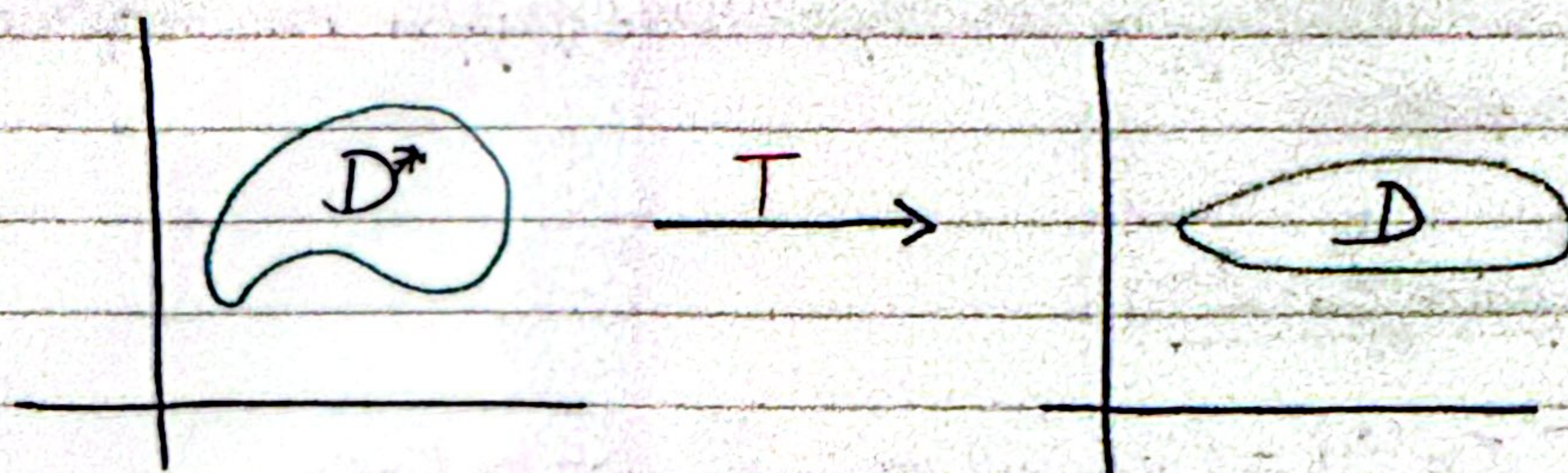


Διάλεξη 5

06/10/2026

ΑΠΕΙΚΟΝΙΣΤΙΣ ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$)

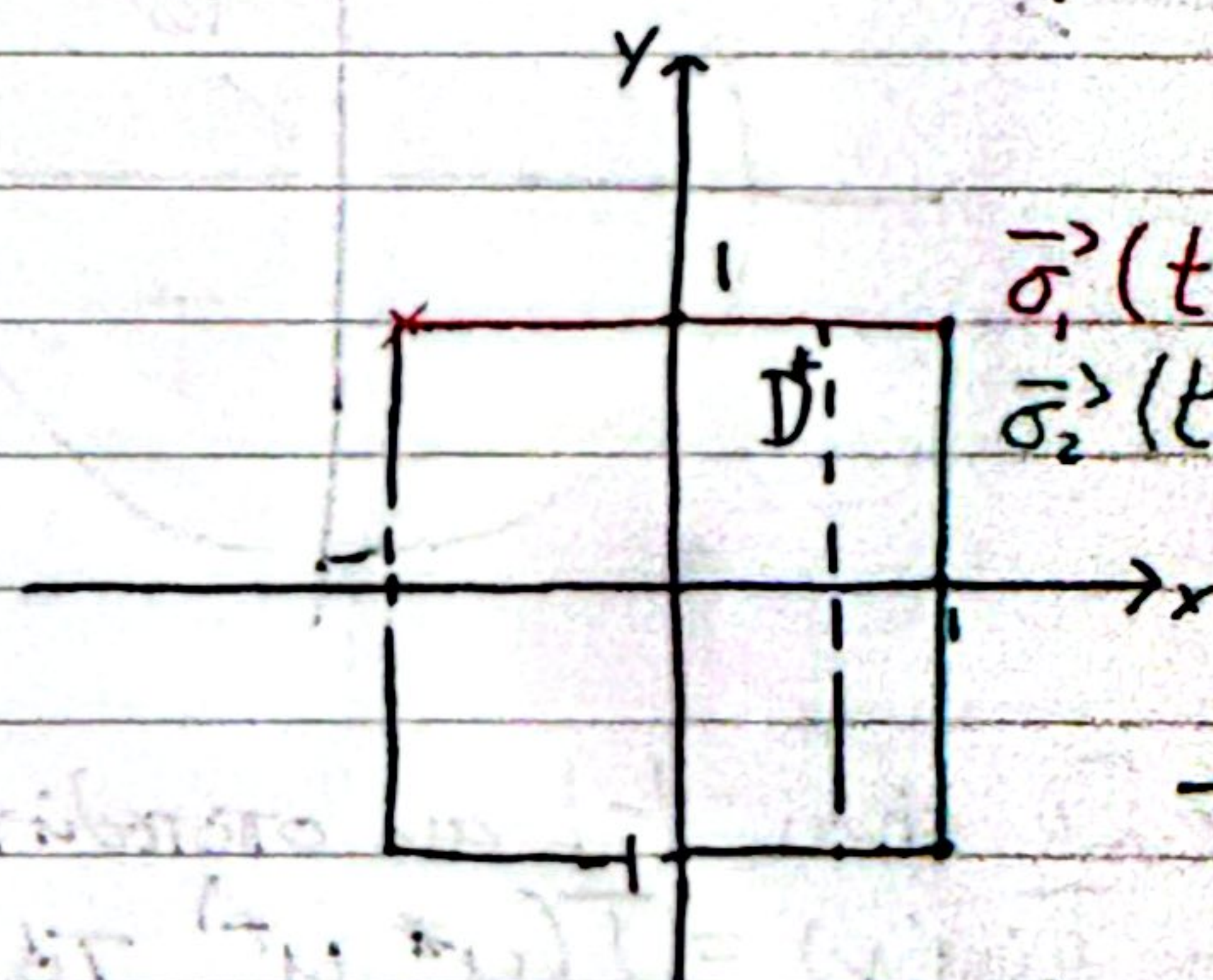
$$T: \mathbb{R}^2 \supset D^* \rightarrow D \subset \mathbb{R}^2$$



Παράδειγμα 1

$$T(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right), \quad T(D^*) = ?$$

