac{|\nabla\phi|^{p-1}(|x|-1)^{p-1}}{\phi^{p-1}}dx&\\\le c\int_{B_R\setminus B_\rho}\frac{|\nabla f|}{(|x|-1)^{\frac{s-p}{p}}}\frac{f^{p-1}}{(|x|-1)^{\frac{s(p-1)}{p}}}dx&\\\le c\left(\int_{\bar{B}_1^c}\frac{|\nabla f|^p}{(|x|-1)^{s-p}}dx\right)^{\frac{1}{p}}\left(\int_{\bar{B}_1^c}\frac{|f|^p}{(|x|-1)^s}dx\right)^{\frac{p-1}{p}}&<+\infty.\end{aligned}$$

Therefore we can pass to the limit in (7.11) to obtain

$$-p\int_{B_1^c}\frac{f^{p-1}\nabla f}{\phi^{p-1}}\cdot\frac{\nabla\phi|\nabla\phi|^{p-2}}{(|x|-1)^{s-p}}dx+(p-1)\int_{B_1^c}\frac{f^p}{\phi^p}\frac{|\nabla\phi|^p}{(|x|-1)^{s-p}}dx$$
$$+\left(\frac{s-1}{p}\right)^p\int_{B_1^c}\frac{f^p}{(|x|-1)^s}dx=0.$$

In view of (7.10) we also have

$$-p\int_{B_1^c}\frac{f^{p-1}\nabla f}{\phi^{p-1}}\cdot\frac{\nabla\phi|\nabla\phi|^{p-2}}{(|x|-1)^{s-p}}dx+(p-1)\int_{B_1^c}\frac{f^p}{\phi^p}\frac{|\nabla\phi|^p}{(|x|-1)^{s-p}}dx$$
$$+\int_{\bar{B}_1^c}\frac{|\nabla f|^p}{(|x|-1)^{s-p}}\,dx=0,$$

or equivalently

$$\int_{B_1^c}\frac{1}{(|x|-1)^{s-p}}\left(|\nabla f|^p-pf^{p-1}\nabla f\cdot\frac{\nabla\phi|\nabla\phi|^{p-2}}{\phi^{p-1}}+(p-1)f^p\left|\frac{\nabla\phi|}{\phi}\right|^p\right)dx=0.$$

We next use Lemma 7.1, with $\mathbf{a}=\nabla f$ and $\mathbf{b}=\frac{f}{\phi}\nabla\phi$, to conclude that $\frac{\nabla f}{f}=\frac{\nabla\phi}{\phi}$ and therefore $f=k\phi$ for some constant k. But this a contradiction, since ϕ is not energetic.

Part (ii). In case $s_1(p,n)<s\le\frac{n+1}{2}$, combining Theorem 5.2 and Theorem 3.1(iii) we conclude the existence of a radial positive energetic minimizer ϕ that corresponds to the best constant $c(p,n,s)<\left(\frac{s-1}{p}\right)^p$. In the case $\frac{n+1}{2}<s<s_2(p,n)$ the existence of a radial positive energetic minimizer ϕ follows by combining Theorem 1.6 and Lemma 6.4. The simplicity of the minimizer follows by an argument similar and simpler to the one used in part (i).

□

We finally have

Proof of Theorem 1.5: Integrating (1.5) in the y-variables we obtain the inequality in $\bar{B}_1^c\times\mathbb{R}^m$. We next establish the optimality of the constant.

Since $c(p,n,s)$ is the best constant, given any $\varepsilon>0$ there exists $\eta(x)\in C_c^\infty(\bar{B}_1^c)$ such that

$$c(p,n,s)\le\frac{\int_{\bar{B}_1^c}\frac{|\nabla\eta|^p}{(|x|-1)^{s-p}}dx}{\int_{\bar{B}_1^c}\frac{|\eta|^p}{(|x|-1)^s}dx}\le c(p,n,s)+\varepsilon.$$

We also consider a function $\psi(y)\in C_c^\infty(B_R)$, $B_R\subset\mathbb{R}^m$. Function ψ will be suitably chosen in the sequel. Then $\eta\psi\in C_c^\infty(B_1^c\times B_R)$. We initially consider the case $p>2$ and we use the inequality of Lemma 7.2(ii),

$$\frac{\int_{\mathbb{R}^m}\int_{\bar{B}_1^c}\frac{|\nabla_{(x,y)}|\eta\psi|^p}{(|x|-1)^{s-p}}\,dx\,dy}{\int_{\mathbb{R}^m}\int_{\bar{B}_1^c}\frac{|\eta\psi|^p}{(|x|-1)^s}\,dx\,dy}=\frac{\int_{B_R}\int_{\bar{B}_1^c}\frac{(|\nabla_x\eta(x)|^2\psi^2(y)+\eta^2(x)|\nabla_y\psi(y)|^2)^{\frac{p}{2}}}{(|x|-1)^{s-p}}\,dx\,dy}{\int_{\bar{B}_1^c}\frac{|\eta|^p(x)}{(|x|-1)^s}\,dx\cdot\int_{B_R}|\psi|^p(y)\,dy}$$
$$\le\frac{\int_{\bar{B}_1^c}\frac{|\nabla\eta|^p}{(|x|-1)^{s-p}}dx}{\int_{\bar{B}_1^c}\frac{|\eta|^p}{(|x|-1)^s}dx}+c(p)\frac{\int_{\bar{B}_1^c}\frac{\eta^2|\nabla\eta|^{p-2}}{(|x|-1)^{s-p}}dx}{\int_{\bar{B}_1^c}\frac{|\eta|^p}{(|x|-1)^s}dx}\cdot\frac{\int_{B_R}|\psi|^{p-2}|\nabla_y\psi|^2\,dy}{\int_{B_R}|\psi|^p\,dy}$$
$$+c(p)\frac{\int_{\bar{B}_1^c}\frac{|\eta|^p}{(|x|-1)^{s-p}}dx}{\int_{\bar{B}_1^c}\frac{|\eta|^p}{(|x|-1)^s}dx}\cdot\frac{\int_{B_R}|\nabla_y\psi|^p\,dy}{\int_{B_R}|\psi|^p\,dy}.$$