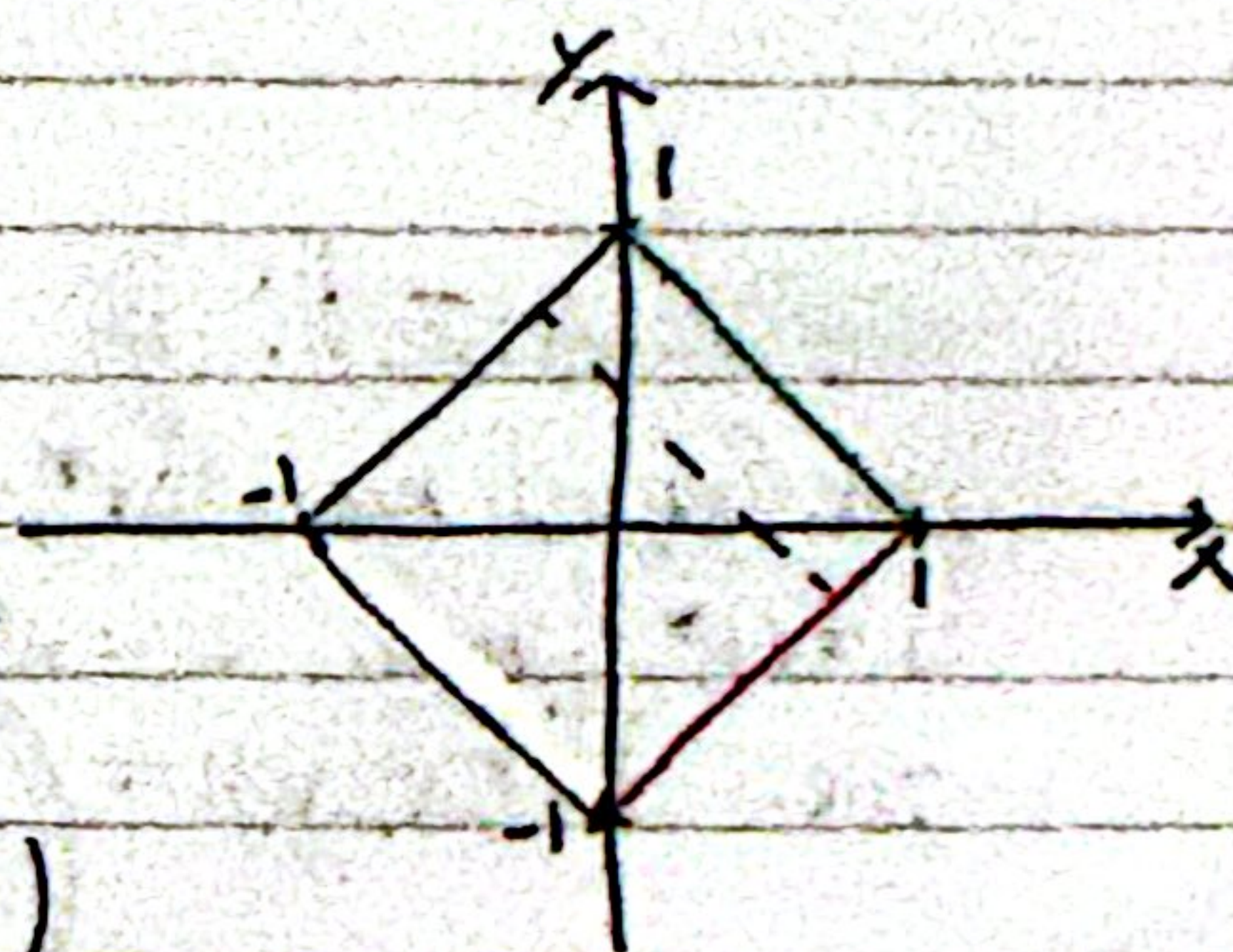
$$D^* = [-1, 1] \times [-1, 1]$$



$$\vec{\sigma}_1(t) = (t, 1), \quad -1 \leq t \leq 1$$

$$\vec{\sigma}_2(t) = (1, t), \quad -1 \leq t \leq 1$$

$\xrightarrow{T}$



$$T(\vec{\sigma}(t)) = \left(\frac{t+1}{2}, \frac{t-1}{2} \right), \quad -1 \leq t \leq 1$$

$$T(1, -1) = \left(\frac{1-1}{2}, \frac{1-(-1)}{2} \right) = (0, 1)$$

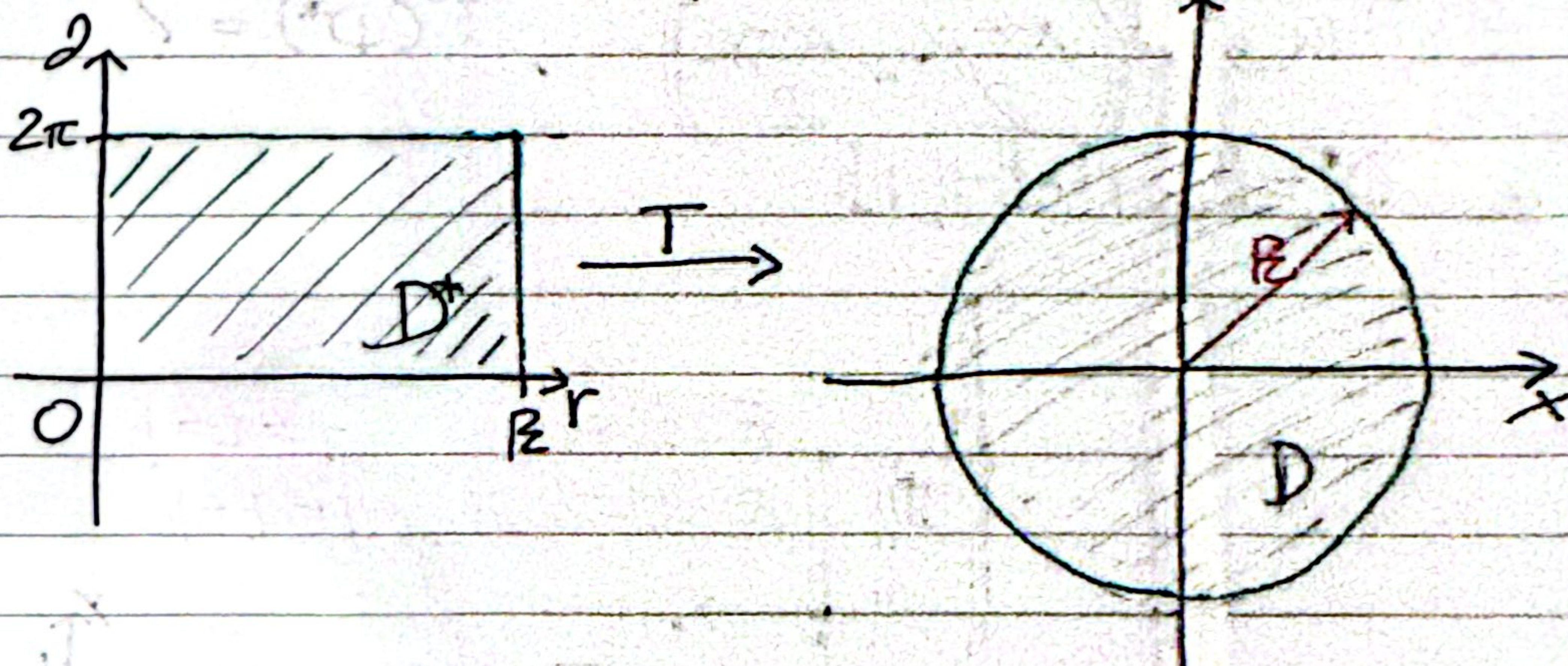
ΘΕΩΡΗΜΑ

Αν $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ - γραμμική απεικόνιση που περιγράφεται απ' τον πίνακα A , $\det A \neq 0$, τότε η T στελνει παράλληλόγραφα σε παράλληλόγραφα και κορυφές σε κορυφές

Παράδειγμα 2

$$T(r, \vartheta) = (\underset{x}{r \cos \vartheta}, \underset{y}{r \sin \vartheta}) \leftarrow \text{πολικές}$$

$$\text{Έστω } D^* = \{0 \leq \vartheta \leq 2\pi, 0 \leq r \leq R\}$$



ΟΡΙΣΜΟΣ 1: Η $T: D^* \rightarrow D$ είναι 1-1 αν οποτεδήποτε για (u, v) και (u^*, v^*) έχουμε $T(u, v) = T(u^*, v^*)$ τότε $u = u^*$ και $v = v^*$

Πχ $T(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right)$ είναι 1-1

"Απόδειξη"

$$\text{Έστω ότι } T(x, y) = T(x', y') \Leftrightarrow \frac{x+y}{2} = \frac{x'+y'}{2},$$

$$\frac{x-y}{2} = \frac{x'-y'}{2}$$

$\leadsto$ πρόσθεση: $x = x'$

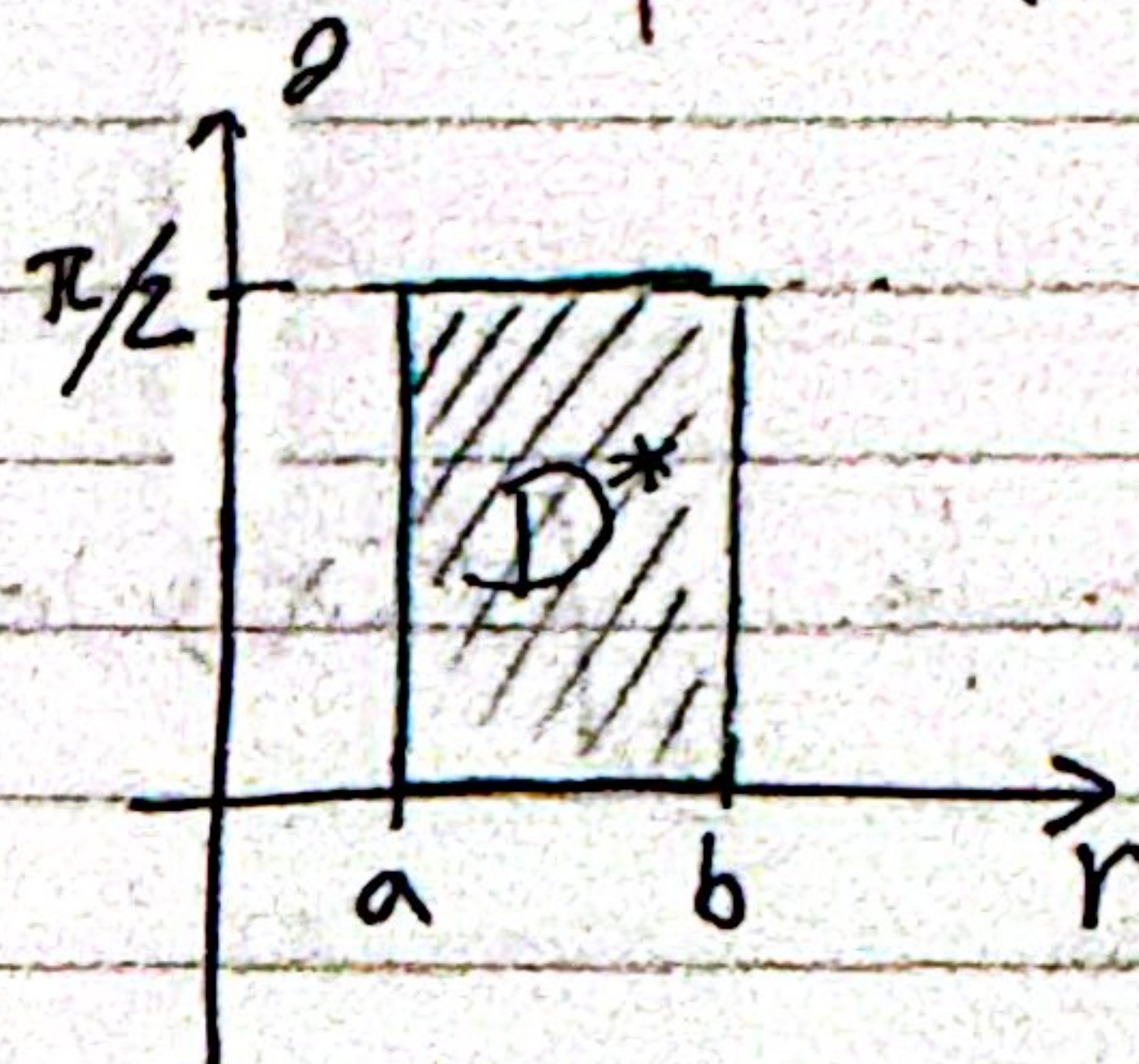
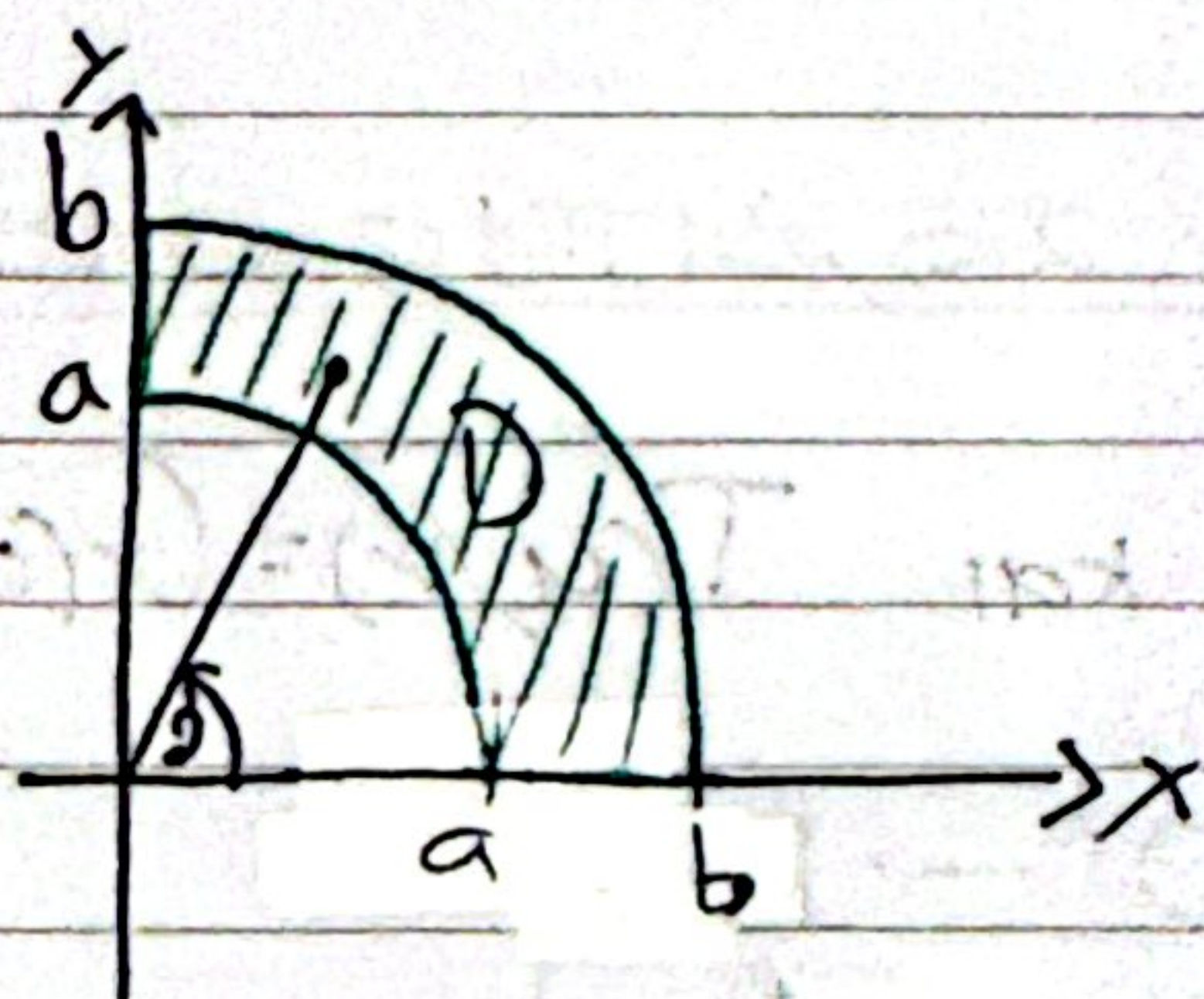
$\leadsto$ αφαίρεση: $y = y'$