Now by Holder's inequality we have

$$\frac{\int_{B_R} |\psi|^{p-2} |\nabla_y \psi|^2 \, dy}{\int_{B_R} |\psi|^p \, dy} \le \left(\frac{\int_{B_R} |\nabla_y \psi|^p \, dy}{\int_{B_R} |\psi|^p \, dy} \right)^{\frac{2}{p}} \tag{7.12}$$

By choosing R large enough and ψ close to the first Dirichlet eigenfunction in B_R, we make the right hand side of (7.12) as small as we like. Hence with these choices we end up with

$$c(p,n,s) \le \frac{\int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)} |\eta\psi|^p}{(|x|-1)^{s-p}} \, dx \, dy}{\int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|\eta\psi|^p}{(|x|-1)^s} \, dx \, dy} \le c(p,n,s) + 2\varepsilon,$$

which shows the optimality of the constant. The case where $1 < p \le 2$ is similar and simpler.

It remains to show the non existence of minimizers. We first recall that for all values of the parameters there always exist a radial positive smooth solution $\phi(r)$, $r = |x|$, of te Euler Lagrange equation

$$\left(\frac{r^{n-1} |\phi'|^{p-2} \phi'}{(r-1)^{s-p}} \right)' + c(p,n,s) \, \frac{r^{n-1} |\phi|^{p-2} \phi}{(r-1)^s} = 0, \quad r > 1.$$

In addition we recall that

$$\frac{(r-1)^{p-1} |\phi'(r)|^{p-2} \phi'(r)}{\phi^{p-1}(r)},$$

is uniformly bounded for $r \in (1, +\infty)$.

Suppose now that there exists a positive minimizer $f(x,y)$ that realizes the best constant $c(p,n,s)$ that is

$$\int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)} f|^p}{(|x|-1)^{s-p}} \, dxdy = c(p,n,s) \int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|f|^p}{(|x|-1)^s} \, dxdy. \tag{7.13}$$

In addition it solves the Euler Lagrange equation (2.1), that is

$$\nabla_{(x,y)} \cdot \left(\frac{|\nabla_{(x,y)} f|^{p-2} \nabla f}{(|x|-1)^{s-p}} \right) + c(p,n,s) \, \frac{|f|^{p-2} f}{(|x|-1)^s} = 0, \qquad (x,y) \in \bar{B}_1^c \times \mathbb{R}^m.$$

We multiply (7.13) by f^p / ϕ^{p-1} and integrate over $(B_R \setminus B_\rho) \times \mathbb{R}^m$, $1 < \rho < R$. Working as in the proof of Theorem 1.4 we eventually arrive at

$$\int_{\mathbb{R}^m} \int_{B_1^c} \frac{1}{(|x|-1)^{s-p}} \left(|\nabla f|^p - p f^{p-1} \nabla f \cdot \frac{\nabla\phi |\nabla \phi|^{p-2}}{\phi^{p-1}} + (p-1) f^p \left| \frac{\nabla \phi|}{\phi} \right|^p \right) dxdy = 0;$$

here $\nabla f = \nabla_{(x,y)} f$ and $\nabla \phi = (\nabla_x \phi, 0)$. We note that all integrals are well defined, by similar arguments as in the proof of Theorem 1.4. Applying Lemma (7.1), we conclude that $f(x,y) = k\phi(|x|)$. However, since ϕ is independent of y it is not in the energy space in $\bar{B}_1^c \times \mathbb{R}^m$.

□

By essentially the same arguments as in the proof of Theorem 1.5, one can prove the following more general result. If Ω is a proper subset of $\mathbb{R}^n$ and $C_\Omega(p,n,s) > 0$ is the best constant of the inequality

$$\int_\Omega \frac{|\nabla u|^p}{d^{s-p}} dx \ge C_\Omega(p,n,s) \int_\Omega \frac{|u|^p}{d^s} dx, \qquad \forall u \in C_c^\infty(\Omega), \tag{7.14}$$

then

Theorem 7.3. *For $p > 1$, $n \geq 2$, $m \geq 1$ and $s > 1$, the following Hardy inequality holds true*

$$\int_{\mathbb{R}^m}\int_{\Omega} \frac{|\nabla_{(x,y)}u(x,y)|^p}{(|x|-1)^{s-p}}\,dx\,dy \geq C_\Omega(p,n,s)\int_{\mathbb{R}^m}\int_{\Omega} \frac{|u|^p(x,y)}{(|x|-1)^s}\,dxdy, \quad \forall u \in C_c^\infty(\Omega\times\mathbb{R}^m),$$

where the constant $C_\Omega(p,n,s)$ is the same as in (7.14) and is sharp.

We note in particular that $C_\Omega(p,n,s)$ depends on the space dimension n of Ω and not on the space dimension $m+n$ of $\Omega\times\mathbb{R}^m$.

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