ΟΡΙΣΜΟΣ 2: Η $T: D^* \rightarrow D$ είναι επί του D αν $\forall (x,y) \in D \exists (u,v) \in D^*$ τέω $T(u,v) = (x,y)$

Παράδειγμα 1

$$D = \left\{ (x,y) : a \leq \sqrt{x^2+y^2} \leq b, \begin{matrix} x > 0 \\ y > 0 \end{matrix} \right\} \quad T(r,\theta) = (r\cos\theta, r\sin\theta)$$

Βρείτε D^* τέω $T(D^*) = D$ | $\sqrt{x^2+y^2} = r$



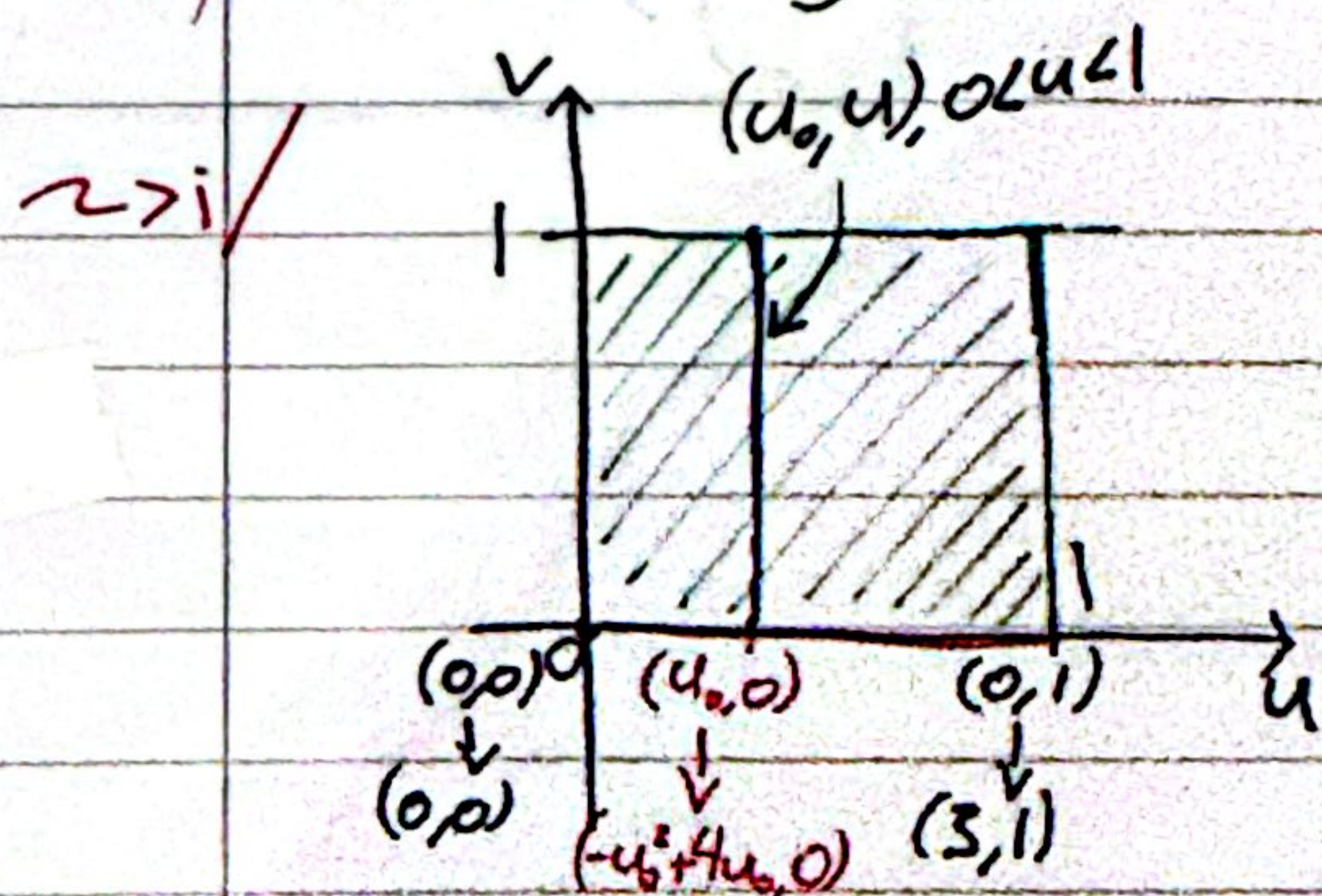
Η $T: D^* \rightarrow D$ είναι 1-1 και επί

Παράδειγμα 2

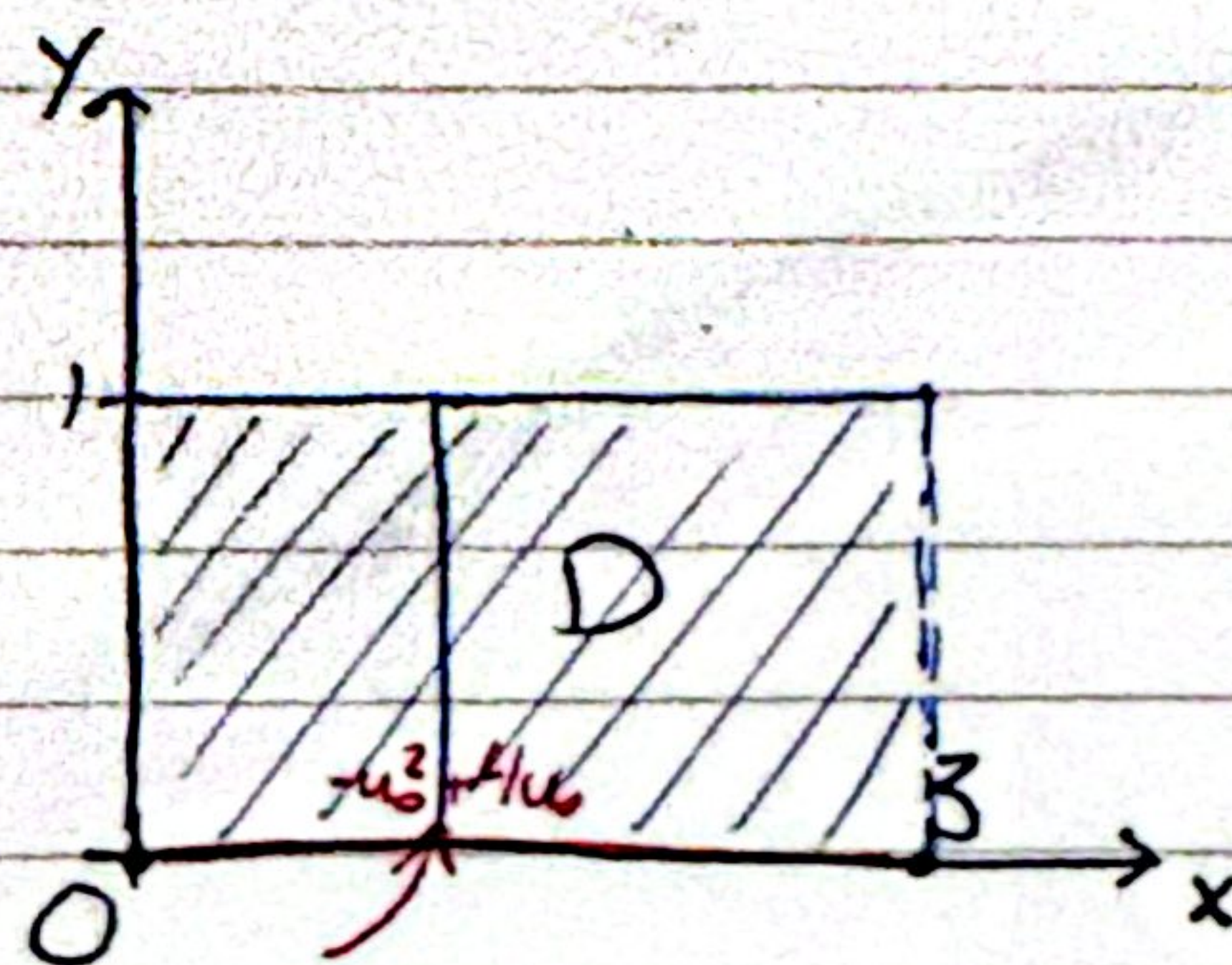
$$D^* = [0,1] \times [0,1], \quad T(u,v) = (-u^2+4u, v)$$

i// Βρείτε $T(D^*) = D$

ii// Είναι 1-1;



$\xrightarrow{T}$



$0 \leq u_0 \leq 1$, $0 \leq -u^2+4u \leq 3$ άρα η "κάτω" πλευρά

πάει στο διάστημα $0 \leq x \leq 3$

$$T(u,v) = (-u^2+4u, v), \quad 0 \leq v \leq 1 \quad \therefore T(D^*) = D = [0,3] \times [0,1]$$

~> ii/ Με τον ορισμό

$$\text{Έστω } T(u, v) = T(u', v') \Leftrightarrow \begin{cases} -u^2 + 4u = -u'^2 + 4u' \\ v = v' \end{cases}$$

$$\leadsto u'^2 - u^2 + 4u - 4u' = 0 \Leftrightarrow (u' - u)(u' + u) - 4(u' - u) = 0 \Leftrightarrow$$

$$\Leftrightarrow (u' - u)(u' + u - 4) = 0 \Leftrightarrow u' = u \text{ ή } u' + u = 4 \text{ απορ.}$$

∴ η T είναι 1-1

$$\begin{aligned} 0 \leq u' \leq 1 \\ 0 \leq u \leq 1 \\ \Rightarrow 0 \leq u' + u \leq 2 \end{aligned}$$

ΑΜΑΤΗ ΜΕΤΑΒΛΗΤΩΝ ΣΕ ΔΙΤΙΟ ΟΛΟΚΛΗΡΩΜΑ

Έχετε $\int_D f(x, y) dx dy$ και $T(u, v) = (x(u, v), y(u, v))$

είναι 1-1 και επί $T: D^* \rightarrow D$

$$\int_D f(x, y) dx dy = \int_D f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

ακρωβανή
ορίζουσα

Στη 1η μεταβλητή

$$I = \int_a^b f(x) dx$$

$$x = x(u) \Rightarrow dx = x'(u) du$$

$$A = x(a)$$

$$B = x(b)$$

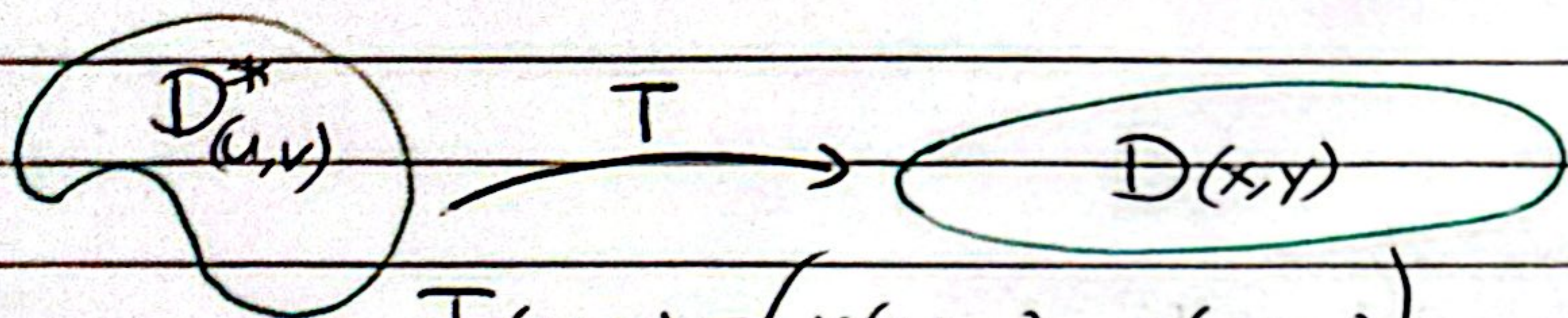
$$I = \int_a^b f(x(u)) x'(u) du$$

Διάλεξη 6

ΑΜΑΤΗ ΜΕΤΑΒΛΗΤΩΝ

08/10/2026

$$T(D^*) = D$$



$$T(u,v) = (x(u,v), y(u,v))$$

$$f(x,y) \rightarrow f(x(u,v), y(u,v))$$

$$\iint_D f(x,y) dx dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

Ιακωβιανή Ορίζουσα: $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

Ιακωβιανή Πολικών συντεταγμένων (r, θ)

$$x = r \cos \theta$$

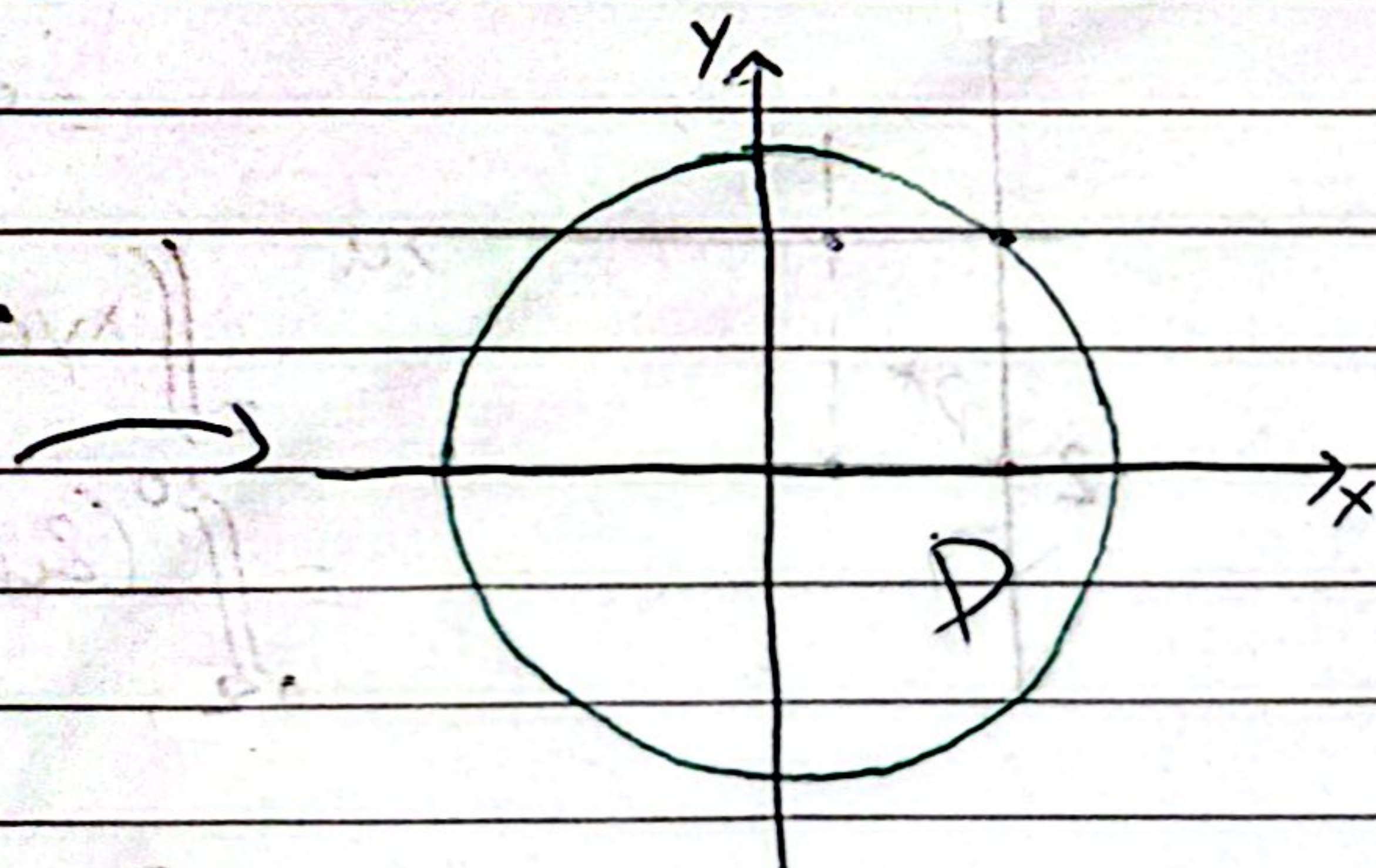
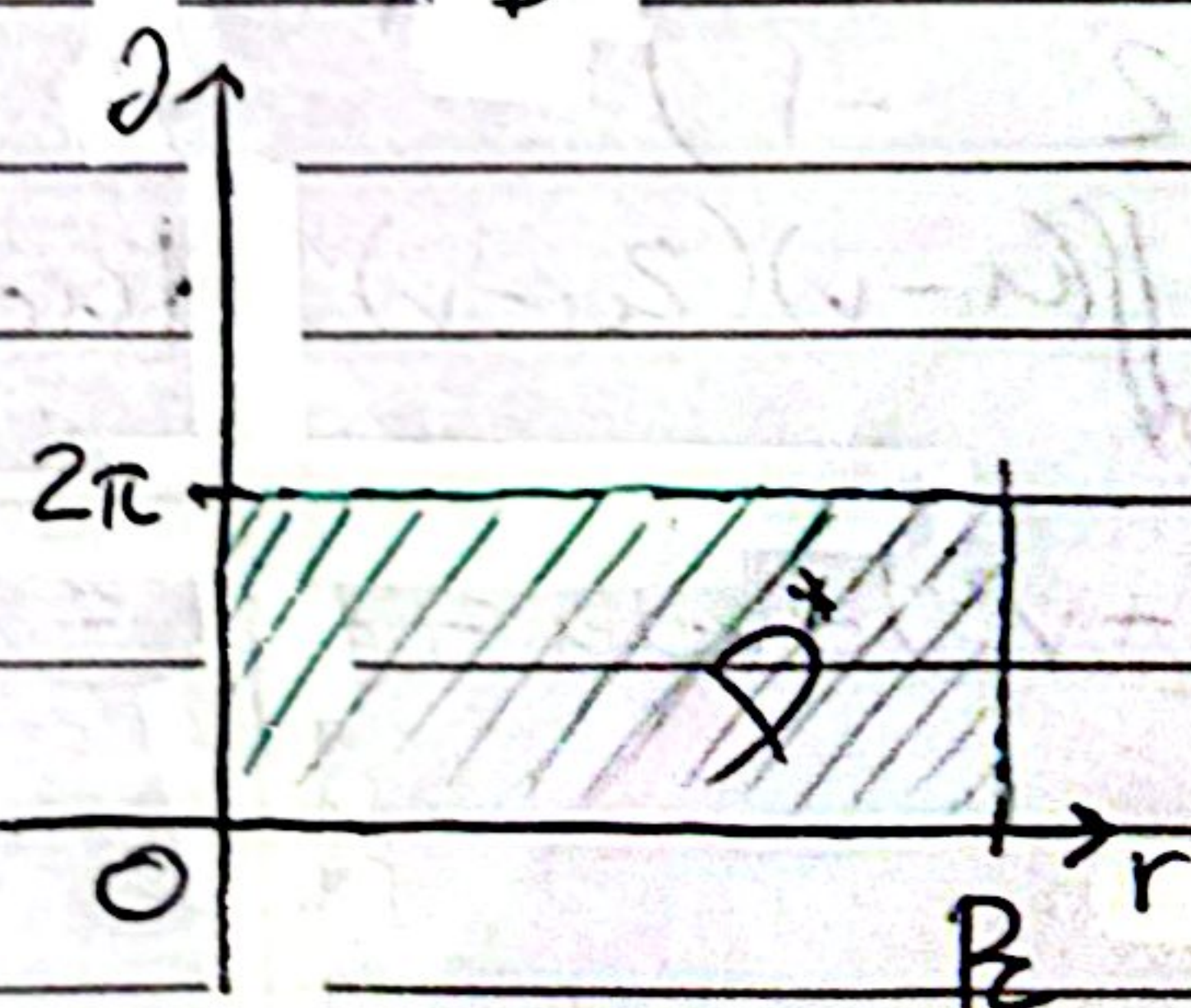
$$y = r \sin \theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

Παράδειγμα

Υπολογίστε το εμβαδόν κύκλου ακτίνας R . $D: \{(x,y): x^2 + y^2 \leq R^2\}$

$$\iint_D 1 dx dy = \iint_{D^*} 1 r dr d\theta = \iint 1 r d\theta dr$$



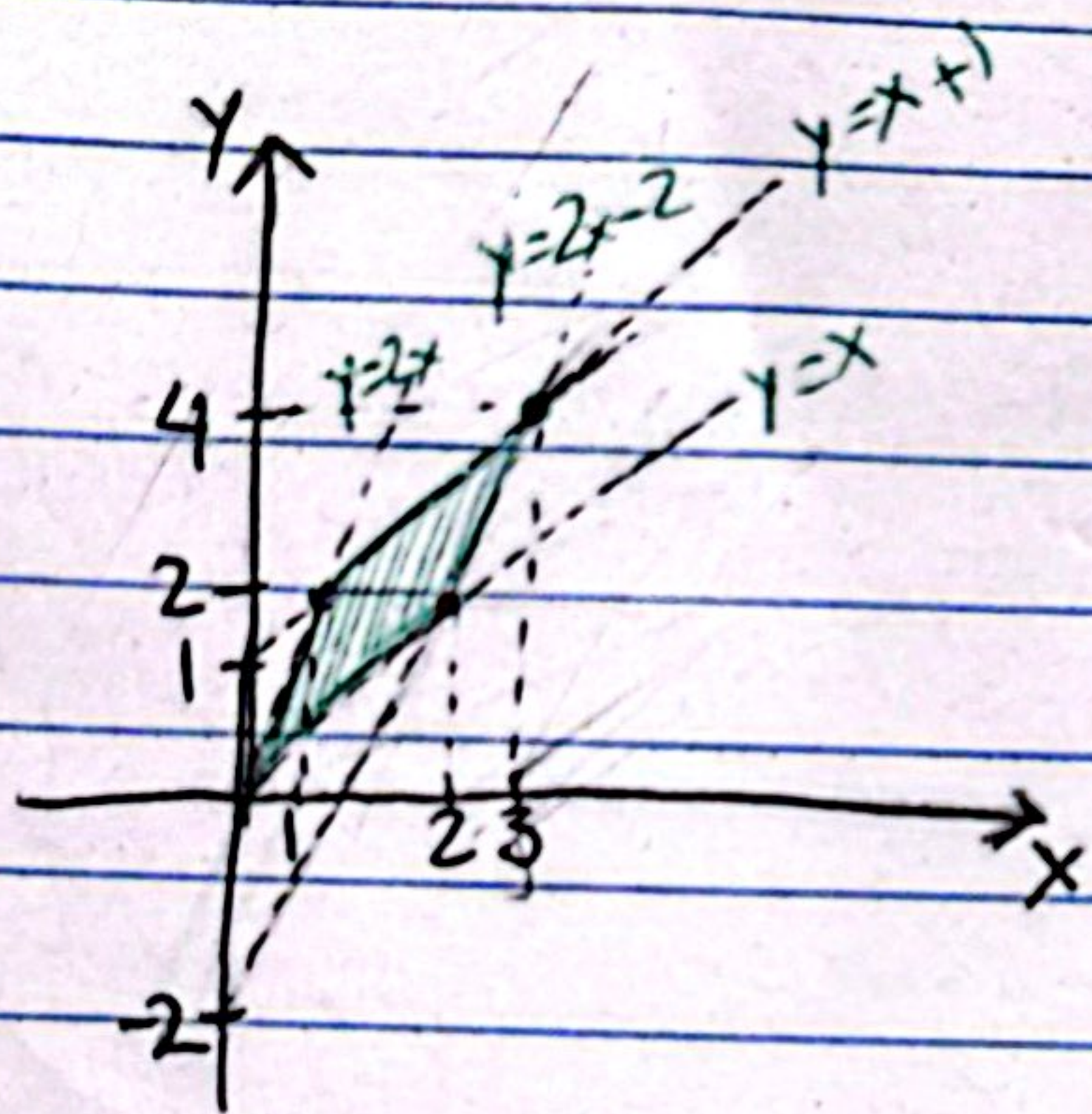
$$\therefore = 2\pi \int_0^R r dr = \pi R^2$$

Παράδειγμα 2

P - παραλληλόγραφο που φράσσεται από:

$$y = 2x, y = 2x - 2, y = x, y = x + 1$$

Υπολογίστε το $I = \int xy dx dy$. Κάπε την αλλαγή μεταβλητών $x = u - v$, $y = 2u - v$ (*)



$$\begin{aligned} y = 2x - 2 &\} \Rightarrow 2x - 2 = x + 1 \Leftrightarrow x = 3 \\ y = x + 1 &\} \text{ και } y = 4 \end{aligned}$$

$$\begin{aligned} y = 2x &\} \Rightarrow 2x = x + 1 \Leftrightarrow x = 1 \\ y = x + 1 &\} \text{ και } y = 2 \end{aligned}$$

Αντιστρέφουμε τις (*):

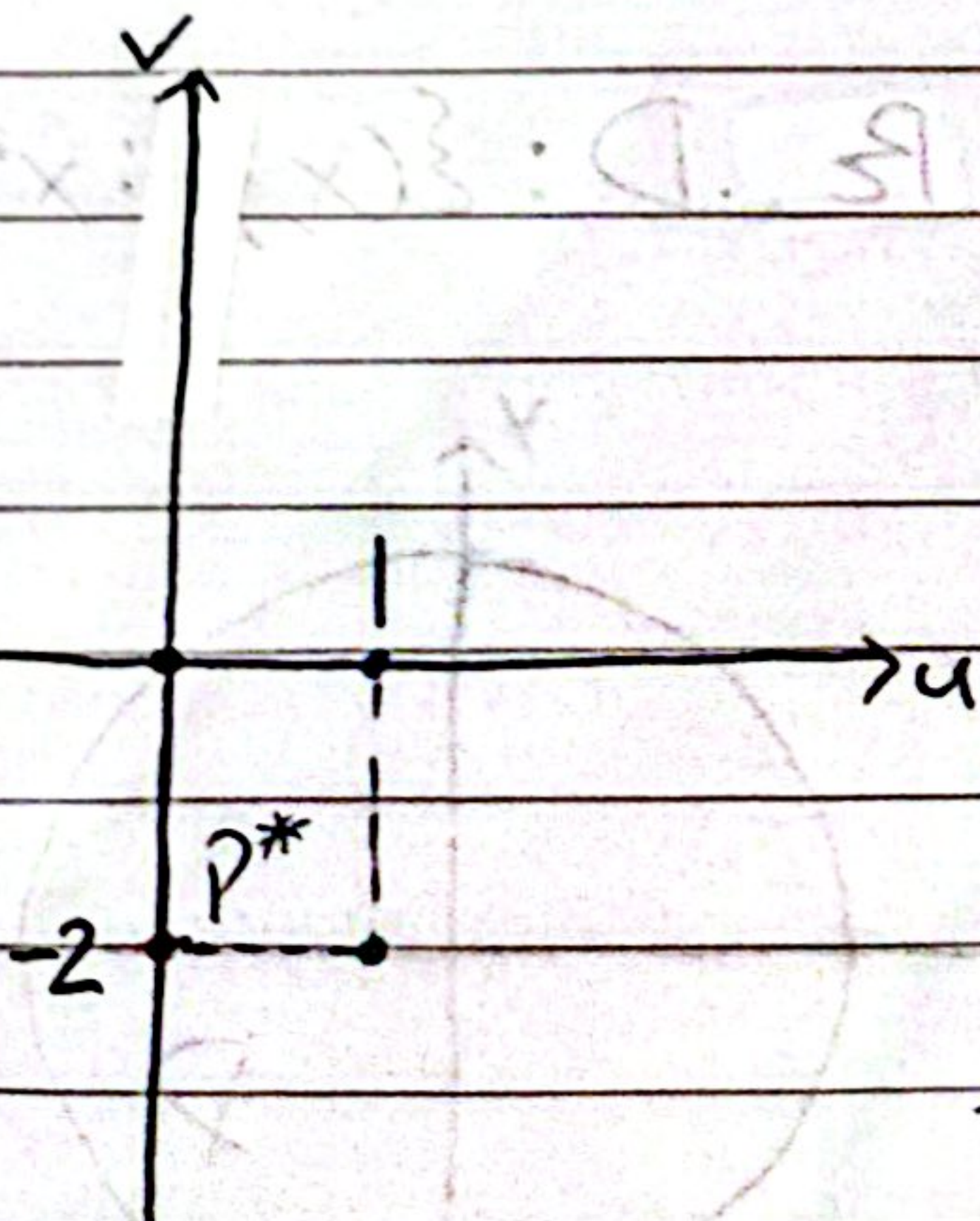
Αφαιρούμε απ'τη (2) την (1):

$$u = y - x \text{ οπότε}$$

$$(1) \Rightarrow v = u - x = y - 2x$$

$$\therefore (u, v) = T^{-1}(y - x, y - 2x)$$

Βρίσκουμε τώρα τις 4 κορυφές του P^*

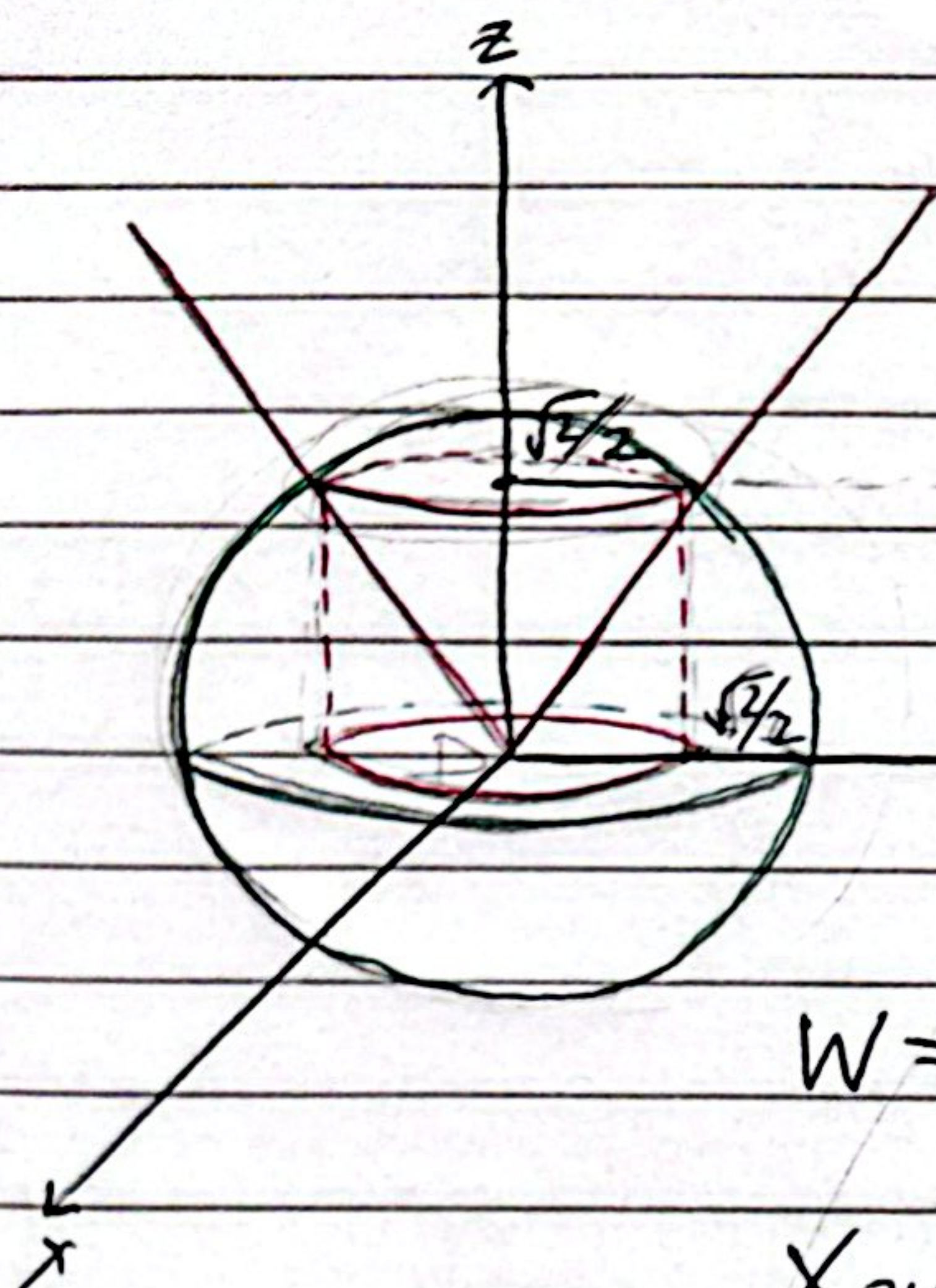


$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \\ &= \det \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = 1 \end{aligned}$$

$$\begin{aligned} \iint_P xy dx dy &= \iint_{P^*} (u - v)(2u - v) |du dv| = \\ &= \int_{-2}^0 \int_0^1 (2u^2 - 3uv + v^2) dv du = \dots = 7 \end{aligned}$$

Παράδειγμα 3

Βρείτε τον όγκο του στεγχείου μεταξύ $z^2 \geq \frac{1}{2}$, $x^2 + y^2 + z^2 = 1$ και $z \geq 0$



$$\left. \begin{aligned} \text{Την τομή } z^2 &= x^2 + y^2 \\ x^2 + y^2 + z^2 &= 1 \end{aligned} \right\} \Rightarrow 2(x^2 + y^2) = 1$$

$$x^2 + y^2 = \frac{1}{2}$$

$$W: \frac{\sqrt{2}}{2} \leq z \leq \sqrt{1 - x^2 - y^2}$$

$$D: 0 \leq x^2 + y^2 \leq \frac{1}{2}$$

$$W = \iiint_W 1 \, dv = \iint_D \int_{\frac{\sqrt{2}}{2}}^{\sqrt{1-x^2-y^2}} 1 \, dz \, dx \, dy = \iint_D \left(\sqrt{1-x^2-y^2} - \frac{\sqrt{2}}{2} \right) dx \, dy$$

Χρησιμοποιούμε πολικές: $\int_0^{2\pi} \int_0^{\sqrt{2}/2} \left(\sqrt{1-r^2} - \frac{\sqrt{2}}{2} \right) r \, dr \, d\theta =$

$$= 2\pi \int_0^{\sqrt{2}/2} \left(r(1-r^2)^{1/2} - \frac{\sqrt{2}}{2} r \right) dr = 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{\sqrt{2}}{4} r^2 \right]_0^{\sqrt{2}/2} = \dots$$

$$W_2: \begin{aligned} x^2 + y^2 + z^2 &\leq 1 \\ z^2 &\geq x^2 + y^2 \\ z &\geq 0 \end{aligned}$$

$$V_2 = \iiint_{W_2} 1 \, dv = \iiint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz \, dx \, dy = \iint_D (\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}) dx \, dy$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}/2} ((1-r^2)^{1/2} - r) r \, dr \, d\theta = 2\pi \int_0^{\sqrt{2}/2} (r(1-r^2)^{1/2} - r^2) dr = 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{r^3}{3} \right]_0^{\sqrt{2}/2} =$$

$$= \frac{2\pi}{3} \left(1 - \frac{\sqrt{2}}{2} \right)$$